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Czechoslovak Mathematical Journal, Vol. 46 (1996), No. 4, 599–605

Persistent URL: http://dml.cz/dmlcz/127322

Czechoslovak Mathematical Journal, 46 (121) 1996, Praha

SOME INCLUSION THEOREMS FOR ABSOLUTE SUMMABILITY

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(Received March 14, 1994)

1. INTRODUCTION

Let $\sum x_n$ be an infinite series with partial sums s_n, and let $A = (a_{nv})$ be a lower semi-matrix with nonzero diagonal entries. By (T_n) we denote the A-transform of the sequence $s = (s_n)$, i.e.,

$$T_n = \sum_{v=0}^{n} a_{nv} s_v \quad (n = 0, 1, 2, \ldots). \tag{1}$$

The series $\sum x_n$ is said to summable $|A|_k (k \geqslant 1)$, if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty \tag{2}$$

(see e.g. [4]).

In the special case of $A = (a_{nv})$ being a Riesz matrix, i.e., weighted mean matrix, we shall write $|R, p_n|_k$ for summability $|A|_k$. The case in which $k = 1$ reduces to the usual absolute weighted mean summability $|R, P_n|$. Recall that a weighted mean matrix is defined by

$$a_{nv} = p_v / P_n \quad \text{for } 0 \leqslant v \leqslant n$$

and

$$a_{nv} = 0 \quad \text{for } v > n$$

where (p_n) is a sequence of positive real numbers and

$$P_n = p_0 + p_1 + \ldots + p_n, \quad P_{-1} = 0.$$

[1] This paper was supported by the Scientific and Technical Research Council of Turkey (TBAG-ÇG 2).

Throughout the paper, we suppose that $P_n \to \infty$ as $n \to \infty$.

In this paper, using functional analytic techniques, we give necessary and sufficient conditions for the series $\sum x_n$ to be summable $|A|_k$ $(k \geqslant 1)$, whenever it is summable $|R, p_n|$, from which we deduce some knonw results.

2. The main result

Given a lower semi-matrix $A = (a_{nv})$, we introduce two lower semi-matrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\begin{aligned}
\bar{a}_{nv} &= \sum_{i=v}^{n} a_{ni}; \ n, v = 0, 1, 2, \ldots, \\
\hat{a}_{nv} &= \bar{a}_{nv} - \bar{a}_{n-1,v}; \ n = 1, 2, \ldots, \\
\hat{a}_{00} &= \bar{a}_{00} = a_{00}, \\
\hat{a}_{nv} &= \bar{a}_{nv} = 0 \text{ if } v \geqslant n.
\end{aligned}$$

Since A is a lower semi-matrix, so is $\hat{A}$.

We also note that

$$T_n = (As)_n = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \sum_{i=v}^{n} = a_{ni} x_v = \sum_{v=0}^{n} \bar{a}_{nv} x_v$$

and

$$T_n - T_{n-1} = \sum_{v=0}^{n} (\bar{a}_{nv} - \bar{a}_{n-1,v}) x_v = \sum_{v=0}^{n} \Delta \hat{a}_{nv} x_v \tag{3}$$

where $s_v = x_0 + x_1 + \ldots + x_v$ and $\bar{a}_{n-1,n} = 0$.

Using this notation, we have

Theorem. *$|R, p_n|$ summability implies $|A|_k (k \geqslant 1)$ summability if and only if*

$$\text{(i)} \quad |\hat{a}_{vv}| \frac{P_v}{p_v} = O\left(v^{\frac{1}{k}-1}\right),$$

$$\text{(ii)} \quad \left(\sum_{n=v+1}^{\infty} n^{k-1} |\Delta \hat{a}_{nv}|^k \right)^{1/k} = O\left(\frac{p_v}{P_v}\right),$$

$$\text{(iii)} \quad \left(\sum_{n=v+1}^{\infty} n^{k-1} |\hat{a}_{n,v+1}|^k \right)^{1/k} = O(1)$$

where $\Delta \hat{a}_{nv} = \hat{a}_{nv} - \hat{a}_{n,v+1}$.

Proof. Necessity. Let t_n be the Riesz means of $\sum x_v$, i.e.

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})x_v.$$

Now we have

$$(4)\qquad \begin{aligned} c_n &:= t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} p_{v-1}x_v, \quad n \geqslant 1,\\ c_0 &:= x_0 \end{aligned}$$

and

$$(5)\qquad \begin{aligned} C_n &:= T_n - T_{n-1} = \sum_{v=0}^{n} \hat{a}_{nv}x_v, \quad n \geqslant 1,\\ C_0 &:= x_0. \end{aligned}$$

We are given that $|R, p_n| \Longrightarrow |A|_k$, $k \geqslant 1$. Hence

$$(6)\qquad \sum_{n-1}^{\infty} n^{k-1}|C_n|^k < \infty$$

whenever

$$(7)\qquad \sum |c_n| < \infty.$$

The spaces of sequences (x_v) satisfying (6) and (7) are BK-spaces (i.e., Banach spaces with continuous coordinates) if normed by

$$(8)\qquad \|C\| = \left(|C_0|^k + \sum_{n=1}^{\infty} n^{k-1}|C_n|^k\right)^{1/k} \quad \text{and} \quad \|c\| = \sum_{n=1}^{\infty}|c_n|,$$

respectively.

Observe that (5) transforms the space of sequences satisfying (7) into the space of sequences satisfying (6). Applying the Banach-Steinhaus theorem, we find that there is a constant $M > 0$ such that

$$(9)\qquad \|C\| \leqslant M\|c\|$$

for all sequences satisfying (7). Applying (4) and (5) to the sequence $x = e_v - e_{v+1}$, where e_v is the v^{th} coordinate vector, we see that

$$c_n = \begin{cases} 0; & n < v \\ \frac{p_v}{P_v}; & n = v \\ \frac{-p_v p_n}{P_n P_{n-1}}; & n > v \end{cases} \quad \text{and} \quad C_n = \begin{cases} 0; & n < v \\ \hat{a}_{vv}; & n = v \\ \hat{a}_{nv} - \hat{a}_{n,v+1}; & n > v. \end{cases}$$

By (8), it follows that

$$\|c\| = \frac{2p_v}{P_v} \text{ and } \|C\| = \left(v^{k-1}|\hat{a}_{vv}|^k + \sum_{n=v+1}^{\infty} n^{k-1}|\Delta\hat{a}_{nv}|^k\right)^{1/k}.$$

By (9), we have

$$v^{k-1}|\hat{a}_{vv}|^k + \sum_{n=v+1}^{\infty} n^{k-1}|\Delta\hat{a}_{nv}|^k \leqslant (2M)^k\left(\frac{p_v}{P_v}\right)^k.$$

Since this holds for any $v \geqslant 1$, we get the necessity of (i) and (ii). To prove the necessity of (iii), we again apply (4) and (5) to the sequence $x = e_{v+1}$. Hence we get that

$$c_n = 0 \quad \text{if} \quad n < v + 1$$

and

$$c_n = \frac{P_v p_n}{P_n P_{n-1}} \quad \text{if} \quad n \geqslant v + 1,$$

and also

$$C_n = 0 \quad \text{if} \quad n < v + 1$$

and

$$C_n = \hat{a}_{n,v+1} \quad \text{if} \quad n \geqslant v + 1.$$

By (8) we have

$$\|c\| = 1 \text{ and } \|C\| = \left(\sum_{n=v+1}^{\infty} n^{k-1}|\hat{a}_{n,v+1}|^k\right)^{1/k}.$$

It follows from (9) that

$$\left(\sum_{n=v+1}^{\infty} n^{k-1}|\hat{a}_{n,v+1}|^k\right)^{1/k} = O(1),$$

which implies the necessity of (iii).

Sufficiency. By (4), we have

(10) $$x_v = \frac{P_v}{p_v}c_v - \frac{P_{v-2}}{p_{v-1}}c_{v-1}; \quad P_{-1} = p_{-1} = 0.$$

Inserting (10) in to (5), we may write

$$C_n = \sum_{v=0}^{n} \hat{a}_{nv} x_v = \hat{a}_{n0} c_0 + \sum_{v=1}^{n} \hat{a}_{nv} \Big(\frac{P_v}{p_v} c_v - \frac{P_{v-2}}{p_{v-1}} c_{v-1} \Big)$$

$$= \hat{a}_{n0} c_0 + \hat{a}_{nn} \frac{P_n}{p_n} c_n + \sum_{v=1}^{n-1} (\hat{a}_{nv} P_v - \hat{a}_{n,v+1} P_{v-1}) \frac{c_v}{p_v}$$

$$= \sum_{v=0}^{n-1} (\hat{a}_{nv} P_v - \hat{a}_{n,v+1} P_{v-1}) \frac{c_v}{p_v} + \hat{a}_{nn} \frac{P_n}{p_n} c_n.$$

Since

$$\hat{a}_{nv} P_v - \hat{a}_{n,v+1} P_{v-1} = P_v \Delta \hat{a}_{nv} + p_v \hat{a}_{n,v+1},$$

we have

$$C_n = \sum_{v=0}^{n-1} \Big(\frac{P_v}{p_v} \Delta \hat{a}_{nv} + \hat{a}_{n,v+1} \Big) c_v + \hat{a}_{nn} \frac{P_n}{p_n} c_n.$$

Now set $H_n := n^{1-\frac{1}{k}} C_n$, $n \geqslant 1$. Then we get

$$H_n = \sum_{v=1}^{n} u_{nv} c_v$$

where

$$u_{nv} = \begin{cases} n^{(1-\frac{1}{k})} \cdot \Big(\frac{P_v}{p_v} \Delta \hat{a}_{nv} + \hat{a}_{n,v+1} \Big); & 1 \leqslant v \leqslant n-1, \\ n^{(1-\frac{1}{k})} \cdot \frac{P_n}{p_n} \hat{a}_{nn}; & v = n, \\ 0; & v > n. \end{cases}$$

Hence, $\sum x_v$ is summable $|A|_k$, $k \geqslant 1$, whenever $\sum x_v$ is summable $|R, p_n|$ if and only if

$$\sum |H_n|^k < \infty \quad \text{whenever} \quad \sum |c_n| < \infty$$

or equivalently, if and only if the matrix $U = (u_{nv})$ maps l_1 into l_k, $k \geqslant 1$, where

$$l_k = \Big\{ x = (x_v) : \sum_v |x_v|^k < \infty \Big\}.$$

Nonetheless, it is well-known that the matrix U maps l_1 into l_k, $k \geqslant 1$, if and only if

$$\sup_v \sum_{n=1}^{\infty} |u_{nv}|^k < \infty$$

(see e.g. [3], Theorem 5, p. 167). □

By the definition of $U = (u_{nv})$, we have

$$\sum_{n=v}^{\infty} |u_{nv}|^k = v^{k-1}\Big(\frac{P_n}{p_n}|\hat{a}_{nn}|\Big)^k + \sum_{n=v+1}^{\infty} n^{k-1}\Big|\frac{P_v}{p_v}\Delta\hat{a}_{nv} + \hat{a}_{n,n+1}\Big|^k.$$

Hence the conditions (i)–(iii) imply that $\sum\limits_{n=v}^{\infty} |u_{nv}|^k = O(1)$ as $v \to \infty$, whence the result.

Taking the matrix $A = (a_{nv})$ to be the weighted mean matrix (R, q_n) where $q_v > 0$ for each v and $Q_n = q_0 + q_1 + \ldots + q_n \to \infty$ as $n \to \infty$, we deduce some known results and list them below:

Corollary 1 ([5]). $|R, p_n| \Rightarrow |R, q_n|_k$, $k \geqslant 1$ *if and only if*

(i) $$\frac{q_v P_v}{Q_v p_v} = O(v^{\frac{1}{k}-1}),$$

(ii) $$q_v\Big(\sum_{n=v+1}^{\infty} n^{k-1}\Big(\frac{q_n}{Q_n Q_{n-1}}\Big)^k\Big)^{1/k} = O\Big(\frac{p_v}{P_v}\Big),$$

(iii) $$Q_v\Big(\sum_{n=v+1}^{\infty} n^{k-1}\Big(\frac{q_n}{Q_n Q_{n-1}}\Big)^k\Big)^{1/k} = O(1).$$

P r o o f. Apply Theorem with $A = (a_{nv})$ a weighted mean matrix (R, q_n). Observe that, in this case,

$$\hat{a}_{nv} = \frac{q_n Q_{v-1}}{Q_n Q_{n-1}} \quad \text{and} \quad \Delta\hat{a}_{nv} = \hat{a}_{nv} - \hat{a}_{n,v+1} = \frac{-q_n q_v}{Q_n Q_{n-1}}.$$

□

Corollary 2 ([2]). $|R, p_n| \Rightarrow |R, q_n|$ *if and only if*

(11) $$q_v P_v = O(Q_v p_v).$$

P r o o f. Apply Corollary 1 with $k = 1$.

Note that Corollary 2 has been obtained also by Sunouchi [6] in the sufficient form. When reviewing that paper Bosanquet has observed that condition (11) is not only sufficient but also necessary for $|R, p_n| \Rightarrow |R, q_n|$.

When $p_n = 1$ for all n, the $|R, p_n|$ summability is the same as $|C, 1|$ summability. Hence, using Theorem, one can write the necessary and sufficient conditions for $|C, 1| \Rightarrow |A|_k$, $k \geqslant 1$, immediately. So we omit the details. □

3. Concluding remarks

(a) Taking $A = (a_{nv})$ to be the weighted mean matrix (R, q_n) and defining

$$p_v = a^v \quad \text{and} \quad q_v = (v+1)^\alpha$$

where $a > 1$ and $\alpha > -1$, one can see that

$$\frac{P_v}{p_v} \sim \frac{a}{a-1} \quad \text{and} \quad Q_v \sim \frac{vq_v}{\alpha}.$$

Hence conditions (i)–(iii) of Theorem hold.

(b) If we take that matrix $A = (a_{nv})$ to be the weighted mean matrix (R, p_n), then by the condition (i) of Theorem, we must have

$$v^{1-\frac{1}{k}} = O(1),$$

which is impossible when $k > 1$. This means that there is a series $\sum x_n$ which is $|R, p_n|$ summable but not $|R, p_n|_k$, $k > 1$ summable. Actually, such a series is constructed in [5].

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