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ON THE STRUCTURE AND ARITHMETIC OF FINITELY PRIMARY MONOIDS

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1. INTRODUCTION

T. Tamura initiated the investigations of commutative cancellative archimedian semigroups which were later called N-semigroups by M. Petrich (cf. [Pe, Ta] and the literature cited there). Motivated by problems in ring theory M. Satyanarayana studied commutative semigroups where all proper ideals are primary ([Sa, An]); he called these semigroups primary.

In order to generalize the classical concept of a divisor theory, F. Halter-Koch considered commutative cancellative monoids where all non-units are primary ([HK1]); he also used the notion primary monoid. In particular, one-dimensional noetherian domains allow a divisor homomorphism into a coproduct of primary monoids. This makes it possible to study the arithmetic of such domains by studying the arithmetic of primary monoids (cf. [HK2]). It was this application which was the main motivation for the present paper and in particular our arithmetical results should be seen in this respect.

First, we show that the concept of N-semigroups and both notions of primary semigroups coincide for commutative cancellative monoids. In section 3 we deal with the integral closure and complete integral closure of primary monoids. In [HK2] finitely primary monoids were defined as certain submonoids of finitely generated factorial monoids. In section 4 we derive a characterization of finitely primary monoids in terms of their complete integral closure (Theorem 1). This allows us to characterize integral domains whose multiplicative monoids are finitely primary (Theorem 2). The final section is devoted to the arithmetic of primary monoids. We show that finitely primary monoids have finite catenary degree (Theorem 3); as a consequence we obtain that the multiplicative monoid of invertible ideals of a one-dimensional noetherian domain $\mathfrak{o}$, whose integral closure is a finitely generated $\mathfrak{o}$-module, has finite catenary degree (Theorem 4).

2. Preliminaries

Throughout this paper, a monoid means a commutative cancellative semigroup with identity. In general monoids will be written multiplicatively except when we consider additive submonoids of $(\mathbb{N}^s, +)$. $\mathbb{N}_+$ denotes the set of positive integers and $\mathbb{N} = \mathbb{N}_+ \cup \{0\}$. For basic notions concerning commutative semigroups we refer to [Gi2; chapter I].

Let H be a monoid and $H^\times$ its group of invertible elements. H is said to be reduced if $H^\times = \{1\}$. A non-empty subset $I \subseteq H$ is called an ideal if $HI \subseteq I$; an ideal I is said to be proper if $I \neq H$. Let $I \subset H$ be a proper ideal. I is called *maximal*, if it is not contained in any proper ideal; I is said to be *prime*, if $a, b \in H$ and $ab \in I$ implies that $a \in I$ or $b \in I$. I is called *primary*, if $a, b \in H$, $ab \in I$ and $a \notin I$ implies that $b^n \in I$ for some $n \in \mathbb{N}_+$.

Obviously $H \setminus H^\times$ is the unique maximal ideal of H and $H \setminus H^\times$ is a prime ideal; by definition every prime ideal is primary.

If $I \subset H$ is an ideal, then

$$\sqrt{I} = \{a \in H \mid a^n \in I \text{ for some } n \in \mathbb{N}_+\}$$

is called the *radical* of I. The following facts may be proved as in commutative ring theory: let I be a proper ideal.

a) If I is primary, then $\sqrt{I}$ is prime.
b) If $\sqrt{I}$ is maximal, then I is primary.
c) $\sqrt{I}$ is the intersection of all prime ideals containing I.

Let H be a monoid; an element $q \in H$ is said to be *primary*, if the ideal qH is a primary ideal (equivalently: $q \notin H^\times$ and $a, b \in H$, $q \mid ab$ and $q \nmid a$ implies $q \mid b^n$ for some $n \in \mathbb{N}_+$). Obviously, every prime element is primary; however there are primary elements which are not irreducible and irreducible elements which are not primary.

Let H be a monoid and $\mathcal{Q}(H)$ a quotient group of H with $H \subseteq \mathcal{Q}(H)$. For a submonoid $S \subseteq H$ we define the congruence modulo S in H by

$$a \equiv b \bmod S \quad \text{if} \quad a^{-1}b \in \mathcal{Q}(S)$$

(or equivalently, $aS \cap bS \neq \emptyset$). We denote by H/S the factor monoid of H with respect to the congruence modulo S. In particular we set $H_{\mathrm{red}} = H/H^\times$.

A submonoid $S \subseteq H$ is called *divisor closed*, if $a \in H$, $b \in S$ and $a \mid b$ implies $a \in S$. Obviously, a proper subset $S \subset H$ is a divisor closed submonoid if and only if $H \setminus S$ is a prime ideal of H. We say that a submonoid $S \subseteq H$ is *saturated*, if $S = H \cap \mathcal{Q}(S)$. Obviously, every divisor closed submonoid is saturated.

Lemma 1. *Let H be a monoid with $H \neq H^\times$. Then the following conditions are equivalent:*

1. *$H^\times$ and H are the only divisor closed submonoids of H.*
2. *H has exactly one prime ideal.*
3. *All proper ideals are primary.*
4. *All non-units are primary.*
5. *If $a, b \in H$ and $b \notin H^\times$, then $a \mid b^n$ for some $n \in \mathbb{N}_+$.*

Proof. 1. $\Longrightarrow$ 2. This follows from the above remark.

2. $\Longrightarrow$ 3. Let $I \subset H$ be a proper ideal. Since $\sqrt{I}$ is the intersection of all prime ideals containing I, 2. implies that $\sqrt{I} = H \setminus H^\times$. $H \setminus H^\times$ is the unique maximal ideal of H and hence I is primary.

3. $\Longrightarrow$ 4. Obvious.

4. $\Longrightarrow$ 5. Let $a, b \in H$ and $b \notin H^\times$. Then ab is primary and $ab \mid ab$; since $ab \nmid a$, it follows that $ab \mid b^{n+1}$ and hence $a \mid b^n$.

5. $\Longrightarrow$ 1. Let $S \subseteq H$ be a divisor closed submonoid with $S \neq H^\times$. For $b \in S \setminus H^\times$ we have

$$\{a \in H \mid a \mid b^n \text{ for some } n \in \mathbb{N}_+\} \subseteq S \subseteq H$$

and the first set equals H by 5. □

Definition 1. A monoid H with $H \neq H^\times$, which satisfies the equivalent conditions of the previous Lemma is called *primary*.

Remark. Parts of the above Lemma may be found in [Sa; Theorem 2.11], [HK1; Theorem 1.8] and [Ge1; Proposition 3].

Let $\varphi \colon H \to D$ be a monoid homomorphism; φ is called a *divisor homomorphism* if $a, b \in H$ and $\varphi(a) \mid \varphi(b)$ implies $a \mid b$ (equivalently: $\varphi(H) \subseteq D$ is saturated and the induced homomorphism $\varphi_{\mathrm{red}} \colon H_{\mathrm{red}} \to D_{\mathrm{red}}$ is injective); furthermore φ gives rise to a unique group homomorphism $\mathcal{Q}(\varphi) \colon \mathcal{Q}(H) \to \mathcal{Q}(D)$ (cf. [G-HK; section 2]).

Lemma 2. *Let $\varphi \colon H \to D$ be a monoid homomorphism.*

1. *If H is primary and $\varphi(H) \neq \varphi(H)^\times$, then $\varphi(H)$ is primary.*

2. *If $\varphi(H)$ is primary and* $\operatorname{Ker} \mathcal{Q}(\varphi)H^\times/H^\times$ *a torsion group, then H is primary.*
3. *If D is primary, $H \ne H^\times$ and φ a divisor homomorphism, then H is primary.*
4. *H is primary if and only if H_{red} is primary.*

Proof. 1. Let $\varphi(a), \varphi(b) \in \varphi(H)$ be given with $a, b \in H$ and suppose that $\varphi(b) \notin \varphi(H)^\times$. Then $b \notin H^\times$ and hence there exists some $n \in \mathbb{N}_+$ with $a \mid b^n$ which implies that $\varphi(a) \mid \varphi(b)^n$.

2. Let $a, b \in H$ with $b \notin H^\times$ be given. First we verify that $\varphi(b) \notin \varphi(H)^\times$: assume to the contrary that $\varphi(b) \in \varphi(H)^\times$. Then there exists some $c \in H$ such that $1 = \varphi(b)\varphi(c) = \varphi(bc)$. Since $\operatorname{Ker} \mathcal{Q}(\varphi)H^\times/H^\times$ is a torsion group it follows that $(bc)^m \in H^\times$ for some $m \in \mathbb{N}_+$ and hence $b \in H^\times$, a contradiction.

Since $\varphi(H)$ is primary and $\varphi(b) \notin H^\times$, we have $\varphi(ac) = \varphi(b^n)$ for some $c \in H$ and some $n \in \mathbb{N}_+$. Therefore $\mathcal{Q}(\varphi)(acb^{-n}) = 1$ and thus $(acb^{-n})^m \in H^\times$ for some $m \in \mathbb{N}_+$ which implies that $a \mid b^{nm}$.

3. Let $a, b \in H$ with $b \notin H^\times$ be given. Then $\varphi(b) \notin D^\times$ and hence $\varphi(a) \mid \varphi(b)^n = \varphi(b^n)$ for some $n \in \mathbb{N}_+$ which implies that $a \mid b^n$.

4. Since the canonical homomorphism $\pi\colon H \to H_{\mathrm{red}}$ is a surjective divisor homomorphism, the assertion follows from 1. and 3. $\square$

3. Integral closure and complete integral closure

Let H be a monoid. The integral closure $\widetilde{H} \subseteq \mathcal{Q}(H)$ and the complete integral closure $\widehat{H} \subseteq \mathcal{Q}(H)$ are defined by

$$\widetilde{H} = \{x \in \mathcal{Q}(H) \mid x^n \in H \text{ for some } n \in \mathbb{N}_+\}$$

and

$$\widehat{H} = \{x \in \mathcal{Q}(H) \mid \text{there exists some } c \in H \text{ such that } cx^n \in H \text{ for all } n \in \mathbb{N}_+\}.$$

H is called *integrally closed*, if $H = \widetilde{H}$ and *completely integrally closed (c.i.c.)*, if $H = \widehat{H}$.

Obviously
$$H \subseteq \widetilde{H} = \widetilde{\widetilde{H}} \subseteq \widehat{H} = \widetilde{\widehat{H}} \subseteq \mathcal{Q}(H) \;;$$
in general we have $\widehat{H} \ne \widehat{\widehat{H}}$. If H is primary then $\widehat{\widehat{H}}$ is c.i.c. ([Gel; Theorem 4]).

Since $\mathcal{Q}(H_{\mathrm{red}}) = \mathcal{Q}(H)/H^\times$ we obtain

$$\widetilde{H_{\mathrm{red}}} = \widetilde{H}/H^\times$$

and

$$\widehat{H_{\mathrm{red}}} = \widehat{H}/H^\times.$$

Hence H is integrally closed (resp. c.i.c.) if and only if H_{red} is integrally closed (resp. c.i.c.).

It is well known that $H^\times = \widetilde{H}^\times \cap H$ (cf. [G-HK; Lemma 5.4]). For primary monoids we even have the following result:

Proposition 1. *Let H be a primary monoid. Then*

$$H^\times = \widetilde{H}^\times \cap H = \widehat{H}^\times \cap H.$$

Proof. Obviously we have $H^\times \subseteq \widetilde{H}^\times \cap H \subseteq \widehat{H}^\times \cap H$; to obtain the converse let $a \in \widehat{H}^\times \cap H$ be given. Then there exist a $b \in \widehat{H}$ with $ab = 1$ and a $c \in H$ such that $cb^n \in H$ for all $n \in \mathbb{N}_+$. Assume to the contrary that $a \notin H^\times$. Since H is primary, there is an $r \in \mathbb{N}_+$ with $c \mid a^r$. Hence $a^r b^n \in H$ for all $n \in \mathbb{N}_+$. Setting $n = r + 1$ we infer

$$a^r b^{r+1} = (ab)^r b = b \in H,$$

and thus $a \in H^\times$, a contradiction.

$\square$

Proposition 2. *Let H, S be monoids such that $H \subseteq S \subseteq \widetilde{H}$. Then H is primary if and only if S is primary.*

Proof. First suppose that H is primary and let $a, b \in S$ be given with $b \notin S^\times$. There is an $n \in \mathbb{N}_+$ such that $a^n, b^n \in H$. Since $b' \notin S^\times \cap H = \widetilde{H}^\times \cap H = H^\times$, it follows that $a^n \mid b^{nm}$ (in H) for some $m \in \mathbb{N}_+$ and hence $a \mid b^{nm}$ (in S).

Conversely suppose S to be primary and let $a \in H$ and $b \in H \setminus H^\times$. Then $b \notin H^\times = S^\times \cap H$, and therefore there is an $n \in \mathbb{N}_+$ such that $a \mid b^n$ (in S). Thus $a^{-1}b^n \in S$ and $a^{-m}b^{nm} \in H$ for some $m \in \mathbb{N}_+$, which implies that $a \mid a^m \mid b^{nm}$ (in H). $\square$

Remark. In [HK1; Theorem 3.1] it was proved that $\widetilde{H}$ is primary, if H is primary.

The next lemma relates the complete integral closure of two monoids to their conductor. Let H and D be monoids contained in some common group. The conductor $\mathfrak{f}_{D/H}$ of H in D is defined as

$$\mathfrak{f}_{D/H} = \{c \in H \mid cD \subseteq H\}.$$

If $\mathfrak{f}_{D/H} \neq \emptyset$, then $D \subseteq \mathcal{Q}(H)$ and $\mathfrak{f}_{D/H}$ is an ideal in H.

Lemma 3. *Let $H \subseteq D$ be monoids.*

1. *If $\mathfrak{f}_{D/H} \neq \emptyset$, then $\widehat{H} = \widehat{D}$.*
2. *Let D be finitely generated. Then $\mathfrak{f}_{D/H} \neq \emptyset$ if and only if $\widehat{H} = \widehat{D}$.*

Proof. 1. Suppose $\mathfrak{f}_{D/H} \neq \emptyset$; obviously, we have $\widehat{H} \subseteq \widehat{D}$. Conversely, let $x \in \widehat{D}$ and $c \in D$ be such that $cx^n \in D$ for all $n \in \mathbb{N}_+$. Then for some $f \in \mathfrak{f}_{D/H}$ we have $fcx^n \in H$ and hence $x \in \widehat{H}$.

2. We suppose $\widehat{H} = \widehat{D}$ and have to verify that $\mathfrak{f}_{D/H} \neq \emptyset$. Let D be generated by $u_1, \dots, u_s$; since $D \subseteq \widehat{H}$ there are $f_i \in H$ such that $f_i u_i^k \in H$ for every $k \in \mathbb{N}$ and every $1 \leqslant i \leqslant s$. We set $f = \prod_{i=1}^{s} f_i$; then for every $u = \prod_{i=1}^{s} u_i^{k_i} \in D$ we have $fu = \prod_{i=1}^{s} (f_i u_i^{k_i}) \in H$. □

4. The structure of finitely primary monoids

In [HK2] finitely primary monoids were introduced. Their relevance lies in their appearance in the theory of one-dimensional domains (cf. Theorem 2 and the example at the end of section 5). We recall the definition.

Definition 2. A monoid H is called *finitely primary* (of rank $s \in \mathbb{N}_+$ and of exponent $\alpha \in \mathbb{N}_+$) if it is a submonoid of a factorial monoid D containing exactly s mutually non-associated prime elements $p_1, \dots, p_s$,

$$H \subseteq D = [p_1, \dots, p_s] \times D^\times,$$

satisfying the following two conditions:

a) $H^\times = H \cap D^\times$.

b) For any $a = \varepsilon p_1^{\alpha_1} \dots p_s^{\alpha_s} \in D$ (where $\alpha_1, \dots, \alpha_s \in \mathbb{N}$ and $\varepsilon \in D^\times$), the following two assertions hold true:
   i) If $a \in H \setminus H^\times$, then $\alpha_1 \geqslant 1, \dots, \alpha_s \geqslant 1$.
   ii) If $\alpha_1 \geqslant \alpha, \dots, \alpha_s \geqslant \alpha$, then $a \in H$.

The situation is especially simple if the factorial monoid D in the above definition is reduced. Then D is isomorphic to $(\mathbb{N}^s, +)$ and obviously we have the following condition: a submonoid $H \subseteq (\mathbb{N}^s, +)$ with $\mathcal{Q}(H) = \mathbb{Z}^s$ is finitely primary if and only if

$$f + \mathbb{N}_+^s \subseteq H \subseteq \mathbb{N}_+^s \cup \{0\}$$

for some $f \in \mathbb{N}_+^s$.

We return to the general situation. Clearly, the existence of an α such that ii) holds is equivalent to $\mathfrak{f}_{D/H} \neq \emptyset$.

Our aim is to characterize a finitely primary monoid H by its inner properties.

Theorem 1. *Let H be a monoid. Then the following conditions are equivalent:*

1. *H is a finitely primary monoid of rank s.*
2. *H is primary, $\widehat{H} \simeq \mathbb{N}^s \times \widehat{H}^\times$ and $\mathfrak{f}_{\widehat{H}^\times/H} \neq \emptyset$.*

In particular, the factorial monoid D of Definition 2 is just the complete integral closure of H.

Proof. 1. $\Longrightarrow$ 2. Suppose H is a finitely primary monoid of rank s with exponent α and let all notations be as in Definition 2. To show that H is primary, let $a, b \in \dot{H}$ be given with $b \notin H^\times$. Then $a = \varepsilon p_1^{\alpha_1} \dots p_s^{\alpha_s}$, $b = \mu p_1^{\beta_1} \dots p_s^{\beta_s}$ with $\varepsilon, \mu \in D^\times, \alpha_i \geqslant 0$ and $\beta_i \geqslant 1$ for $1 \leqslant i \leqslant s$. Setting $n = \alpha + \max\{\alpha_i \mid 1 \leqslant i \leqslant s\}$ we obtain

$$b^n = a\Big(\varepsilon^{-1}\mu^n \prod_{i=1}^r p_i^{n\beta_i - \alpha_i}\Big).$$

Since $n\beta_i - \alpha_i \geqslant \alpha$ for every $1 \leqslant i \leqslant s$, the second factor lies in H and hence $a \mid b^n$ (in H).

Since $f = \prod_{i=1}^{s} p_i^\alpha \in \mathfrak{f}_{D/H}$ and since the factorial monoid D is c.i.c., we have $\widehat{H} = D$ by Lemma 3, which implies the remaining assertions.

2. $\Longrightarrow$ 1. Let $p_1, \dots, p_s$ be non-associated prime elements of $\widehat{H}$; then $\widehat{H} = [p_1, \dots, p_s] \times \widehat{H}^\times$. We set $D = \widehat{H}$ and have to verify conditions a) and b) of Definition 2.

It follows from Proposition 1 that $D^\times \cap H = \widehat{H}^\times \cap H = H^\times$, whence a) holds. To verify b i), let $a = \varepsilon \prod_{i=1}^{s} p_i^{\alpha_i} \in H \setminus H^\times$ be given. Choose an arbitrary $b = \eta \prod_{i=1}^{s} p_i^{\beta_i} \in H$ with $\beta_i \geqslant 1$ for all $1 \leqslant i \leqslant s$. Since H is primary, there is an $n \in \mathbb{N}_+$ such that $b \mid a^n$ in H, and hence in $\widehat{H}$. Therefore $\alpha_i \geqslant 1$ for all $1 \leqslant i \leqslant s$.

For every prime p_i there exists a $c_i \in H$ such that $c_i p_i^n \in H$ for all $n \in \mathbb{N}_+$. We set $f_0 = f \prod_{i=1}^{s} c_i$ where $f \in H$ satisfies $f\widehat{H}^\times \subseteq H$. Then by construction we have $f_0\widehat{H} \subseteq H$ (i.e. $\mathfrak{f}_{\widehat{H}/H} \neq \emptyset$), and hence ii) holds. □

Corollary 1. *Let H be a monoid and $s \in \mathbb{N}_+$.*

1. *H is finitely primary of rank s if and only if H_{red} is finitely primary of rank s.*

2. *Suppose that H is finitely primary of rank s. Then $\widetilde{H}$ is finitely primary of rank s and $\widetilde{H} = p_1 \dots p_s \widehat{H} \cup \widetilde{H}^\times$ where $p_1, \dots, p_s$ are non-associated prime elements of $\widehat{H}$.*

Proof. 1. From Lemma 2 we infer that H is primary if and only if H_{red} is primary. $\widehat{H}$ is factorial and has s non-associated prime elements if and only if this holds true for $\widehat{H}/H^\times = \widehat{H_{\mathrm{red}}}$. Finally for some $f \in H$, we have $f\widehat{H}^\times \subseteq H$ if and only if $(fH^\times)(\widehat{H}^\times/H^\times) \subseteq H/H^\times$.

2. Proposition 2 implies that $\widetilde{H}$ is primary. Since $\widehat{H}$ is factorial, it is c.i.c. and hence $\widehat{\widetilde{H}} = \widehat{H}$; thus $\widetilde{H}$ is a finitely primary monoid. It can be verified immediately that $\widetilde{H} = p_1 \dots p_s \widehat{H} \cup \widetilde{H}^\times$. □

Remark. There are primary monoids H which are not finitely primary but for which $\widetilde{H}$ is finitely primary (cf. [G-HK-L; Theorem 3]).

Corollary 2. *Let H be a monoid. Then the following conditions are equivalent:*

1. *H_{red} is finitely generated and finitely primary.*
2. *H_{red} is finitely generated and primary.*
3. *$\widetilde{H}$ is a primary Krull monoid and $(\widetilde{H}^\times : H^\times) < \infty$.*
4. *H is finitely primary of rank 1 and $(\widehat{H}^\times : H^\times) < \infty$.*

Proof. 1. $\Longrightarrow$ 2. This is a consequence of Theorem 1.

2. $\Longrightarrow$ 3. Since H_{red} is primary, it follows from Lemma 2 and Proposition 2 that $\widetilde{H}$ is primary. By [HK3; Theorem 4] we infer that $\widetilde{H_{\mathrm{red}}} = \widetilde{H}/H^\times$ is finitely generated; therefore $\widetilde{H}/\widetilde{H}^\times$ is finitely generated and $(\widetilde{H}^\times : H^\times) < \infty$. Since $\widetilde{H}_{\mathrm{red}}$ is a finitely generated integrally closed monoid, it is a Krull monoid (see [HK3; Theorem 5 and Remark 2]) and whence $\widetilde{H}$ is Krull.

3. $\Longrightarrow$ 4. By [HK1; Corollary 2.10] primary Krull monoids are factorial and each pair of prime elements are associated; hence $\widetilde{H} \simeq \mathbb{N} \times \widetilde{H}^\times$. In particular $\widetilde{H}$ is c.i.c. and thus $\widetilde{H} = \widehat{H}$. Since $(\widetilde{H}^\times : H^\times) = (\widehat{H}^\times : H^\times) < \infty$, there is an element $f \in H$ with $f\widehat{H}^\times \subseteq H$. Therefore, by Theorem 1, H is finitely primary of rank 1.

4. $\Longrightarrow$ 1. H_{red} is finitely primary by Corollary 1 and it remains to show that it is finitely generated. However, if H is finitely primary of exponent α and $p \in \widehat{H}$ is a prime element, then

$$\{p^k \varepsilon H^\times \mid 0 \leqslant k \leqslant \alpha \text{ and } \varepsilon \in \widehat{H}^\times\}$$

is a generating system of $H/H^\times$, which is finite since $(\widehat{H}^\times : H^\times) < \infty$. □

As a final result of this section, we show that every integrally closed monoid D with D_{red} finitely generated may be realized as the complete integral closure of a primary submonoid.

Proposition 3. *Let D be an integrally closed monoid such that D_{red} is finitely generated. Then there exists a primary submonoid $H \subseteq D$ with $\widehat{H} = D$.*

Proof. Let $u_1, \ldots, u_n \in D$ be such that D is generated by $\{u_1, \ldots, u_n\} \cup D^\times$. We define

$$H = \{\varepsilon u_1^{k_1} \ldots u_n^{k_n} \in D \mid \varepsilon \in D^\times, k_1, \ldots, k_n \in \mathbb{N}_+\} \ \cup \{1\};$$

obviously $H \subseteq D$ is a primary submonoid.

Since D_{red} is finitely generated and integrally closed, it follows that D_{red} is c.i.c. by [Gi2; Theorems 12.4 and 7.8]. Hence D is c.i.c. which implies that $\widehat{H} \subseteq \widehat{D} = D$.

In order to verify that $D \subseteq \widehat{H}$ let $v = \varepsilon u_1^{l_1} \ldots u_n^{l_n} \in D$ be given with $\varepsilon \in D^\times$ and $l_1, \ldots, l_n \in \mathbb{N}$. Setting $c = u_1 \ldots u_n$, it follows that $cv^n \in H$ for all $n \in \mathbb{N}_+$ i.e. $v \in \widehat{H}$. □

Remark. Saturated submonoids of finitely primary monoids are primary by Lemma 2.3. However, in general a saturated submonoid of a finitely primary monoid may not be finitely primary as can be seen by the following.

Let $S \hookrightarrow D$ be a divisor theory where D is free abelian of finite rank and S is not factorial. Hence there is an element $a \in S$ with $p \mid a$ for all primes $p \in D$. By Proposition 3 there exists a primary submonoid $H \subseteq D$ with $\widehat{H} = D$. Since D is reduced and factorial, H is finitely primary. We set

$$T = H \cap \mathcal{Q}(S).$$

Suppose that H is finitely primary of exponent $\alpha \in \mathbb{N}_+$; then for every $s \in S$ we have

$$s = \frac{sa^\alpha}{a^\alpha} \in \mathcal{Q}(H \cap S) \subseteq \mathcal{Q}(T) \subseteq \mathcal{Q}(S)$$

which implies $\mathcal{Q}(S) = \mathcal{Q}(T)$. Furthermore, we obtain

$$\widehat{T} = \widehat{H} \cap \mathcal{Q}(S) = D \cap \mathcal{Q}(S) = S$$

and $T = H \cap \mathcal{Q}(T)$. Thus $T \subseteq H$ is saturated, but T is not finitely primary since $\widehat{T} = S$ is not factorial.

Let R be an integral domain. We denote by $R^\bullet = R \setminus \{0\}$ its multiplicative monoid; we set $R^\times = R^{\bullet\times}$ and $\widehat{R} = \widehat{R^\bullet} \cup \{0\}$. Then $\widehat{R}$ is the usual complete integral closure of the domain R and if $\overline{R}$ denotes the integral closure of R we have

$$R \subseteq \overline{R} \subseteq \widehat{R}.$$

If R is noetherian, then $\overline{R} = \widehat{R}$. This means in particular, that for noetherian domains there is a purely multiplicative description of their integral closure. R is said to be a local domain if it has a unique maximal ideal.

If S/R is a ring extension then

$$\mathrm{Ann}_R(S/R) = \{r \in R \mid rS \subseteq R\}$$

denotes the annihilator of the factor module S/R. By definition we have

$$\mathrm{Ann}_R(S/R) = \mathfrak{f}_{S^\bullet/R^\bullet} \cup \{0\}$$

and this coincides with the classical definition of the conductor of a ring extension.

Lemma 4. *Let S/R be a ring extension such that S is contained in a quotient field of R.*

1. *If $\mathrm{Ann}_R(S/R) \neq (0)$, then $\widehat{R} = \widehat{S}$.*
2. *If S is a finitely generated R-module, then $\mathrm{Ann}_R(S/R) \neq (0)$.*
3. *If R is noetherian and $\mathrm{Ann}_R(S/R) \neq (0)$, then S is a finitely generated R-module.*

Proof. 1. This follows from Lemma 3.

2. Let $s_1, \ldots, s_n \in S$ be such that $S \subseteq \sum_{i=1}^{n} Rs_i$; then there is some $0 \neq f \in R$ with $fs_i \in R$ for $1 \leqslant i \leqslant n$ and hence $fS \subseteq \sum_{i=1}^{n} Rfs_i \subseteq R$.

3. Let $0 \neq f \in \mathrm{Ann}_R(S/R)$. Since R is noetherian, the submodule $fS \subseteq R$ is finitely generated. Being isomorphic to fS (as an R-module), S is a finitely generated R-module. $\square$

Lemma 5. *Let R be an integral domain, X the set of non-zero prime ideals of R and Y the set of all prime ideals of $R^\bullet$ (in the semigroup theoretical sense). Then*

$$\bigcap_{\mathfrak{p} \in X} (\mathfrak{p} \setminus \{0\}) = \bigcap_{\mathfrak{q} \in Y} \mathfrak{q}.$$

P r o o f. For every $\mathfrak{p} \in X$ we have $\mathfrak{p} \setminus \{0\} \in Y$ and hence $\bigcap_{\mathfrak{p}\in X} (\mathfrak{p} \setminus \{0\}) \supseteq \bigcap_{\mathfrak{q}\in Y} \mathfrak{q}$. Conversely, let $s \in \bigcap_{\mathfrak{p}\in X} (\mathfrak{p} \setminus \{0\})$ be given; then $\mathcal{Q}(R^\bullet) = \{s^{-n}a \mid a \in R^\bullet\}$ by [Kap; Theorem 19] and hence $s \in \bigcap_{\mathfrak{q}\in Y} \mathfrak{q}$ by [Ge1: Lemma 4]. □

Remark. The intersection of all non-zero prime ideals of R is usually called the pseudoradical of R and domains with non-zero pseudoradical are called G-domains (cf. [Gi1] or [Kap]). The above Lemma implies that the property of being a G-domain is a purely multiplicative one (cf. [Ge1; section 4]).

Theorem 2. *Let R be an integral domain.*

1. *The following are equivalent:*
   a) *$R^\bullet$ is primary monoid.*
   b) *R is a one-dimensional local domain.*
2. *The following are equivalent:*
   a) *$R^\bullet$ is a finitely primary monoid.*
   b) *R is a one-dimensional local domain, $\widehat{R}$ is a semilocal principal ideal domain and the conductor of $\widehat{R}/R$ is non-zero.*
3. *The following are equivalent:*
   a) *$R^\bullet$ is a primary monoid and $R^\bullet/R^\times$ is finitely generated.*
   b) *R is a one-dimensional local noetherian domain, $\overline{R}$ is a discrete valuation ring which is finitely generated as R-module, and $(\overline{R}^\times : R^\times) < \infty$.*

P r o o f. 1. We use the notations of Lemma 5.

a) $\Longrightarrow$ b) Let $R^\bullet$ be primary. Then by Lemma 1 $R^\bullet$ has only one prime ideal. Since for every $\mathfrak{p} \in X$ we have $\mathfrak{p} \setminus \{0\} \in Y$, it follows that R has exactly one non-zero prime ideal.

b) $\Longrightarrow$ a) Suppose that $R \setminus R^\times$ is the only non-zero prime ideal of R and let $\mathfrak{q}_0 \in Y$ be given. By Lemma 5 we infer

$$R^\bullet \setminus R^\times = \bigcap_{\mathfrak{p}\in X} (\mathfrak{p} \setminus \{0\}) = \bigcap_{\mathfrak{q}\in Y} \mathfrak{q} \subseteq \mathfrak{q}_0.$$

Thus $\mathfrak{q}_0 = R^\bullet \setminus R^\times$ and $\mathfrak{q}_0$ is the only prime ideal of $R^\bullet$.

2. a) $\Longrightarrow$ b) Since $R^\bullet$ is a finitely primary monoid, it is primary and hence R is one-dimensional and local by 1. $\widehat{R}$ is a factorial domain having only finitely many non-associated primes and hence it is a semilocal principal ideal domain. Clearly the conductor of $\widehat{R}/R$ is non-zero (cf. the observation after Definition 2).

b) $\Longrightarrow$ a) $R^\bullet$ is primary by 1. $\widehat{R}$ is a factorial domain having only finitely many non-associated prime elements. If $0 \neq f \in R$ lies in the conductor of $\widehat{R}/R$ then $f\widehat{R}^\times \subseteq f\widehat{R} \subseteq R$. Therefore $R^\bullet$ is finitely primary by Theorem 1.

3. a) $\Longrightarrow$ b) Corollary 2 implies that $R^\bullet$ is finitely primary of rank 1 and $(\widehat{R}^\times : R^\times) < \infty$. Thus for some prime element $p \in \widehat{R^\bullet}$ we have $\widehat{R^\bullet} = \{\varepsilon p^n \mid \varepsilon \in \widehat{R}^\times$ and $n \in \mathbb{N}\}$, and whence $\widehat{R}$ is a discrete valuation ring. Since $p^\alpha \in R$ for some $\alpha \in \mathbb{N}_+$ and because $(\widehat{R}^\times : R^\times) < \infty$, $\widehat{R}$ is a finitely generated R-module. Therefore R is noetherian by the theorem of Eakin-Nagata ([Ma; Theorem 3.7]); furthermore $\overline{R} = \widehat{R}$, $\dim R = \dim \overline{R} = 1$ and R is local.

b) $\Longrightarrow$ a) $R^\bullet$ is primary by 1. Since R is noetherian we infer $\overline{R} = \widehat{R}$ and hence by 2b) R is finitely primary of rank 1. Thus the assertion follows from Corollary 2. □

Corollary 3. *Let $\mathfrak{o}$ be a one-dimensional noetherian domain, $\mathfrak{p} \subseteq \mathfrak{o}$ a non-zero prime ideal and $\Omega(\mathfrak{p})$ the set of all invertible $\mathfrak{p}$-primary ideals which are multiplicatively irreducible (i.e. not a product of two invertible ideals). Then the following conditions are equivalent:*

1. *$\mathfrak{o}_\mathfrak{p}^\bullet/\mathfrak{o}_\mathfrak{p}^\times$ is finitely generated.*
2. *$\#\Omega(\mathfrak{p}) < \infty$.*
3. *There is exactly one prime ideal $\overline{\mathfrak{p}} \subseteq \overline{\mathfrak{o}}$ with $\overline{\mathfrak{p}} \cap \mathfrak{o} = \mathfrak{p}$. $\overline{\mathfrak{o}_\mathfrak{p}}$ is a finitely generated $\mathfrak{o}_\mathfrak{p}$-module and $(\overline{\mathfrak{o}_\mathfrak{p}}^\times : \mathfrak{o}_\mathfrak{p}^\times) < \infty$.*

P r o o f. 1. $\Leftrightarrow$ 2. Let $\mathcal{I}(\mathfrak{o})$ denote the multiplicative monoid of invertible ideals in $\mathfrak{o}$ and $\mathcal{I}(\mathfrak{o}, \mathfrak{p}) \subseteq \mathcal{I}(\mathfrak{o})$ the submonoid generated by $\mathfrak{p}$-primary invertible ideals. Then $\Omega(\mathfrak{p})$ is a generating system of $\mathcal{I}(\mathfrak{o}, \mathfrak{p})$ consisting of irreducible elements. $\mathcal{I}(\mathfrak{o}, \mathfrak{p})$ is isomorphic to $\mathcal{I}(\mathfrak{o}_\mathfrak{p})$, the monoid of invertible ideals in $\mathfrak{o}_\mathfrak{p}$. Since invertible ideals in a local ring are principal we have

$$\mathcal{I}(\mathfrak{o}_\mathfrak{p}) = \{\alpha \mathfrak{o}_\mathfrak{p} \mid 0 \neq \alpha \in \mathfrak{o}_\mathfrak{p}\} \simeq \mathfrak{o}_\mathfrak{p}^\bullet/\mathfrak{o}_\mathfrak{p}^\times .$$

1. $\Leftrightarrow$ 3. This follows from Theorem 2.3 □

Remarks. 1. Part 1 of Theorem 2 was first established (with a different proof) in [HK1; Theorem 4.1].

2. In [G-HK-L; Theorem 4] a characterization of integral domains R is derived for which $R^\bullet$ is primary and the group of divisibility $\mathcal{Q}(R^\bullet)/R^\times$ is a finitely generated abelian group.

3. Suppose that $R^\bullet$ is finitely primary. Then the rank of $R^\bullet$ equals the number of prime ideals of $\widehat{R}$. Note that in general R is not noetherian (cf. [Hu; Example 68]).

4. The equivalence of 2. and 3. in Corollary 3 was first proved in [HK5; Satz 2].

Proposition 4. *Let R be an integral domain.*

1. *$R^\bullet$ is a finitely generated monoid if and only if R is a finite field.*
2. *If $R^\bullet$ is primary, then $R^\bullet$ is not finitely generated.*

Proof. 1. If R is finite then $R^\bullet$ is finitely generated. Conversely, suppose that $R^\bullet$ is a finitely generated monoid. Let K denote the quotient field of R and $k \subseteq K$ its prime field. Since $K^\times$ is a finitely generated abelian group, k is a finite field and K/k is a finitely generated abelian extension (cf. [Kar; Ch. 4, Theorem 1.4 and Lemma 5.1]). Therefore K is finite, R is a finite integral domain which implies that R is a finite field.

2. If $R^\bullet$ is primary then $R^\bullet \neq R^\times$ and hence the assertion follows from 1. □

Example. Let $\mathfrak{o}$ be a one-dimensional noetherian domain and suppose that $\bar{\mathfrak{o}}$ is a finitely generated $\mathfrak{o}$-module. Then for every non-zero prime ideal $\mathfrak{p} \subseteq \mathfrak{o}$ $\overline{\mathfrak{o}_\mathfrak{p}} = \widehat{\mathfrak{o}}_\mathfrak{p}$ is a semilocal Dedekind domain and hence a principal ideal domain; furthermore $\overline{\mathfrak{o}_\mathfrak{p}}$ is a finitely generated $\mathfrak{o}_\mathfrak{p}$-module. Thus Theorem 2 implies that $\mathfrak{o}_\mathfrak{p}^\bullet$ is a finitely primary monoid. Indeed, it was this example which motivated the introduction of finitely primary monoids ([HK2; Proposition 6]).

We reconsider the situation of Corollary 3. For this we suppose further that $\bar{\mathfrak{o}}$ has the finite norm property (e.g. $\mathfrak{o}$ an order in a global field) and let $\mathfrak{f}$ denote the conductor of $\bar{\mathfrak{o}}/\mathfrak{o}$. Then $(\overline{\mathfrak{o}_\mathfrak{p}}^\times : \mathfrak{o}_\mathfrak{p}^\times) \leqslant (\bar{\mathfrak{o}} : \mathfrak{f}) < \infty$ by [Ne; Kap. I, Satz 12.11], and hence $\mathfrak{o}_\mathfrak{p}^\bullet/\mathfrak{o}_\mathfrak{p}^\times$ is finitely generated if and only if there is exactly one prime ideal $\bar{\mathfrak{p}} \subseteq \bar{\mathfrak{o}}$ lying over $\mathfrak{p}$.

6. The arithmetic of finitely primary monoids

Let H be a monoid and $\varrho\colon H \to H_{\mathrm{red}}$ the canonical epimorphism. We denote by $\mathcal{U}(H)$ the set of irreducible elements of H. The *factorization monoid* $\mathcal{Z}(H)$ of H is defined as the free abelian monoid with basis $\mathcal{U}(H_{\mathrm{red}})$. The elements $z \in \mathcal{Z}(H)$ are written in the form

$$z = \prod_{u \in \mathcal{U}(H_{\mathrm{red}})} u^{n_u}$$

with $n_u \in \mathbb{N}$ and $n_u = 0$ for all but finitely many $u \in \mathcal{U}(H_{\mathrm{red}})$. Furthermore we call

$$\sigma(z) = \sum_{u \in \mathcal{U}(H_{\mathrm{red}})} n_u \in \mathbb{N}$$

the size of z. Next let $\pi\colon \mathcal{Z}(H) \to H_{\mathrm{red}}$ denote the canonical homomorphism; we say H is *atomic*, if π is surjective.

Suppose that H is an atomic monoid and let $a \in H$ be given. The elements of $\pi^{-1}(\varrho(a)) \subseteq \mathcal{Z}(H)$ are called *factorizations* of a and $L(a) = \{\sigma(z) \mid z \in \pi^{-1}(\varrho(a))\} \subseteq$

$\mathbb{N}$ is the *set of lengths* of a. For two factorizations $z, z' \in \mathcal{Z}(H)$ we call

$$d(z, z') = \max\left\{\sigma\left(\frac{z}{\gcd(z, z')}\right), \sigma\left(\frac{z'}{\gcd(z, z')}\right)\right\} \in \mathbb{N}$$

the *distance* between z and z'.

H is said to be a *BF-monoid* (bounded factorization monoid) if it is atomic and if $L(a)$ is finite for all $a \in H$; H is said to be an *FF-monoid* (finite factorization monoid) if it is atomic and if $\pi^{-1}(\varrho(a))$ is finite for all $a \in H$. Hence, by definition every FF-monoid is a BF-monoid.

Suppose that H is atomic; the *catenary degree*

$$c(H) \in \mathbb{N} \cup \{\infty\}$$

of H is the minimal $N \in \mathbb{N} \cup \{\infty\}$ such that for every $a \in H$ and every $z, z' \in \pi^{-1}(\varrho(a))$ there exist factorizations $z = x_0, x_1, \dots, x_l = z' \in \pi^{-1}(\varrho(a))$ such that $d(x_{i+1}, x_i) \leqslant N$ for $0 \leqslant i < l$.

Concerning BF-monoids and FF-monoids the reader is referred to [HK4]; the catenary degree of atomic monoids was first studied in [G-L; cf. Definition 2] (in a different terminology). Our main aim in this section is to prove that finitely primary monoids have finite catenary degree (Theorem 3).

Proposition 5. *Let H be an atomic primary monoid.*

1. *If H contains a prime element, then $H \simeq \mathbb{N} \times H^\times$.*
2. *If $\widehat{H}$ is a BF-monoid, then H is a BF-monoid.*
3. *If H_{red} is uncountable, then H is not an FF-monoid.*

Proof. 1. Let p be a prime element and $a \in H$ arbitrary. Then there is a minimal $m \in \mathbb{N}$ such that $a \mid p^m$ and hence $p^m = a\varepsilon$ for some $\varepsilon \in H$ with $p \nmid \varepsilon$. Because p is prime it follows that $p \nmid \varepsilon^k$ for any $k \in \mathbb{N}_+$. This implies that $\varepsilon \in H^\times$ since H is primary. Therefore we have $a = p^m\varepsilon^{-1}$ where m and ε are uniquely determined.

2. Since $\widehat{H}^\times \cap H = H^\times$ by Proposition 1, the result follows from [HK4; Theorem 3].

3. We may suppose that H is reduced and uncountable. Let $1 \neq a \in H$ be given; for every $k \in \mathbb{N}_+$ we set $M_k(a) = \{b \in H \mid b \mid a^k\}$. Since H is primary we have $H = \bigcup\limits_{k \in \mathbb{N}_+} M_k(a)$. Since H is uncountable, $M_k(a)$ has to be infinite for some $k \in \mathbb{N}_+$. Therefore [HK4; Corollary 2] implies the assertion. $\square$

Proposition 6. *Let H be a finitely primary monoid. Then H is a BF-monoid. Furthermore H is an FF-monoid if and only if $(\widehat{H}^\times : H^\times) < \infty$.*

Proof. We have $\widehat{H}^\times \cap H = H^\times$ by Proposition 1; furthermore Theorem 1 implies that $\widehat{H}$ is a factorial monoid and that there is an $f \in H$ such that $f\widehat{H} \subseteq H$. The assertion now follows from [HK4; Theorems 3 and 4]. □

Let H be a finitely primary monoid of rank s with exponent α, say

$$H \subseteq \widehat{H} = [p_1, \dots, p_s] \times \widehat{H}^\times.$$

For every $a = \varepsilon p_1^{k_1} \dots p_s^{k_s} \in \widehat{H}$, with $\varepsilon \in \widehat{H}^\times$ and $k_i \in \mathbb{N}$, we set $v_{p_i}(a) = k_i$ for every $1 \leqslant i \leqslant s$.

Lemma 6. *Let all notations be as above and let $a \in H$. Then we have*

1. $\max L(a) \leqslant \min\{v_{p_i}(a) \mid 1 \leqslant i \leqslant s\}$.
2. *If $s \geqslant 2$, then* $\min L(a) \leqslant 2\alpha$.

Proof. Let $a = \varepsilon p_1^{k_1} \dots p_s^{k_s} \in H$.

1. Suppose $a = u_1 \dots u_l$ with $u_j \in \mathcal{U}(H)$. Since for every $b \in H \setminus H^\times$ we have $v_{p_i}(b) \geqslant 1$ for all $1 \leqslant i \leqslant s$, we infer that for every $1 \leqslant i \leqslant s$,

$$l \leqslant \sum_{j=1}^{l} v_{p_i}(u_j) = v_{p_i}(a).$$

Hence $\max L(a) \leqslant \min\{v_{p_i}(a) \mid 1 \leqslant i \leqslant s\}$.

2. Let $s \geqslant 2$; if $\min\{k_i \mid 1 \leqslant i \leqslant s\} \leqslant 2\alpha$, then by part 1. it follows that

$$\min L(a) \leqslant \max L(a) \leqslant 2\alpha.$$

So suppose that $k_i \geqslant 2\alpha$ for all $1 \leqslant i \leqslant s$. We set $a_1 = \varepsilon p_1^{k_1-\alpha} p_2^\alpha \dots p_s^\alpha$ and $a_2 = p_1^\alpha p_2^{k_2-\alpha} \dots p_s^{k_s-\alpha}$. Then $a_1, a_2 \in H$ and $\max L(a_1) \leqslant \alpha$, $\max L(a_2) \leqslant \alpha$ by 1. Therefore

$$\min L(a) \leqslant \max L(a_1) + \max L(a_2) \leqslant 2\alpha.$$

□

Theorem 3. *Let H be a finitely primary monoid of exponent α. Then $c(H) \leqslant \max\{3, 4\alpha - 2\}$.*

Proof. Suppose that H is finitely primary of rank $s \in \mathbb{N}_+$ and that

$$H \subseteq \widehat{H} = [p_1, \dots, p_s] \times \widehat{H}^\times.$$

By Corollary 1 we may assume without restriction that H is reduced. Since $(p_1 \dots p_s)^\alpha \in H$, there exists an irreducible element $u \in H$ with $1 \leqslant v_{p_i}(u) \leqslant \alpha$ for all $1 \leqslant i \leqslant s$.

Now let $1 \neq a \in H$ be given. We write a in the form

$$a = u^\kappa a_0$$

with $\kappa \in \mathbb{N}$ being maximal such that $a_0 \in H$. This implies

$$\min\{v_{p_i}(a_0) \mid 1 \leqslant i \leqslant s\} \leqslant 2\alpha - 1$$

and hence $\max L(a_0) \leqslant 2\alpha - 1$ by Lemma 6.

We define a subset $Z \subseteq \pi^{-1}(a)$ as

$$Z = \{u^\kappa x \in \mathcal{Z}(H) \mid x \in \pi^{-1}(a_0)\}.$$

If $z = u^\kappa x \in Z$ and $z' = u^\kappa x' \in Z$, then

$$d(z, z') = d(x, x') \leqslant \max L(a_0) \leqslant 2\alpha - 1.$$

Hence it remains to verify that for every $z \in \pi^{-1}(a)$ there exists a $\max\{3, 4\alpha - 2\}$-chain of factorizations from z to some $z' \in Z$. Let

$$z = u^\varrho u_1 \dots u_\lambda \in \pi^{-1}(a)$$

be given with $0 \leqslant \varrho \leqslant \kappa$ and $u_j \in \mathcal{U}(H)$. We do the proof by induction on ϱ from $\varrho = \kappa$ to $\varrho = 0$. If $\varrho = \kappa$, then $z \in Z$ and we are done. Now let $\varrho < \kappa$; we set $b = u_1 \dots u_\lambda$ and distinguish two cases:

Case 1: $\min\{v_{p_i}(b) \mid 1 \leqslant i \leqslant s\} < 2\alpha$. Then, by Lemma 6

$$\lambda \leqslant \max L(b) < 2\alpha.$$

Furthermore, if $v_{p_j}(b) = \min\{v_{p_i}(b) \mid 1 \leqslant i \leqslant s\}$, then

$$\kappa v_{p_j}(u) \leqslant v_{p_j}(a) < \varrho v_{p_j}(u) + 2\alpha,$$

which implies that

$$\kappa - \varrho < 2\alpha.$$

Therefore for some $x \in \pi^{-1}(a_0)$ we have

$$d(u^\kappa x,\ u^\varrho u_1 \dots u_\lambda) \leqslant \max\{\kappa - \varrho + \max L(a_0), \lambda\} \leqslant 4\alpha - 2.$$

Case 2: $v_{p_i}(b) \geqslant 2\alpha$ for all $1 \leqslant i \leqslant s$. Then there exists some $\nu \in \{1, \dots, \lambda\}$ with $\nu \leqslant 2\alpha$ such that for $b_0 = \prod_{i=1}^{\nu} u_i$ we have

$$v_{p_i}(b_0) \geqslant 2\alpha \quad \text{for all} \quad 1 \leqslant i \leqslant s.$$

If $s = 1$, we may require in addition that $v_{p_1}(b_0) \leqslant 4\alpha - 2$. Then $b_0 = uc$ for some $c \in H$. If $s = 1$, then

$$\max L(c) \leqslant v_{p_1}(c) \leqslant v_{p_1}(b_0) - 1 \leqslant 4\alpha - 3$$

and hence there is some $y \in \pi^{-1}(c)$ with $\sigma(y) \leqslant 4\alpha - 3$. If $s \geqslant 2$, then by Lemma 6.2 there is some factorization $y \in \pi^{-1}(c)$ with $\sigma(y) \leqslant 2\alpha$. Setting

$$z' = u^{e+1} y \prod_{i=\nu+1}^{\lambda} u_i$$

we have

$$d(z, z') \leqslant \max\{\nu, 1 + \sigma(y)\} \leqslant \max\{2\alpha + 1, 4\alpha - 2\} = \max\{3, 4\alpha - 2\}.$$

Now the assertion follows by induction hypothesis. □

Theorem 4. *Let $\mathfrak{o}$ be a one-dimensional noetherian domain and suppose that $\overline{\mathfrak{o}}$ is a finitely generated $\mathfrak{o}$-module.*

1. *The multiplicative monoid of invertible ideals of $\mathfrak{o}$ has finite catenary degree.*
2. *Suppose that for every prime ideal $\mathfrak{p}$ of $\mathfrak{o}$ containing the conductor of $\overline{\mathfrak{o}}/\mathfrak{o}$ there is exactly one prime ideal of $\overline{\mathfrak{o}}$ lying over $\mathfrak{p}$ and that $(\overline{\mathfrak{o}_\mathfrak{p}}^\times : \mathfrak{o}_\mathfrak{p}^\times) < \infty$. If $\mathfrak{o}$ has a finite Picard group, then $c(\mathfrak{o}^\bullet) < \infty$.*

P r o o f. 1. Let $\mathcal{I}(\mathfrak{o})$ (resp. $\mathcal{I}(\mathfrak{o}_\mathfrak{p})$ for some non-zero $\mathfrak{p} \in \operatorname{spec}(\mathfrak{o})$) denote the multiplicative monoid of invertible ideals of $\mathfrak{o}$ (resp. of $\mathfrak{o}_\mathfrak{p}$). Then $\mathcal{I}(\mathfrak{o}_\mathfrak{p}) \simeq \mathfrak{o}_\mathfrak{p}^\bullet/\mathfrak{o}_\mathfrak{p}^\times$ for every non-zero $\mathfrak{p} \in \operatorname{spec}(\mathfrak{o})$ and by [Ne; Kap. I, 12.6]

$$\begin{array}{rcl} \varphi\colon \mathcal{I}(\mathfrak{o}) & \to & \coprod\limits_{(0)\neq\mathfrak{p}\in\operatorname{spec}(\mathfrak{o})} \mathcal{I}(\mathfrak{o}_\mathfrak{p}) \\ \mathfrak{a} & \to & (\mathfrak{a}\mathfrak{o}_\mathfrak{p})_{(0)\neq\mathfrak{p}\in\operatorname{spec}(\mathfrak{o})} \end{array}$$

is an isomorphism.

Let $\mathfrak{f}$ denote the conductor of $\overline{\mathfrak{o}}/\mathfrak{o}$; further let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ denote the prime ideals containing the conductor and let $\mathcal{P}$ be the set of non-zero prime ideals prime to the conductor. For $\mathfrak{p} \in \mathcal{P}$ the localization $\mathfrak{o}_\mathfrak{p} = \overline{\mathfrak{o}_\mathfrak{p}}$ is a discrete valuation ring ([Ne; Kap. I, 12.10]) and hence $\mathfrak{o}_\mathfrak{p}^\bullet/\mathfrak{o}_\mathfrak{p}^\times \simeq (\mathbb{N}, +)$. Thus we have

$$\mathcal{I}(\mathfrak{o}) \simeq \mathcal{F}(\mathcal{P}) \times \prod_{i=1}^{r} \mathfrak{o}_{\mathfrak{p}_i}^\bullet/\mathfrak{o}_{\mathfrak{p}_i}^\times$$

where $\mathcal{F}(\mathcal{P})$ denotes the free abelian monoid with basis $\mathcal{P}$. In the example in section 5 we showed that all $\mathfrak{o}_{\mathfrak{p}_i}^\bullet$, $1 \leqslant i \leqslant r$, are finitely primary and hence they have finite catenary degree by Theorem 3. Therefore

$$c(\mathcal{I}(\mathfrak{o})) = \max\{c(\mathfrak{o}_{\mathfrak{p}_i}^\bullet) \mid 1 \leqslant i \leqslant r\} < \infty .$$

2. The monoid homomorphism

$$\begin{aligned} \partial \colon \mathfrak{o}^\bullet &\to \mathcal{I}(\mathfrak{o}) \\ a &\to a\mathfrak{o} \end{aligned}$$

is a cofinal divisor homomorphism whose class group is just the Picard group of the domain $\mathfrak{o}$ ([Ge2; Proposition 6]). Corollary 3 implies that $\prod_{i=1}^{r} \mathfrak{o}_{\mathfrak{p}_i}^\bullet/\mathfrak{o}_{\mathfrak{p}_i}^\times$ is a finitely generated monoid. Now the assertion follows from [G-L; Theorem 2 and Proposition 2]. □

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