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RADON-NIKODYM DERIVATIVES IN VECTOR INTEGRATION

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0. INTRODUCTION

By means of localization techniques of the type used in [9], a Radon-Nikodym theorem was stated in [1] in the context of the locally convex spaces, using an integration with a deep bornological character, defined in 1983 by R. Rao and A. S. Sastry. This vector integration cannot bee in general compared with the bilinear vector integral used in [2] ([3], and in other papers appearing in their references). In the context of the Banach spaces the integrable functions of [2] coincide with the functions of L^1 following Dobrakov, if the space of operators is endowed with the strong topology. The integral introduced in Definition 1 extends to the integral of [2] and coincides with Dobrakov's one [5] in the context of the Banach spaces, being the space of operators endowed with the strong topology. By means of this integral, a weak vector integration is introduced in Definition 2 which extends the integral of [10] and allows us to give a Radon-Nikodym theorem about the derivation of a measure with values in a locally convex space Y with respect to a measure valued in the space of the linear and continuous functions from a Banach space X into Y, endowed with the pointwise convergence topology. This theorem extends the corresponding one given in [9] for Banach spaces and measures of bounded variation.

Clearly every integrable function according to Definition 1 is weak integrable (Definition 2) and the Radon-Nikodym type theorem, proved here in Theorem 8, allow to state (Corollary 9) that for a given weak integrable function there exists an integrable function (according to Definition 1) such that their integrals on every measurable set coincide. The application of this result to the integrals of [5] and [10] would be doubtless fruitful for both theories.

1. Preliminaries and notation

Let Σ be a σ-algebra of subsets of a set Ω and $L(X, Y)$ the space of the linear and continuous functions from a Banach space X into a locally convex Hausdorff and complete space Y, endowed with the topology of the pointwise convergence. From now on, $\alpha: \Sigma \to Y$ will be a countably additive vector measure, and $\mu: \Sigma \to L(X, Y)$ will denote a countable additive measure of bounded continuous semivariation, and as usual $|\mu|_q$ and $\|\mu\|_q$ will be the q-variation and the q-semivariation of μ respectively, for every $q \in Q$, where Q is a saturated generating family of seminorms on Y.

Σ^+ will be the family of all subsets $A \in \Sigma$ such that $\|\mu\|_q(A) > 0$ for some $q \in Q$, and we will assume that every pairwise disjoint family contained in Σ^+ is at most countable (or finite). (Let us remark that this condition holds trivially if μ verifies the **-condition in the known Bartle' sense—see for instance [1].)

Definition 1. A function $f: \Omega \to X$ is said to be *μ-integrable* if the following conditions are verified:

1.1 For every $q \in Q$ and $\varepsilon > 0$ there exists $A \in \Sigma$ such that $\|\mu\|_q(A) < \varepsilon$ and $f \cdot X_{\Omega - A}$ is a uniform limit of simple functions (simple functions and their integrals are defined as usual, and the integral of $f \cdot X_{\Omega - A}$ is defined from the integral of simple functions in a standard way).

1.2 For every $q \in Q$ we have

$$\lim_{\substack{\|\mu\|_q(A) \to 0 \\ a \in \Sigma_f}} q\left(\int_A f \, \mathrm{d}\mu \right) = 0.$$

Σ_f being the set of all $A \in \Sigma$ such that $f \cdot X_A$ is a uniform limit of simple functions.

If 1.1 and 1.2 hold, then

$$\int_B f \, \mathrm{d}\mu = \lim_{A \in \Sigma_f} \int_{B \cap A} f \, \mathrm{d}\mu.$$

Definition 2. A function $f: \Omega \to X$ is said to be a *weakly μ-integrable* function if the following conditions hold:

2.1 f is $y'\mu$-integrable in the sense of Definition 1 for every $y' \in Y'$ (as usual, Y' denotes the dual space of Y).

2.2 For every $A \in \Sigma$ there exists $y_A \in Y$ such that

$$y'(y_A) = \int_A f \, \mathrm{d}(y'\mu)$$

for every $y' \in Y'$.

If 2.1 and 2.2 hold then we will write $y_A = \int_A f \, d\mu$.

Definition 3. We say that a set $B \in \Sigma^+$ *is localized on a subset* $K \subseteq X$ (with respect to α and μ) if for every $D \in \Sigma^+$ with $D \subseteq B$ and $\varepsilon > 0$ there exist $E \in \Sigma^+$ and $b \in K$ such that $E \subseteq D$ and

$$|y'(\alpha(A) - \mu(A)(b)) \leqslant \varepsilon \cdot |y'\mu|(A),$$

for every $y' \in Y'$ and $A \in \Sigma$ with $A \subseteq E$.

3. Main results

Proceeding like in [9] the following lemma and proposition can be proved.

Lemma 4. *If a set $B \in \Sigma^+$ is localized on a compact subset $K \subseteq X$, then for every $\varepsilon > 0$ there exist a compact $K_1 \subseteq X$ with diameter less than (or equal to) ε and $D \in \Sigma^+$ such that $K_1 \subseteq K, D \subseteq B$ and D is localized on K_1.*

Proposition 5. *Suppose that the following conditions are verified:*

5.1 *For every $A \in \Sigma^+$ there exist $B \in \Sigma^+$ and a compact subset $K \subseteq X$ such that $B \subseteq A$ and B is localized on K.*

5.2 *$q(\alpha(A)) = 0$ if $\|\mu\|_q(A) = 0$ for every $A \in \Sigma$ and $q \in Q$.*

Then, for every $n \geqslant 1$ there exists a measurable partition $\{A^n_m\}_{m \in I_n} \subseteq \Sigma^+$ of Ω (where $I_n \subseteq \mathbb{N}$) and a family $\{K^n_m\}_{m \in I_n}$ of compact subsets of X with diameter less than or equal to $1/n$, such that

5.3 *For every $n \in \mathbb{N}$ and $m \in I_n$ there exists $x^n_m \in K^n_m$ such that*

$$|y'(\alpha(B) - \mu(B)(x^n_m))| \leqslant \frac{1}{n}|y'\mu|(B),$$

for every $y' \in Y'$ and $B \in \Sigma$ with $B \subseteq A^n_m$.

5.4 *For every $m \in I_{n+1}$ there exists $j \in I_n$ such that $A^{n+1}_m \subseteq A^n_j$ and $K^{n+1}_m \subseteq K^n_j$.*

Moreover, it can be assumed that A^n_m is localized on K^n_m for every $m \in I_n$ and $n \in \mathbb{N}$.

Proposition 6. *If $\lim\limits_{\substack{\|\mu\|_q(A) \to 0 \\ a \in \Sigma}} |y'\alpha(A)| = 0$ for every $q \in Q$ and every $y' \in Y'$ with $\|y'\|_q = \sup\{|y'(y)\|: y \in Y$ with $q(y) \leqslant 1$, and for every $A \in \Sigma^+$ there exist $B \in \Sigma^+$*

and a compact subset $K \subseteq X$ such that $B \subseteq A$ and B is localized on K, then α is of bounded semivariation, $y'\alpha$ is of bounded semivariation for every $y' \in Y'$, and

6.1 $$\lim_{\substack{\|\mu\|_q(A)\to 0\\ a\in\Sigma}} q(\alpha(A)) = 0$$

for every $q \in Q$.

Proof. Let be $y' \in Y'$, then there exists $q \in Q$ such that $\|y'\|_q \leqslant 1$, and Proposition 5 implies the existence of a measurable partition (that we can assume to be countable) $\{A_n\}_{n\in\mathbb{N}} \subseteq \Sigma^+$ of Ω and $\{x_n\}_{n\in\mathbb{N}} \subseteq K$ such that the inequality $|y'(\alpha(B) - \mu(B)(x_n))| \leqslant |y'\mu|(B)$ holds for every $n \in \mathbb{N}$ and $B \in \Sigma$ with $B \subseteq A_n$. Then there exists $\delta > 0$ such that $|y'\alpha(A)| < 1$ for every $A \in \Sigma$ with $\|\mu\|_q(A) < \delta$, and since $\|\mu\|_q$ is continuous we can find $j \in \mathbb{N}$ such that $\|\mu\|_q(\bigcup\limits_{n>j} A_n) < \delta$. So for every $B \in \Sigma$ we have that

$$\begin{aligned}
|y'\alpha(B)| &\leqslant \sum_{n=1}^{j} |y'(\alpha(B\cap A_n) - \mu(B\cap A_n)(x_n))| + \Big|y'\Big(\sum_{n=1}^{j} \mu(B\cap A_n)(x_n)\Big)\Big| + 1\\
&\leqslant \sum_{n=1}^{j} |y'\mu|(B\cap A_n) + [\max_{1\leqslant n\leqslant j} \|x_n\|]\, |y'\mu|(B) + 1\\
&\leqslant [1 + \max_{1\leqslant n\leqslant j} \|x_n\|\,]\, |y'\mu|(\Omega) + 1\\
&\leqslant [1 + \max_{1\leqslant n\leqslant j} \|x_n\|\,]\, \|\mu\|_q(\Omega) + 1 = M < +\infty.
\end{aligned}$$

And therefore, $\|y'\alpha\|(\Omega) \leqslant 2M < +\infty$.

Moreover, it follows from the Nikodym boundedness theorem that $\|\alpha\|_q(\Omega) < +\infty$. So $\{y'\alpha : y' \in Y', \|y'q\| \leqslant 1\}$ is a family of uniformly bounded and uniformly countably additive measures, and it follows from Theorem 1.2.4 of [4] (p. 11) that if $(C_n)_{n\in\mathbb{N}} \subseteq \Sigma$ is a pairwise disjoint sequence, then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$q(\alpha(B)) = \sup_{\substack{y'\in Y'\\ \|y'\|_q\leqslant 1}} |y'\alpha(B)| \leqslant \varepsilon$$

for every $B \in \Sigma$ with $B \subseteq \bigcup\limits_{n\geqslant n_0} C_n$: this yields (6.1) since $q(\alpha(A)) = 0$ if $A \in \Sigma$ verifies the identity $\|\mu\|_q(A) = 0$, as it is easily deduced from the hypothesis and the Hahn-Banach theorem. □

Proposition 7. *If there exists a weakly μ-integrable function $f : \Omega \to X$ such that $\alpha(A) = \int_A f \,\mathrm{d}\mu$ for every $A \in \Sigma$, then we have*

7.1 $$\lim_{\substack{\|\mu\|_q(A)\to 0\\ a\in\Sigma}} |y'\alpha(A)| = 0$$

for every $q \in Q$ and $y' \in Y'$ with $\|y'\|_q \leqslant 1$.

7.2 *For every $A \in \Sigma^+$ there exist a set $B \in \Sigma^+$ and a compact subset $K \subseteq X$ such that $B \subseteq A$ and B is localized on K.*

Proof. 7.1 Let $q \in Q$ and $y' \in Y'$ verify $\|y'\|_q \leqslant 1$. Then

$$(y'\alpha)(A) = \int_A f \,\mathrm{d}(y'\mu) \qquad (A \in \Sigma);$$

and, therefore, $y'\alpha$ is $|y'\mu|$-continuous and so 7.1 is satisfied.

7.2 If $A \in \Sigma^+$, then there exist $q_0 \in Q$ and $y'_0 \in Y'$ such that $\|\mu\|_{q_0}(A) > 0$, $\|y'_0\|_{q_0} \leqslant 1$ and $|y'_0\mu|(A) > 0$. Since the function f is $y'_0\mu$-integrable, there exists $\Omega_1 \in \Sigma$ such that

$$|y'_0\mu|(\Omega - \Omega_1) < \frac{|y'_0\mu|(A)}{2}$$

and $f \cdot X_{\Omega_1}$ is a uniform limit of simple functions. Let us consider $B = A \cap \Omega_1$ and $K = \overline{f(B)}$. Clearly K is compact since $f \cdot X_{\Omega_1}$ is a uniform limit of simple functions, and moreover

$$\|\mu\|_{q_0}(B) \geqslant |y'_0\mu|(B) \geqslant |y'_0\mu|(A) - |y'_0\mu|(\Omega - \Omega_1) \geqslant \frac{|y'_0\mu|(A)}{2} > 0$$

and $B \in \Sigma^+$. Let us prove now that B is localized on K. In fact, if $D \in \Sigma^+$, $\varepsilon > 0$ and $D \subseteq B$, then $f(D) \subseteq K$ and there exist $d_1, \ldots, d_k \in D$ such that $f(D) \subseteq \bigcup_{i=1}^{k} B(f(d_i), \varepsilon)$, so $D = \bigcup_{i=1}^{k} D_i$ with $D_i = D \cap f^{-1}(B(f(d_i), \varepsilon))$; and since $D \in \Sigma^+$ there exists $j \in \{1, \ldots, k\}$ such that $D_j \in \Sigma^+$. If we take $b = f(d_j) \in f(D) \subseteq K$, for every $A \in \Sigma$ with $A \subseteq D_j$ we have that $f(A) \subseteq B(f(d_j), \varepsilon)$; and therefore,

$$\begin{aligned}|y'(\alpha(A) - \mu(A)(b))| &= \left| \int_A (f - f(d_j)) \,\mathrm{d}(y'\mu) \right| \\ &\leqslant [\sup_{a \in A} \|f(a) - f(d_j)\|] |y'\mu|(A) \leqslant \varepsilon |y'\mu|(A)\end{aligned}$$

for every $y' \in Y'$ and the proof is complete. □

Theorem 8. *α has a Radon-Nikodym derivative with respect to μ (i. e. there exists a μ-integrable function $f \colon \Omega \to X$ such that $\int_A f \,\mathrm{d}\mu = \alpha(A)$ for every $A \in \Sigma$) if and only if the following conditions hold:*

8.1 *For every $q \in Q$ and $y' \in Y'$ with $\|y'\|_q \leqslant 1$ we have*

$$\lim_{\substack{\|\mu\|_q(A) \to 0 \\ a \in \Sigma}} |y'\alpha(A)| = 0.$$

8.2 *For every $A \in \Sigma^+$, there exist $B \in \Sigma^+$ and a compact subset $K \subseteq X$ such that $B \subseteq A$ and B is localized on K.*

Proof. Clearly the conditions 8.1 and 8.2 are necessary, as follows from Proposition 7, since every μ-integrable function is weakly μ-integrable (and the values of the integrals coincide).

Suppose now that the conditions 8.1 and 8.2 are fulfilled. For every $n \geqslant 1$ there exist a countable (or finite) measurable partition $\{A^n_m\}_{m \in I_n} \subseteq \Sigma^+$ of Ω and a family $\{K^n_m\}_{m \in I_n}$ of compact subsets of X with diameter less than or equal to $\frac{1}{n}$, verifying 5.3 and 5.4. Keeping the notation of Proposition 5, let us consider

$$g_n = \sum_{m \in I_n} x^n_m X_{A^n_m} \qquad (n \in \mathbb{N});$$

then $(g_n(w))_{n \in \mathbb{N}}$ is a Cauchy sequence for every $w \in \Omega$ and there exists

$$f(w) = \lim_n g_n(w).$$

If $\varepsilon > 0$ and $q \in Q$, for every $n \in \mathbb{N}$ there exists $r_n \in \mathbb{N}$ such that

$$\|\mu\|_q\Big(\bigcup_{\substack{m \in I_n \\ m > r_n}} A^n_m\Big) < \frac{\varepsilon}{2^{n+1}};$$

and if $A_\varepsilon = \bigcup_{n \in \mathbb{N}} \Big(\bigcup_{\substack{m \in I_n \\ m > r_n}} A^n_m\Big)$, we have $\|\mu\|_q(A_\varepsilon) < \varepsilon$ and the sequence of simple functions $f_n = g_n \cdot X_{\bigcup_{\substack{m \in I_n \\ m \leqslant r_n}} A^n_m} = \sum_{\substack{m \in I_n \\ m \leqslant r_n}} x^n_m X_{A^n_m}$ is uniformly convergent to f on $\Omega - A_\varepsilon$.

Moreover, for every $\varepsilon > 0$ and $q \in Q$ Proposition 6 implies the existence of $\delta > 0$ such that $q(\alpha(A)) < \frac{\varepsilon}{3}$ if $A \in \Sigma$ verifies $\|\mu\|_q(A) < \delta$.

If $B \in \Sigma$ is such that $\|\mu\|_q(B) < \min(\delta, \frac{\varepsilon}{6})$ and $f \cdot X_B$ is a uniform limit of simple functions, then we can find a simple function $h = \sum_{j=1}^{k} x_j X_{D_j}$ such that $\|f(w) - h(w)\| \leqslant 1$ for every $w \in B$. Then for every $n \in \mathbb{N}$ and $w \in B$ we have that $\|g_n(w) - h(w)\| \leqslant 2$ (i.e., $\|x^n_m - x_j\| \leqslant 2$ if $A^n_m \cap D_j \cap B \neq \emptyset$, where $m \in I_n$ and $1 \leqslant j \leqslant k$); and therefore, for every $y' \in Y'$ with $\|y'\|_q \leqslant 1$ and for every $n \in \mathbb{N}$ we have that

$$
\begin{aligned}
y'\Big(\int_B f\,\mathrm{d}\mu\Big) &= \int_B (f-h)\,\mathrm{d}(y'\mu) + y'\Big(\int_B h\,\mathrm{d}\mu\Big) \\
&\leqslant |y'\mu|(B) + y'\Big(\sum_{j=1}^{k} \mu(D_j\cap B)(x_j)\Big) \\
&\leqslant \|y'\|_q(B) + \sum_{j=1}^{k} \lim_{r\to+\infty} \sum_{m=1}^{r} y'\mu(D_j\cap B\cap A_m^n)(x_j) \\
&\leqslant \frac{\varepsilon}{6} + \sup_r \sum_{m=1}^{r} \Bigg| \sum_{\substack{j=1 \\ D_j\cap B\cap A_m^n\neq\emptyset}}^{k} y'\mu(D_j\cap B\cap A_m^n)(x_j - x_m^n)\Bigg| \\
&\quad + \sup_r \Bigg| \sum_{m=1}^{r} \sum_{\substack{j=1 \\ D_j\cap B\cap A_m^n\neq\emptyset}}^{k} y'\mu(D_j\cap B\cap A_m^n)(x_m^n)\Bigg| \\
&\leqslant \frac{\varepsilon}{6} + 2|y'\mu|(B) + \sup_r \sum_{m=1}^{r} |y'(\mu(B\cap A_m^n)(x_m^n) - \alpha(B\cap A_m^n))| \\
&\quad + \sup_r \Big| \sum_{m=1}^{r} y'\alpha(B\cap A_m^n)\Big| \\
&\leqslant \frac{\varepsilon}{6} + 2|y'\mu|(B) + \sup_r \sum_{m=1}^{r} \frac{1}{n}|y'\mu|(B\cap A_m^n) \\
&\quad + \sup_r \Big|y'\alpha\Big(B\cap\Big(\bigcup_{m=1}^{r} A_m^n\Big)\Big)\Big| \\
&\leqslant \frac{\varepsilon}{6} + 3|y'\mu|(B) + \frac{\varepsilon}{3} \leqslant \frac{\varepsilon}{6} + 3\|\mu\|_q(B) + \frac{\varepsilon}{3} \leqslant \varepsilon
\end{aligned}
$$

and therefore, $q\big(\int_B f\,\mathrm{d}\mu\big) \leqslant \varepsilon$; the function f is μ-integrable.

Moreover, for every $A\in\Sigma$, $n\in\mathbb{N}$, $q\in Q$ and $y'\in Y'$ with $\|y'\|_q\leqslant 1$ we have

$$
\begin{aligned}
y'\Big(\int_A f\,\mathrm{d}\mu - \alpha(A)\Big) &= \lim_{r\to+\infty} \sum_{m=1}^{r} y'\Big(\int_{A\cap A_m^n} (f-g_n)\,\mathrm{d}\mu + \int_{A\cap A_m^n} g_n\,\mathrm{d}\mu - \alpha(A\cap A_m^n)\Big) \\
&\leqslant \sup_r \sum_{m=1}^{r} \Big[\frac{1}{n}|y'\mu|(A\cap A_m^n) + y'(\mu(A\cap A_m^n)(x_m^n) - \alpha(A\cap A_m^n))\Big] \\
&\leqslant \sup_r \Big[\frac{1}{n}|y'\mu|\Big(A\cap\Big(\bigcup_{m=1}^{r} A_m^n\Big)\Big) + \frac{1}{n}\sum_{m=1}^{r} |y'\mu|(A\cap A_m^n)\Big] \\
&\leqslant \frac{2}{n}|y'\mu|(A) \leqslant \frac{2}{n}\|\mu\|_q(A) \leqslant \frac{2}{n}\|\mu\|_q(\Omega)
\end{aligned}
$$

and, therefore, $\alpha(A) = \int_A f \, \mathrm{d}\mu$. $\square$

Corollary 9. *If $f\colon \Omega \to X$ is a weakly μ-integrable function, then there exists a μ-integrable function $g\colon \Omega \to X$ such that*

$$\int_A f \, \mathrm{d}\mu = \int_A g \, \mathrm{d}\mu$$

for every $A \in \Sigma$.

Proof. It follows trivially from Theorem 8 and Proposition 7. $\square$

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