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k-COMMON CONSEQUENTS IN BOOLEAN MATRICES[1]

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1. INTRODUCTION

Let M_n denote the set of all $n \times n$ matrices over the Boolean algebra $\{0,1\}$, and let $V = \{a_1, \ldots, a_n\}$ be a finite set with $n \geqslant 2$. By a binary relation on V we mean a subset Q of $V \times V$. The set of all binary relations on V (including the empty relation) is denoted by $B_n(V)$. The map

$$Q \to M(Q) = (m_{ij})$$

where $m_{ij} = 1$ if $(a_i, a_j) \in Q$ and $m_{ij} = 0$ otherwise, is an isomorphism of $B_n(V)$ onto M_n.

Let $G_n(V)$ be the set of all directed graphs with n vertices $\{a_1, \ldots, a_n\}$. Then each matrix in M_n can be regarded as the adjacency matrix of $G \in G_n(V)$.

It is well known that there is a one to one correspondence between $B_n(V)$, M_n and $G_n(V)$:

$$Q \longleftrightarrow M(Q) \longleftrightarrow G(Q),$$

where $G(Q)$ is the graph corresponding to the matrix $M(Q)$.

In 1983, Š. Schwarz ([1]) introduced a concept of the common consequent as follows.

Definition 1.1. Let $Q \in B_n(V)$. We say that a pair of vertices (a_i, a_j), $a_i \neq a_j$, has a common consequent (c.c.) if there is a n integer $l > 0$ such that

$$a_i Q^l \cap a_j Q^l \neq \emptyset. \tag{1.1}$$

If a_i, a_j have a c.c. then the least integer $l > 0$ for which (1.1) holds is denoted by $L_Q(a_i, a_j)$.

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In 1990, we ([2]) introduced a concept of the generalized vertex exponent (G.V.E.) for $M(Q)$.

Definition 1.2. Let $Q \in B_n(V)$. The generalized vertex exponent of Q, denoted by $\exp_Q(1)$, is the least integer $l > 0$ such that

$$\bigcap_{i=1}^{n} a_i Q^l \neq \emptyset. \tag{1.2}$$

In terms of Boolean matrices, the common consequent in [1] means that the rows corresponding to a_i and a_j in $M(Q^l)$ have a 1 in the same column, while G.V.E. in [2] means that there is a column of all 1's in $M(Q^l)$.

Naturally we can extend the common consequent to the k common consequent (k-c.c.) as follows.

Definition 1.3. Let $Q \in B_n(V)$. We say that a group of vertices $\{a_{i_1}, \ldots, a_{i_k}\} \subset V = \{a_1, \ldots, a_n\}$, $2 \leqslant k \leqslant n$, $a_{i_t} \neq a_{i_u}$, $t \neq u$, has a k-common consequent (k-c.c.) if there is an integer $l > 0$ such that

$$\bigcap_{j=1}^{k} a_{i_j} Q^l \neq \emptyset. \tag{1.3}$$

If a_{i_1}, ..., a_{i_k} have a k-c.c. then the least integer $l > 0$ for which (1.3) holds is denoted by $L_Q(a_{i_1}, \ldots, a_{i_k})$.

If there is at least one group $(a_{i_1}, \ldots, a_{i_k})$ for which $L_Q(a_{i_1}, \ldots, a_{i_k})$ exists, we define $L_Q(k) = \max L_Q(a_{i_1}, \ldots, a_{i_k})$, where $(a_{i_1}, \ldots, a_{i_k})$ runs through all groups with k elements for which $L_Q(a_{i_1}, \ldots, a_{i_k})$ exists. If $M = M(Q)$, then we write $L_Q(k) = L_M(k)$. If there is no group $(a_{i_1}, \ldots, a_{i_k})$ for which $L_Q(a_{i_1}, \ldots, a_{i_k})$ exists, we define $L_Q(k) = L_M(k) = 0$.

In terms of Boolean matrices, k-c.c. means that the rows corresponding to a_{i_1}, ..., a_{i_k} in $M(Q^l)$ have a 1 in the same column.

Clearly, 2-c.c. is the common consequent in [1] while n-c.c. is the generalized vertex exponent in [2], which was obtained by Schwarz ([3]).

It is well known that a relation Q is called primitive if there is an integer $t > 0$ such that $Q^t = V \times V$. Let $P_n(V)$ be the set of all primitive relations in $B_n(V)$. Then it is easy to see that if $Q \in P_n(V)$, then $L_Q(a_{i_1}, \ldots, a_{i_k})$ exists for any group $(a_{i_1}, \ldots, a_{i_k})$, $2 \leqslant k \leqslant n$. We define

$$L(k) = \max\{L_Q(k) \mid Q \in P_n(V)\}.$$

As we know, a Boolean square matrix A is called reducible if there is a permutation matrix P such that PAP^{-1} is of the form

$$\begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where B, D are square matrices. Otherwise it is called irreducible. Let $IR_n(V)$ be the set of all irreducible relations in $B_n(V)$. For $Q \in B_n(V)$, we define

$$\tilde{L}(k) = \max\{L_Q(k) \mid Q \in IR_n(V)\}.$$

Up to now, we have known the following results:

$$L(2) = \begin{cases} \frac{1}{2}n^2 - n + 1 & \text{if } n \text{ is even,} \\ \frac{1}{2}n^2 - n + \frac{3}{2} & \text{if } n \text{ is odd,} \end{cases} \qquad \text{(Š. Schwarz 1985 [1])}$$

(or $L(2) = \frac{1}{2}n^2 - \frac{1}{2}n + 1 - \left[\frac{n}{2}\right]$),

$$L(n) = n^2 - 3n + 3. \qquad \text{(Š. Schwarz 1986 [3])}$$

In this paper we investigate $L(k)$ and $\tilde{L}(k)$, $2 \leqslant k \leqslant n-1$, and obtain some special bounds for $L(K)$ and $\tilde{L}(k)$. Generally, we have

$$L(k) \leqslant \tilde{L}(k) \leqslant \left[\frac{k-1}{k}n\right](n-1) + 1, \qquad 2 \leqslant k \leqslant n-1.$$

In many cases this result is the best possible.

2. Preliminaries

By the first projection $\Pi(Q)$ of Q we mean the subset of V consisting of all $a_i \in V$ for which $a_i Q \neq \emptyset$.

The following lemmas are obvious.

Lemma 2.1. *If* $\Pi(Q) = V$, *then* $\bigcap_{j=1}^{k} a_{i_j} Q^l \neq \emptyset$, $\{a_{i_1}, \ldots, a_{i_k}\} \subseteq V$, *implies* $\bigcap_{j=1}^{k} a_{i_j} Q^{l+t} \neq \emptyset$ *for any integer* $t > 0$.

Lemma 2.2. *If* $2 \leqslant k_1 \leqslant k_2 \leqslant n$, *then*

$$L_Q(k_1) \leqslant L_Q(k_2), \qquad Q \in B_n(V).$$

$Q \in B_n(V)$ is irreducible if and only if $G(Q)$ is strongly connected. (See, e.g., [1].)

If Q is irreducible, then for any $a_i \in V$ there is a least integer $h_i = h(a_i)$, $1 \leqslant h_i \leqslant n$, such that $a_i \in a_i Q^{h_i}$. Moreover, $M(Q)$ is permutation cogredient to a matrix of the form

$$\begin{pmatrix} 0 & A_1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & A_{d-1} \\ A_d & 0 & \dots & 0 & 0 \end{pmatrix}$$

where A_1 is a $v_i \times v_{i+1}$ submatrix, $d = (h_1, \dots, h_n)$. It is equivalent to the assertion that the set $V = \Pi(Q)$ admits a decomposition into d disjoint nonempty subsets $V = V_1 \cup \dots \cup V_d$ such that

$$Q \subset (V_1 \times V_2) \cup (V_2 \times V_3) \cup \dots \cup (V_d \times V_1),$$

where $|V_i| = v_i$ and $v_{d+1} = v_1$. The number d $(1 \leqslant d \leqslant n)$ is called the index of imprimitivity of Q. The sets V_1, ..., V_d are called the sets of imprimitivity of Q. Q is primitive iff it is irreducible and $d(Q) = 1$ (see, e.g., [1]).

The following lemma is known.

Lemma 2.3 ([1]). *Let Q be irreducible, $d \geqslant 1$ and let V' be one of the sets of imprimitivity of Q. If $a_i \in V'$, then there is an integer $k_0 \geqslant 0$ such that for any $k \geqslant k_0$ we have $a_i Q^{kd} = V'$.*

For k-c.c. we have

Theorem 2.4. *Let $Q \in B_n(V)$. Suppose that Q is irreducible and $d(Q) > 1$. Then $L_Q(a_{i_1}, \dots, a_{i_k})$ exists iff $a_{i_1}, \dots, a_{i_k}$ are contained in the same set of imprimitivity of Q.*

Proof. a) Suppose that $a_{i_j} \in V'$, $j = 1, \dots, k$. Then (by Lemma 2.3) there is an integer k_0 such that for any $k \geqslant k_0$ we have $a_{i_j} Q^{dk} = V'$, $j = 1$, ..., k. Hence $L_Q(a_{i_1}, \dots, a_{i_k})$ exists.

b) Let $a_{i_1} \in V'$, $a_{i_j} \notin V'$, $j = 2, \dots, k$, say $a_{i_2} \in V''$, $V' \neq V''$. By Lemma 1.1 [1] $L_Q(a_{i_1}, a_{i_2})$ does not exist. Hence $L_Q(a_{i_1}, \dots, a_{i_k})$ does not exist, either. □

According to Lemma 2.2 and the results of [1] and [3], we have

$$L(2) \leqslant L(k) \leqslant L(n),$$

namely $\frac{1}{2}n^2 - \frac{n}{2} + 1 - \left[\frac{n}{2}\right] \leqslant L(k) \leqslant n^2 - 3n + 3$, $2 \leqslant k \leqslant n$.

3. Estimations of $L(k)$ for a primitive relation

We need the following lemma in [1] to derive a better estimate of $L(k)$.

Lemma 3.1 ([1]). *Let Q be irreducible, $Q \in B_n(V)$, $n \geqslant 2$ and let V_1 be a nonempty proper subset of V. Then $V_1 Q$ contains at least one element of V which is not contained in V_1.*

Corollary 3.2. *Let Q be primitive, $Q \in B_n(V)$, $n \geqslant 2$ and $a_i \in V$. If $a_i Q^s = a_i Q^t$ for some $1 \leqslant s < t$, then $a_i Q^s = V$.*

Lemma 3.3. *Let $V = \{a_1, \dots, a_n\}$ and let $V_1, \dots, V_k$ $(2 \leqslant k \leqslant n)$ be the subsets of V with $|V_i| \geqslant r > 0$, $i = 1, \dots, k$. If $r \geqslant \left[\frac{k-1}{k} n\right] + 1$, then $\bigcap_{i=1}^{k} V_i \neq \emptyset$.*

P r o o f. First of all, we prove that

$$\left|\bigcup_{i=1}^{k} V_i\right| \geqslant kr - (k-1)n, \qquad 2 \leqslant k < n. \tag{3.1}$$

If $k = 2$, $\left|\bigcap_{i=1}^{2} V_i\right| \geqslant |V_1| + |V_2| - |V| \geqslant 2r - 3n$.
If $k = 3$, $\left|\bigcap_{i=1}^{3} V_i\right| \geqslant |V_3| - \left(|V| - \left|\bigcap_{i=1}^{2} V_i\right|\right) \geqslant r - n + (2r - n) = 3r - 2n$.
Suppose that $\left|\bigcap_{i=1}^{k-1} V_i\right| \geqslant (k-1)r - (k-2)n$, $2 \leqslant k \leqslant n-1$. Then

$$\begin{aligned}\left|\bigcap_{i=1}^{k} V_i\right| &\geqslant |V_k| - \left(|V| - \left|\bigcap_{i=1}^{k} V_i\right|\right) \geqslant r - n + [(k-1)r - (k-2)n] \\ &= kr - (k-1)n, \qquad 2 \leqslant k \leqslant n.\end{aligned}$$

If $r \geqslant \left[\frac{k-1}{k} n\right] + 1$, by (3.1)

$$\left|\bigcap_{i=1}^{k} V_i\right| \geqslant k\left(\left[\frac{k-1}{k} n\right] + 1\right) - (k-1)n. \tag{3.2}$$

Case 1. $k \mid n$.

According to (3.1)

$$\left|\bigcap_{i=1}^{k} V_i\right| \geqslant (k-1)n + k - (k-1)n = k > 0.$$

Case 2. $k \nmid n$.

Let $n = ak + t$, $t = 1, \dots, k-1$, a is an integer, $a > 1$. According to (3.1) we have

$$\Big|\bigcap_{i=1}^{k} V_i\Big| \geqslant k\Big(\Big[(k-1)a + t - \frac{t}{k}\Big] + 1\Big) - (k-1)(ak+t)$$
$$= k[(k-1)a + t - 1 + 1] - (k-1)(ak+t) = t > 0.$$

Hence $\bigcap_{i=1}^{k} V_i \neq \emptyset$. □

Note that if Q is primitive, Q^t is primitive for any $t > 1$. We have

Lemma 3.4. *Suppose that Q is primitive, $Q \in B_n(V)$, $n \geqslant 2$. Recall that h_i is the least integer for which $a_i \in a_i Q^{h_i}$. Then*

$$L_Q(a_{i_1}, \dots, a_{i_k}) \leqslant \Big[\frac{k-1}{k} n\Big] \max(h_{i_1}, \dots, h_{i_k}).$$

P r o o f. Consider the chain

$$a_{i_j} \in a_{i_j} Q^{h_{i_j}} \subset a_{i_j} Q^{2h_{i_j}} \subset \dots \subset a_{i_j} Q^{[\frac{k-1}{k} n] h_{i_j}} \quad (j = 1, \dots, k). \tag{3.3}$$

By Lemma 3.1 and Corollary 3.2 we have

$$\big|a_{i_j} Q^{[\frac{k-1}{k} n] h_{i_j}}\big| \geqslant \Big[\frac{k-1}{k} n\Big] + 1.$$

Let $h = \max(h_{i_1}, \dots, h_{i_k})$. Multiplying each term in (3.3) by $Q^{[\frac{k-1}{k} n](h - h_{i_j})}$ (define $Q^0 = I$), we obtain

$$a_{i_j} Q^{[\frac{k-1}{k} n](h - h_{i_j})} \subset a_{i_j} Q^{h_{i_j} + [\frac{k-1}{k} n](h - h_{i_j})} \subset \dots \subset a_{i_j} Q^{[\frac{k-1}{k} n] h},$$

whence $|a_{i_j} Q^{[\frac{k-1}{k} n] h}| \geqslant [\frac{k-1}{k} n] + 1$, $j = 1, \dots, k$. Therefore by Lemma 3.3

$$\bigcap_{j=1}^{k} a_{i_j} Q^{[\frac{k-1}{k} n] h} \neq \emptyset.$$

Hence $L_Q(a_{i_1}, \dots, a_{i_k}) \leqslant [\frac{k-1}{k} n] \max(h_{i_1}, \dots, h_{i_k})$. □

Let the lengths of the largest circuit and the least circuit in $G(Q)$ be $\overline{h}$ and h_0, respectively. We have

Corollary 3.5. *Let Q be primitive, $Q \in B_n(V)$. If $\overline{h} \leqslant n - 1$, then*

$$L_Q(k) \leqslant \Big[\frac{k-1}{k} n\Big](n-1). \tag{3.4}$$

In order to obtain better estimates of $L(k)$ using h_0, we establish the following lemma.

Lemma 3.6. *Let Q be primitive, $Q \in B_n(V)$ and $n \geqslant 4$. Denote $L_1 = \left(\left[\frac{k-1}{k}n\right] - 1\right)h_0 + n$. Then for any $a_i \in V$ we have*

$$|a_i Q^{L_1}| \geqslant \left[\frac{k-1}{k}n\right] + 1.$$

P r o o f. Let C be a circuit of length h_0. Denote by $V(C)$ the set of vertices of C. For $\forall u \in V(C)$ we have $u \in uQ^{h_0}$.

For any $a_i \in V - V(C)$, there is a path of length k_i, $1 \leqslant k_i \leqslant n - h_0$, joining a_i with some $u_j \in V(C)$. This means: there is $u_j \in V(C)$ such that $u_j \in a_i Q^{k_i}$, where $k_i \leqslant n - h_0$. Consider the chain

$$u_j \in u_j Q^{h_0} \subset u_j Q^{2h_0} \subset \cdots \subset u_j Q^{[\frac{k-1}{k}n]h_0}$$

and for any integer $t \geqslant 1$, then chain

$$u_j Q^t \subset u_j Q^{h_0+t} \subset \cdots \subset u_j Q^{[\frac{k-1}{k}n]h_0+t}.$$

For any $t \geqslant 0$ we have

$$|u_j Q^{[\frac{k-1}{k}n]h_0+t}| \geqslant \left[\frac{k-1}{k}n\right] + 1.$$

Now, since $u_j \in a_i Q^{k_i}$, we have

$$\left[\frac{k-1}{k}n\right] + 1 \leqslant |u_j Q^{[\frac{k-1}{k}n]h_0+t}| \leqslant |a_i Q^{[\frac{k-1}{k}n]h_0+t+k_i}|.$$

Putting $t = n - h_0 - k_i \geqslant 0$, we have

$$|a_i Q^{L_1}| \geqslant \left[\frac{k-1}{k}n\right] + 1.$$

If u belong to C, the chains

$$u \in uQ^{h_0} \subset uQ^{2h_0} \subset \cdots \subset uQ^{[\frac{k-1}{k}n]h_0},$$
$$uQ^t \subset uQ^{h_0+t} \subset uQ^{2h_0+t} \subset \cdots \subset uQ^{[\frac{k-1}{k}n]h_0+t}$$

show that for any $t \geqslant 0$

$$|uQ^{[\frac{k-1}{k}n]h_0+t}| \geqslant \left[\frac{k-1}{k}n\right] + 1.$$

Putting $t = n - h_0$ we obtain $|uQ^{L_1}| \geqslant \left[\frac{k-1}{k}n\right] + 1$. □

Lemma 3.7. *Let Q be primitive, $Q \in B_n(V)$, $n \geqslant 2$. Suppose that $h_0 \leqslant n - 3$. Then*

$$L_Q(k) \leqslant \left(\left[\frac{k-1}{k}n\right] - 1\right)(n-3) + n.$$

P r o o f. Denote $L_1 = [\frac{k-1}{k}n]h_0 + n - h_0$. Since $|a_i Q^{L_1}| \geqslant [\frac{k-1}{k}n] + 1$, we have

$$\bigcap_{i=1}^{k} a_{i_j} Q^{L_1} \neq \emptyset \quad \text{and} \quad L_Q(k) \leqslant L_1 \leqslant \left[\frac{k-1}{k}n\right](n-3) + n.$$

□

Remark. If $n \geqslant 2$, then $[\frac{k-1}{k}n](n-3) + n \leqslant [\frac{k-1}{k}n](n-1) + 1$. By Lemma 3.7 and by (3.4) we need to consider only $h_0 \geqslant n - 2$, $h = n$.

Applying an argument analogous to [1] we treat only two cases as follows.

Case 1. The relation Q given by the graph in Figure 1: $h_0 = n - 2$, $\overline{h} = n$ ($n \geqslant 5$, n is odd).

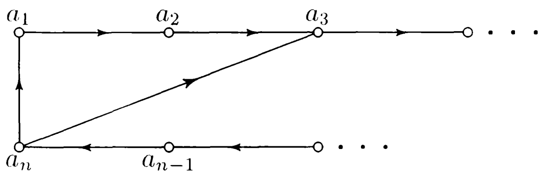

Fig. 1

We shall prove that

$$L_Q(k) \leqslant \left[\frac{k-1}{k}n\right](n-2) + 2. \tag{3.5}$$

Consider the chains

$$a_3 \in a_3 Q^{n-2} \subset a_3 Q^{2(n-2)} \subset \cdots \subset a_3 Q^{[\frac{k-1}{k}n](n-2)}$$

and

$$a_3 Q^t \subset a_3 Q^{n-2+t} \subset a_3 Q^{2(n-2)+t} \subset \cdots \subset a_3 Q^{[\frac{k-1}{k}n](n-2)+t}, \tag{3.6}$$

and denote $L_2 = [\frac{k-1}{k}n](n-2)$. For any integer $t \geqslant 0$, (3.6) implies $|a_3 Q^{L_2+t}| \geqslant [\frac{k-1}{k}n] + 1$.

Since $a_3 = a_1Q^2$, $a_3 = a_2Q$, we have

$$|a_1Q^{L_2+2}| \geqslant \left[\frac{k-1}{k}n\right] + 1, \quad |a_2Q^{L_2+2}| \geqslant \left[\frac{k-1}{k}n\right] + 1.$$

Further, for $3 < i \leqslant n$ we have $a_i = a_3Q^{i-3}$, whence

$$|a_3Q^{L_2+t}| = |a_3Q^{i-3}Q^{L_2-(i-3)+t}| = |a_iQ^{L_2-(i-3)+t}| \geqslant \left[\frac{k-1}{k}n\right] + 1.$$

Putting $t = i - 1$ ($n \geqslant 5$),we have

$$|a_iQ^{L_2+2}| \geqslant \left[\frac{k-1}{k}n\right] + 1, \qquad 3 < i \leqslant n.$$

Hence by Lemma 3.3

$$L_Q(k) \leqslant L_2 + 2 = \left[\frac{k-1}{k}n\right](n-2) + 2.$$

Case 2. The relation Q given by the graph in Figure 2.

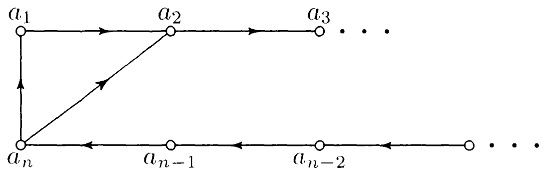

Fig. 2

Using an argument similar to that in the proof of Lemma 2.9 in [1], we can obtain the following conclusion.

If M_0 is the least integer $m > 0$ such that $a_2Q^m \cap a_2Q^{m+s_1} \cap \ldots \cap a_2Q^{m+s_{k-1}} \neq \emptyset$ for $\{s_1, \ldots, s_{k-1}\} \subset \{1, \ldots, n\}$, $s_i \neq s_j$ if $i \neq j$, then

(3.7) $$L_Q(k) = M_0 + 1.$$

In [1], it was proved that

(3.8) $$a_2Q^{n-1} = \{a_2, a_1\},$$
$$a_2Q^{k(n-1)} = \{a_2, a_1, a_n, a_{n-1}, \ldots, a_{n-(k-2)}\}, \quad 2 \leqslant k \leqslant n-1.$$

Let now $L_0 = \left[\frac{k-1}{k}n\right](n-1)$. Since

$$a_2 \subset a_2Q^{n-1} \subset \cdots \subset a_2Q^{[\frac{k-1}{k}n](n-1)}$$

we conclude that $|a_2Q^{L_0}| \geqslant \left[\frac{k-1}{k}n\right]+1$ and also $|a_2Q^{L_0+s}| \geqslant \left[\frac{k-1}{k}n\right]+1$ for any $s>0$. Hence for any $\{s_1,\dots,s_{k-1}\} \subset \{1,\dots,n-2\}$, $\bigcap_{i=0}^{k-1} a_2Q^{L_0+s_i} \neq \emptyset$, where $s_0=0$. This implies $M_0 \leqslant L_0$.

According to (3.7)

$$L_Q(k) \leqslant L_0+1 = \left[\frac{k-1}{k}n\right](n-1)+1. \tag{3.9}$$

Hence we obtain the main result from the above conclusions.

Theorem 3.8. *If Q is a primitive relation, $Q \in B_n(V)$, $n \geqslant 2$, then*

$$L_Q(k) \leqslant L_0+1 = \left[\frac{k-1}{k}n\right](n-1)+1, \quad 2 \leqslant k \leqslant n-1. \tag{3.10}$$

The following example shows that sometimes the bound is sharp for primitive relations given in Figure 2.

Example. Let Q be the relation defined by the graph in Figure 2, $Q \in B_n(V)$, $M=M(Q)$.

If $n=7$, $k=3$, then

$$M_7^{24} = \begin{pmatrix} 0&0&0&1&1&1&1\\ 1&1&0&0&1&1&1\\ 1&1&1&0&0&1&1\\ 1&1&1&1&0&0&1\\ 1&1&1&1&1&0&0\\ 0&1&1&1&1&1&0\\ 0&0&1&1&1&1&1 \end{pmatrix}, \qquad M_7^{25} = \begin{pmatrix} 1&1&0&0&1&1&1\\ 1&1&1&0&0&1&1\\ 1&1&1&1&0&0&1\\ 1&1&1&1&1&0&0\\ 0&1&1&1&1&1&0\\ 0&0&1&1&1&1&1\\ 1&1&0&1&1&1&1 \end{pmatrix}.$$

For M_7^{24} we have $a_1Q^{24} \cap a_3Q^{24} \cap a_5Q^{24} = \emptyset$ while for any a_i, a_j, a_r we have $a_iQ^{25} \cap a_jQ^{25} \cap a_rQ^{25} \neq \emptyset$. Thus $L_Q(3)=25$.

The bound (3.9) gives $\left[\frac{2}{3}\times 7\right](7-1)+1=25$.

If $n=6$, $k=3$, then the bound (3.9) yields

$$\left[\frac{2}{3}\times 6\right](6-1)+1=21.$$

However,

$$M_6^{16} = \begin{pmatrix} 1&1&0&0&1&1\\ 1&1&1&0&0&1\\ 1&1&1&1&0&0\\ 0&1&1&1&1&0\\ 0&0&1&1&1&1\\ 1&1&0&1&1&1 \end{pmatrix}, \qquad M_6^{16} = \begin{pmatrix} 1&1&0&0&1&1\\ 1&1&1&0&0&1\\ 1&1&1&1&0&0\\ 0&1&1&1&1&0\\ 0&0&1&1&1&1\\ 1&1&0&1&1&1 \end{pmatrix}.$$

It is easy to see that $a_1Q^{16} \cap a_3Q^{16} \cap a_5Q^{16} = \emptyset$ while for any a_i, a_j, a_r, we have $a_iQ^{17} \cap a_jQ^{17} \cap a_rQ^{17} \neq \emptyset$. Thus $L_Q(k) = 17 < 21$.

Sometimes the bound in Theorem 3.8 is the best possible. For example when $k = 2$ and n is odd Schwarz had shown that the bound (3.10) is the best possible.

4. Estimations of $\tilde{L}(k)$ for irreducible relation

Since we know the bound of $L(k)$ for a primitive relation, we shall consider only imprimitive relations. Noticing that $\tilde{L}(k)$ does not exist for $n = 2$, we may suppose $n \geqslant 3$.

Theorem 4.1. *Suppose that $Q \in B_n(V)$, $n \geqslant 3$, Q is irreducible and $d(Q) > 1$. Denote $\min\limits_t |V_t| = \beta$.*

a) *If $\beta < k$ and $L_Q(k)$ exists, then $L_Q(k) \leqslant d - 1$.*

b) *If $\beta \geqslant k$ and $L_Q(k)$ exists, then*

$$L_Q(k) \leqslant d - 1 + d\Big(\Big[\frac{k-1}{k}\beta\Big](\beta - 1) + 1\Big).$$

P r o o f. Without loss of generality we may suppose that the matrix representation of Q is of the form

$$\begin{pmatrix} 0 & B_1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & B_{d-1} \\ B_d & 0 & \dots & 0 & 0 \end{pmatrix}.$$

In this case we have

$$M(Q^d) = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_d \end{pmatrix},$$

where A_k are primitive $v_k \times v_k$ Boolean matrices, $\Pi(A_k) = V_k$ are the sets of imprimitivity of Q, and $\bigcup\limits_{t=1}^{d} V_t = V$, $\sum\limits_{i=1}^{d} v_i = n$. By Theorem 2.4, $L_Q(a_{i_1}, \dots, a_{i_k})$ exists iff $a_{i_1}, \dots, a_{i_k}$ are contained in the same set of imprimitivity of Q, say V_t. Suppose that this is the case and $v_t \geqslant 2$. Applying Theorem 3.8 we have

$$L_Q(k) \leqslant d\Big(\Big[\frac{k-1}{k}v_t\Big](v_t - 1) + 1\Big).$$

Let $|V_0| = \beta$. Consider the following two cases.

a) $|V_0| = \beta < k$.

If $|V_t| < k$, $t = 1, \ldots, d$, then no k elements of V have a c.c. In any V_t with $|V_t| \geqslant k$ choose k vertices $a_{i_1}, \ldots, a_{i_k}$. Since $V_0 = V_t Q^u$ for some u, $1 \leqslant u \leqslant d-1$, we have $a_1 Q^u = \ldots = a_k Q^u$, i.e. $L_Q(k)$ exists and $L_Q(k) \leqslant d-1$.

b) $|V_0| = \beta \geqslant k$.

For any $a_1, \ldots, a_k \in V_0$ we have

$$L_Q(k) \leqslant d\Big(\Big[\frac{k-1}{k}\beta\Big](\beta - 1) + 1\Big) = L_3,$$

i.e.

$$\bigcap_{i=1}^{k} a_i Q^{L_3} \neq \emptyset.$$

Let $V_t \neq V_0$ be any set of imprimitivity, $a_1, \ldots, a_k \in V_t$. Since $V_0 = V_t Q^u$ for some u, $1 \leqslant u \leqslant d-1$. Then $a_i Q^u \subset V_0$, $i = 1, \ldots, k$. Therefore $\bigcap_{i=1}^{k} a_i Q^u Q^{L_3} \neq \emptyset$.

$$L_Q(k) \leqslant u + L_3 \leqslant d - 1 + d\Big(\Big[\frac{k-1}{k}\beta\Big](\beta - 1) + 1\Big).$$

□

Write $n = \alpha d + \alpha_1$, where $\alpha \geqslant 1$ is an integer and $0 \leqslant \alpha_1 \leqslant d-1$. Then the least of the number $|V_1|, \ldots, |V_t|$ is $\leqslant \alpha$.

We have $k \leqslant \beta \leqslant \frac{n-\alpha_1}{d}$.

Let $N(\beta, k) = \big[\frac{k-1}{k}\beta\big](\beta - 1) + 1$. This is an increasing function of β. If $L_Q(k)$ exists, we have

$$\begin{aligned} L_Q(k) &\leqslant d - 1 + dN(\beta, k) \leqslant d - 1 + dN(\alpha, k) \\ &= d - 1 + d\Big(\Big[\frac{k-1}{k} \cdot \frac{n-\alpha_1}{d}\Big]\Big(\frac{n-\alpha_1}{d} - 1\Big) + 1\Big). \end{aligned}$$

Putting here $\alpha_1 = 0$ we have

Corollary 4.2. *Let $Q \in B_n(V)$, Q is irreducible, $n \geqslant 3$, $d(Q) > 1$. If $L_Q(k)$ exists, then*

$$\begin{aligned} L_Q(k) &\leqslant d - 1 + d\Big(\Big[\frac{k-1}{k} \cdot \frac{n}{d}\Big]\Big(\frac{n}{d} - 1\Big) + 1\Big) \\ &= d\Big(\Big[\frac{k-1}{k} \cdot \frac{n}{d}\Big]\Big(\frac{n}{d} - 1\Big) + 2\Big) - 1 = \Big[\frac{k-1}{k} \cdot \frac{n}{d}\Big](n - d) + 2d - 1. \end{aligned}$$

Denote $\left[\frac{k-1}{k}\cdot\frac{n}{d}\right](n-d)+2d-1=f(d)$. In order to prove

$$L_Q(k)\leqslant\left[\frac{k-1}{k}n\right](n-1)+1 \tag{4.1}$$

for an irreducible relation, we shall prove

$$f(d)\leqslant\left[\frac{k-1}{k}n\right](n-1)+1. \tag{4.2}$$

Since for $k=2$ Schwarz ([1]) had shown that (4.1) holds, we consider only $k\geqslant 3$. It is easy to prove that $f(d)$ is a decreasing function if $d\in\left(0,\sqrt{\frac{k-1}{2k}n}\right]$, while $f(d)$ is an increasing function if $d\in\left(\sqrt{\frac{k-1}{2k}n},n\right)$ ($d=n$, $M(Q)$ is a permutation matrix, $L_Q(k)$ does not exist.) Thus

$$\begin{aligned} f(d)&\leqslant\max\left(f(2),f(n-1)\right)\\ &=\max\left(\frac{k-1}{2k}n^2-\frac{k-1}{k}n+3,\frac{k-1}{k}\cdot\frac{n}{n-1}+2n-3\right)\\ &\leqslant\begin{cases}6 & n=4,\\ \dfrac{k-1}{2k}n^2-\dfrac{k-1}{k}n+3 & n\geqslant 5.\end{cases}\qquad (3\leqslant k<n).\end{aligned}$$

But if $n=4$, $k=3$, then $\left[\frac{k-1}{k}n\right](n-1)+1=\left[\frac{2}{3}\times 4\right]\times 3+1=7>6$.

If $n\geqslant 5$ then it is not difficult to prove

$$\frac{k-1}{2k}n^2-\frac{k-1}{k}n+3\leqslant\left[\frac{k-1}{k}n\right](n-1)+1.$$

Hence (4.2) holds for $n\geqslant 3$, $2\leqslant k<n$. We have

Theorem 4.3. *Suppose that $Q\in B_n(V)$, $n\geqslant 3$, Q is irreducible. If $L_Q(k)$ exists, $2\leqslant k<n$, we have*

$$L_Q(k)\leqslant\left[\frac{k-1}{k}n\right](n-1)+1. \tag{4.3}$$

Remark. Applying (4.3) for $k=n-1$, we have

$$\tilde{L}(n-1)\leqslant n^2-3n+3,$$

while by the result of Schwarz ([3])

$$L(n)=n^2-3n+3.$$

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