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EXTENSION THEOREMS (VECTOR MEASURES ON QUANTUM LOGICS)

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INTRODUCTION AND PRELIMINARIES

Vector valued measures and their extensions have been widely studied by many authors (see e.g. [1, 2, 4, 5, 7, 8, 10, 11, 12, 16, 19, 20]). The aim of this paper is to complete and generalize known results concerning extensions of various types of vector and group-valued measures defined on Boolean algebras to larger orthomodular structures.

Our discussion falls into three parts. The first part is devoted to extensions of orthogonal measures. (Measures of this type play important role in the non-commutative probability theory and foundations of quantum physics—see e.g. [4, 14] for systematic treatment.) It has been proved in [12] that, if H is a finite dimensional Hilbert space and L is a logic, then the condition that L is H-rich (i.e. L has enough H-valued orthogonal measures) is equivalent to the following extension property: for any Boolean subalgebra B of L and any orthogonal measure $m\colon B \to H$, there exists an orthogonal measure $\overline{m}\colon L \to H$ extending m. We show that, if H is infinite dimensional, then the condition of H-richness is not sufficient for the above extension property. In this connection we prove general extension theorem for orthogonal measures having values in arbitrary Hilbert space. As a corollary we show that every orthogonal measure on a Boolean subalgebra of the projection logic $P(M)$ of a von Neumann algebra M extends to an orthogonal measure on $P(M)$ with values in some (generally larger) Hilbert space.

In the second part of this note we study extensions of vector measures defined on the centre of a given orthomodular lattice L. In this special case we can get extension result without assuming that L has a large set of measures. In particular we prove

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the following extension theorem: Let $\mathscr{A}$ be a Boolean subalgebra of the centre $C(L)$ of an orthomodular lattice L and let X be a Banach space with the Radon-Nikodym property. Then every (finitely additive) measure $\mu\colon \mathscr{A} \to X$ of bounded variation admits bounded extension over L provided that there is a finitely additive measure $\nu\colon L \to [0, \infty[$ such that μ is $\nu|_{\mathscr{A}}$-continuous. Let us remark that a similar result has been proved in [19] in the situation when $X = R$, $\mathscr{A}$ is σ-complete, μ is σ-additive and ν is a valuation.

The concluding part of the paper deals with extensions of vector and group-valued measures defined on set-representable logics. It is proved e.g. that every bounded finitely additive measure $\mu\colon \mathscr{A} \to X$, where $\mathscr{A}$ is a Boolean algebra and X is a normed space with predual, has a bounded extension $\tilde{\mu}\colon \mathscr{B} \to X$ over any Boolean algebra $\mathscr{B}$ containing $\mathscr{A}$ such that $\mathscr{A}$ is μ-dense in $\mathscr{B}$. In particular, if X is a Hilbert space and μ is orthogonal, then the extension $\tilde{\mu}$ is orthogonal, too. Moreover, we prove that every s-bounded measure on a Boolean algebra $\mathscr{A}$ with values in some Banach space, has finitely additive extension to any concrete logic containing $\mathscr{A}$ as subalgebra. These results considerably generalize results of [11] and [12].

Here we fix some notations and recall basic definitions. (For the general theory of quantum logic we refer to [18], for the theory of operator algebras we refer to [15].) A (quantum) logic is a set L endowed with a partial ordering $\leqslant$ and a unary operation $'$, such that the following conditions are satisfied:

(1) L has a least and a greatest element 0 and 1, respectively.
(2) $a \leqslant b \Rightarrow b' \leqslant a'$ for any $a, b \in L$.
(3) $(a')' = a$ for any $a \in L$.
(4) $a \vee b$ exists in L whenever $a, b \in L$ and $a \leqslant b'$.
(5) $a \vee a' = 1$ for any $a \in L$.
(6) $b = a \vee (b \wedge a')$ whenever $a, b \in L$ and $a \leqslant b$.

Elements $a, b \in L$ are said to be orthogonal (in symbol $a \perp b$) if $a \leqslant b'$. A logic K is said to be a sublogic of a logic L if $K \subset L$ and if the ordering, the greatest element, the least element, the orthocomplementation operation and the formation of suprema of orthogonal elements coincide for K and L.

If L is a lattice it is called orthomodular lattice. The centre $C(L)$ of an orthomodular lattice L is the set of all elements $a \in L$ such that $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$ for every $x, y \in L$.

By a measure on a logic L we mean a finitely additive function (i.e. a function which is additive with respect to finite sets of mutually orthogonal elements) with values in some topological group G.

We say that a measure $\mu\colon L \to G$ is completely additive (resp. σ-additive) if, for any system (resp. countable system) of mutually orthogonal elements $\{a_i\colon i \in I\}$

of L, for which the supremum $\bigvee_{i\in I} a_i$ exists in L, the family $\{\mu(a_i)\colon i \in I\}$ is summable and $\mu\Big(\bigvee_{i\in I} a_i\Big) = \sum_{i\in I} \mu(a_i)$ in G. State s (probability measure) on a logic L is defined as a measure with values in the set of positive real numbers such that $s(1) = 1$. A logic L is said to be unital if for every nonzero $a \in L$ there is a state s of L such that $s(a) \neq 0$.

Let $\mathscr{A}$ be a Boolean algebra, (G, τ) a topological group and $\mu\colon \mathscr{A} \to G$ a measure. Then the family $\mathscr{U}_W(0) = \{a \in \mathscr{A}\colon \mu(\mathscr{A} \wedge a) \subseteq W\}$, where W runs through a 0-neighbourhood base of G, is a 0-neighbourhood base of a group topology in $\mathscr{A}$, called the μ-topology. We say that μ is s-bounded if $\mu(a_n) \to 0$ in τ for every disjoint sequence (a_n) in $\mathscr{A}$. If $G = R$, then μ is s-bounded if and only if it is bounded. In the following, we denote by $|\mu|$ the total variation of μ and we set $\|\mu\| = |\mu|(1)$. If L is a logic and H is a Hilbert space, we say that a function $\mu\colon L \to H$ is orthogonal if $a \perp b$ implies $(\mu(a), \mu(b)) = 0$, where $(\cdot, \cdot)$ denotes the inner product in H. We denote by $R(\mu)$ the closed subspace of H generated by the range of μ.

Important examples of measures are measures defined on set representable logics and projection lattices of operator algebras (see e.g. [6] for relevance of these measure in the formalism of quantum theory). By a concrete logic (S, Δ) we mean a system S of subsets of the set Δ ordered by the set inclusion and satisfying the following conditions:

(1) $\emptyset \in \Delta$.
(2) If $A \in \Delta$, then $S \setminus A \in \Delta$.
(3) If $A, B \in \Delta$ and $A \cap B = \emptyset$, then $A \cup B \in \Delta$.

The orthocomplementation in (S, Δ) is given by the set complement.

Throughout the paper let $B(H)$ denote the set of all linear bounded operators on a Hilbert space H and let $L(H)$ be the logic of all the projections on H. (Projection is defined as a self-adjoint idempotent operator.) A $*$-subalgebra $\mathscr{A}$ of $B(H)$ (i.e. a subspace of $B(H)$ closed with respect to the multiplication and adjoints) is called von Neumann algebra if $\mathscr{A} = (\mathscr{A}')'$, where, for $\mathscr{D} \subseteq B(H)$, we set $\mathscr{D}' = \{T \in B(H)\colon TS = ST \text{ for all } S \in \mathscr{D}\}$.

We say that $T \in B(H)$ is a Hilbert-Schmidt operator if $\sum_{\alpha\in I} \|Te_\alpha\|^2 < +\infty$ for some (and therefore for any) orthonormal basis $\{e_\alpha\colon \alpha \in I\}$ of H.

1. Extending orthogonal measures to logics

In this section, we give a counterexample concerning a result of [12] and we prove an extension theorem for orthogonal measures with values in arbitrary Hilbert space. We then apply this theorem to projection logics.

Throughout this section H will denote a Hilbert space and L a logic. The following state-space property of L has been introduced in [12].

Definition 1.1. We say that L is H-rich if, for any finite sequence of non-negative numbers $\{\alpha_i : i \leqslant n\}$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and any set $\{a_i : i \leqslant n\}$ of mutually orthogonal nonzero elements in L, there is an orthogonal measure $s\colon L \to H$ such that $\|s(a_i)\|^2 = \alpha_i$ for each $i \leqslant n$.

For example every concrete logic and the Hilbert-space logic $L(H)$ are H-rich.

The following result ties H-richness with extension property.

Theorem 1.2. ([12]) *Let H be finite dimensional. Then the following conditions are equivalent:*

(1) *L is H-rich.*
(2) *For any Boolean subalgebra B of L and any orthogonal measure $m\colon B \to H$, there is an orthogonal measure $\tilde{m}\colon L \to H$ such that $\tilde{m}(b) = m(b)$ for any $b \in B$.*

If H is infinite dimensional, the condition (1) is again necessary for (2).

The following example shows that, in general, (1) is not sufficient for (2).

Example 1.3. Let H be an infinite dimensional separable Hilbert space. Then there exists a Boolean subalgebra B of $L(H)$ and an orthogonal measure $m\colon B \to H$ which has no extension to an orthogonal measure $\tilde{m} : L(H) \to H$.

P r o o f. Let $\{e_n\}$ be an orthonormal basis of H and $B \subseteq L(H)$ be the Boolean algebra of all projections which project on closed subspaces of the form $\overline{\mathrm{sp}}\,\{e_i : i \in F\}$, where F is either finite or cofinite subset of $\mathbb{N}$.

Let s be a two-valued measure on B defined by $s(P) = 1$ if $P \in B$ projects on cofinite dimensional space and $s(P) = 0$ otherwise. Choose a unit vector $v \in H$ and set $m(P) = s(P)v$ for $P \in B$. Then $m\colon B \to H$ is an orthogonal measure which is not σ-additive. But according to [9], every orthogonal measure on $L(H)$ with values in a separable Hilbert space has to be σ-additive. So m has no extension to an H-valued orthogonal measure on $L(H)$ □

Example 1.3 shows that it is not possible, in general, to extend given orthogonal measure to an orthogonal measure with values in the same Hilbert space. Nevertheless, by allowing enlargement of ranges, we can obtain the following extension theorem for every logic with enough orthogonal measures.

For this let us first observe that every orthogonal measure $m\colon L \to H$ is bounded. Indeed, for every $a \in L$, we have the following inequalities:

$$\|m(a)\|^2 \leqslant \|m(a)\|^2 + \|m(a')\|^2 = \|m(1)\|^2.$$

Theorem 1.4. *Let L be a unital logic such that for every state s on L there is a Hilbert space H and an orthogonal measure $m\colon L \to H$ such that $s(a) = \|m(a)\|^2$ for every $a \in L$.*

Let B be a Boolean subalgebra of L and $m\colon B \to H$ be an orthogonal measure. Then there is a Hilbert space K containing H as a Hilbert subspace and an orthogonal measure $\tilde{m}\colon L \to K$ extending m.

Proof. Since every orthogonal measure is bounded, there is no loss of generality in assuming that $\|m(1)\| = 1$. Let s be a state of B defined via equality $s(a) = \|m(a)\|^2$ $(a \in B)$. By [17], s extends to a state $\tilde{s}$ on L. So that we can find an orthogonal measure $\overline{m}$ on L with values in some Hilbert space F such that $\|\overline{m}(a)\|^2 = \tilde{s}(a)$ for every $a \in L$.

Define a mapping $U\colon \{\overline{m}(P)\colon P \in B\} \to \{m(P)\colon P \in B\}$ by setting

$$U(\overline{m}(P)) = m(P) \quad \text{for } P \in B.$$

We show that U is well defined and extends to a unitary mapping (denoted again by U) of $\overline{\mathrm{sp}}\,\{\overline{m}(P)\colon P \in B\}$ onto $\overline{\mathrm{sp}}\,\{m(P)\colon P \in B\}$.

Indeed, if $\sum\limits_{i=1}^{n} \alpha_i \overline{m}(P_i) = \overline{m}(Q)$ (where $\alpha_i \in C$, $P_i, Q \in B$, $1 \leqslant i \leqslant n$), then, using the equality

$$\begin{aligned}(\overline{m}(P), \overline{m}(Q)) &= \|\overline{m}(P \wedge Q)\|^2 = s(P \wedge Q) \\ &= \|m(P \wedge Q)\|^2 = (m(P), m(Q)) \quad (P, Q \in B),\end{aligned}$$

we obtain

$$\begin{aligned}\left\| \sum_{i=1}^{n} \alpha_i m(P_i) - m(Q) \right\|^2 &= \left\| \sum_{i=1}^{n} \alpha_i m(P_i) \right\|^2 + \|m(Q)\|^2 - 2\,\mathrm{Re}\left(\sum_{i=1}^{n} \alpha_i m(P_i), m(Q) \right) \\ &= \left\| \sum_{i=1}^{n} \alpha_i \overline{m}(P_i) - \overline{m}(Q) \right\|^2 = 0.\end{aligned}$$

Let K be a Hilbert space such that $K \supseteq H$ and $\dim K \geqslant \dim F$. Let us extend U to a unitary mapping of F into K (we put $U\big|_{R(\overline{m})^{\perp}} = V$, where V is a unitary mapping of $R(\overline{m})^{\perp}$ into $R(m)^{\perp}$). Then the mapping $\tilde{m} = U \circ \overline{m}$ is the required extension of m. The proof is complete. □

Let us remark that logics for which every state is representable by an orthogonal measure has been studied and characterized in [5, 13]. This class involves e.g. distributive sum logics investigated in [13].

Modifying slightly the proof of Theorem 4.1 we can also get that extension property for a logic L expressed in Theorem 1.4 is in fact equivalent to the following condition: For every state s on a given Boolean subalgebra B of L there is an orthogonal measure $m\colon L \to K$ such that $\|m(a)\|^2 = s(a)$ for every $a \in B$. This provides an infinite dimensional analogy of Theorem 1.2.

Corollary 1.5. *Let B be a Boolean subalgebra of the projection logic $P(M)$ of a von Neumann algebra M. Let $m\colon B \to H$ be an orthogonal measure. Then there exist a Hilbert space K containing H (as a Hilbert subspace) and an orthogonal measure $\tilde{m}\colon P(M) \to K$ which extends m. Moreover, if B is complete, m is completely additive and H is infinite dimensional, then $\tilde{m}$ can be taken completely additive and with values in H provided that M acts on H.*

Proof. Again we can assume that the measure s defined by $s(P) = \|m(P)\|^2$ $(P \in B)$ is a state. By standard construction, s can be extended to a positive linear functional on $\overline{\text{sp}}\, B$ and then (Hahn-Banach theorem) to a positive linear functional $\tilde{s}$ on M.

Making use of the G.N.S.-construction (see e.g. [15]), we infer that there exists a Hilbert space F and a representation π of M into $B(F)$ such that

$$\tilde{s}(P) = (\pi(P)x, x) \quad \text{for all } P \in M,$$

where x is a suitable unit vector of F. The mapping $\overline{m}\colon P \in M \to \pi(P)x$ is an F-valued orthogonal measure on $P(M)$. So that we obtain an extension $\tilde{s}$ of $\bar{s}$ representable via orthogonal measure $\overline{m}$ in the sense of Theorem 1.4. We can now proceed in the same way as in the proof of Theorem 1.4 and get required extension.

Now let B be complete and m completely additive. Then s extends to a normal functional of the von Neumann algebra generated by B and $\tilde{s}$ above can be taken normal (Hahn-Banach theorem for the weak-$*$ topology). Hence, the G.N.S. representation π of $\tilde{s}$ is normal and $\overline{m}$ defined above is completely additive.

According to [7, 8] every completely additive orthogonal measure is unitarily equivalent to some orthogonal measure with values in $c_2(H) \oplus c_2(H)$, where $c_2(H)$ is the Hilbert space of all Hilbert-Schmidt operators acting on H. Since $c_2(H) \oplus c_2(H)$ is an isomorphic copy of H, we can set (upon obvious identification) $K = H$ in the construction of the proof of Theorem 1.4. The proof is complete. □

As Example 1.3 shows, enlargement of the range in the above corollary is, in general, necessary.

2. Extending vector measures from the centre

In this paragraph we study extensions of Banach-space valued measures defined on the centre of an orthomodular lattice.

Throughout this section let L be an orthomodular lattice and X a Banach space. Let C be a Boolean subalgebra of the centre $C(L)$ of L.

Let $\nu\colon L \to [0,+\infty[$ be a measure. We set $\nu_a(x) = \nu(x \wedge a)$ for each $a, x \in L$.

Moreover, for every Boolean subalgebra $\mathscr{A}$ of L, we put

$$S_\nu(\mathscr{A}) = \left\{ \sum_{i=1}^{n} \alpha_i \nu_{a_i} : n \in \mathbb{N}, \alpha_i \in X, a_i \in \mathscr{A} \right\}.$$

Then $S_\nu(\mathscr{A})$ is a linear subspace in the space of X-valued functions defined on $\mathscr{A}$.

If $\mu\colon \mathscr{A} \to X$ is a measure, we say that μ is $\nu_{|\mathscr{A}}$-continuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\nu(a) < \delta$, with $a \in \mathscr{A}$, then $\|\mu(a)\|_X < \varepsilon$. In the sequel we will use the following notation. If $\gamma \in S_\nu(C)$, we set $|\gamma| = |\gamma_{|C}|$ and $\|\gamma\| = |\gamma|(1)$.

Remark 2.1. For every $c \in C(L)$, $\nu_c\colon L \to [0,+\infty[$ is a measure.

First we remark that

$$(*) \qquad a, b \in L,\ a \perp b \Rightarrow (a \wedge x) \perp (b \wedge y) \quad \text{for every } x, y \in L.$$

Indeed, $a \wedge x \leqslant a \leqslant b' \leqslant b' \vee y' = (b \wedge y)'$.

Now let $c \in C(L)$ and $a, b \in L$ such that $a \perp b$. Then, from $(*)$, we obtain

$$\nu_c(a \vee b) = \nu(c \wedge (a \vee b)) = \nu((a \wedge c) \vee (b \wedge c)) =$$

$$= \nu(a \wedge c) + \nu(b \wedge c) = \nu_c(a) + \nu_c(b).$$

Remark 2.2. If $\gamma \in S_\nu(C)$, then there exist $m \in \mathbb{N}$, $\beta_1, \ldots, \beta_m \in X$ and $b_1, \ldots, b_m \in C$, with $b_1 \vee \ldots \vee b_m = 1$ and $b_i \perp b_j$ for every $i \neq j$, such that $\gamma(x) = \sum\limits_{i=1}^{m} \beta_i \nu_{b_i}(x)$ for every $x \in L$.

This statement can be proved easily by standard rewriting simple function on a Boolean algebra (in its set representation) as a linear combination of characteristic functions of disjoint sets.

Lemma 2.3. *If $\gamma \in S_\nu(C)$, then $\|\gamma(x)\|_X \leqslant \|\gamma\|$ for all $x \in L$.*

Proof. Let $\gamma = \sum_{i=1}^{n} \alpha_i \nu_{a_i} \in S_\nu(C)$. From the Remark 2.2, we can suppose $a_1 \vee \ldots \vee a_n = 1$ and $a_i \perp a_j$ for every $i \neq j$.

Let $\tilde{C}$, h and Ω, be the Stone algebra, the Stone map and the Stone space of C, correspondingly. Moreover, for $A \in \tilde{C}$, let

$$\tilde{\nu}(A) = \nu(h^{-1}(A)) \quad \text{and} \quad \tilde{\gamma}(A) = \gamma(h^{-1}(A)).$$

Set $f = \sum_{i=1}^{n} \alpha_i \chi_{h(a_i)}$. Then f is a $\tilde{C}$-simple function and $\{h(a_i): i \leqslant n\}$ is a partition of Ω in $\tilde{C}$. We have, for $A \in \tilde{C}$,

$$\begin{aligned}\tilde{\gamma}(A) = \gamma(h^{-1}(A)) &= \sum_{i=1}^{n} \alpha_i \nu(a_i \wedge h^{-1}(A)) \\ &= \sum_{i=1}^{n} \alpha_i \tilde{\nu}_{h(a_i)}(A) = \int_A f \, \mathrm{d}\tilde{\nu}.\end{aligned}$$

Then, if $c \in C$, it follows from [3, Lemma 15, page 109], that

$$\begin{aligned}|\gamma|(c) = |\tilde{\gamma}|(h(c)) &= \int_{h(c)} \|f\|_X \, \mathrm{d}\tilde{\nu} \\ &= \sum_{i=1}^{n} \|\alpha_i\|_X \tilde{\nu}(h(a_i) \cap h(c)) = \sum_{i=1}^{n} \|\alpha_i\|_X \nu_{a_i}(c).\end{aligned}$$

Therefore, if $x \in L$,

$$\|\gamma(x)\|_X \leqslant \sum_{i=1}^{n} \|\alpha_i\|_X \nu(a_i \wedge x) \leqslant \sum_{i=1}^{n} \|\alpha_i\|_X \nu(a_i) = \|\gamma\|.$$

□

Proposition 2.4. *Suppose that X has the Radon-Nikodym property. Let $\mathscr{A}$ be a Boolean algebra, $\lambda\colon \mathscr{A} \to [0, +\infty[$ a measure and $\mu\colon \mathscr{A} \to X$ a λ-continuous measure of bounded variation. Then there exists a sequence $\{\mu_n\} \subseteq S_\lambda(\mathscr{A})$ such that $\mu(a) = \lim_n \mu_n(a)$ in X uniformly (with respect to $a \in \mathscr{A}$) and $\lim_{n,m} \|\mu_n - \mu_m\| = 0$.*

Proof. Let $\tilde{\mathscr{A}}$ be the Stone algebra of $\mathscr{A}$ and $h\colon \mathscr{A} \to \tilde{\mathscr{A}}$ the Stone map. We define, for $A \in \tilde{\mathscr{A}}$,

$$\tilde{\mu}(A) = \mu(h^{-1}(A)) \quad \text{and} \quad \tilde{\lambda}(A) = \lambda(h^{-1}(A)).$$

Thus, $\tilde{\mu}$ is a $\tilde{\lambda}$-continuous X-valued measure of bounded variation. Denote by $\sigma(\tilde{\mathscr{A}})$ the σ-algebra generated by $\tilde{\mathscr{A}}$, by $ba(\tilde{\mathscr{A}})$ and $\mathrm{ca}(\sigma(\tilde{\mathscr{A}}))$ respectively the linear space of all bounded real-valued measures on $\tilde{\mathscr{A}}$ and the linear space of all real-valued σ-additive measures on $\sigma(\tilde{\mathscr{A}})$, and by $\mathrm{bva}(\tilde{\mathscr{A}}, X)$, $\mathrm{bvca}(\sigma(\tilde{\mathscr{A}}), X)$ respectively the linear space of all X-valued measures of bounded variation on $\tilde{\mathscr{A}}$ and the linear space of all X-valued σ-additive measures on $\sigma(\tilde{\mathscr{A}})$ of bounded variation. Let

$$T\colon\ \mathrm{bva}(\tilde{\mathscr{A}}, X) \to \mathrm{bvca}(\sigma(\tilde{\mathscr{A}}), X)$$

and

$$S\colon\ \mathrm{ba}(\tilde{\mathscr{A}}) \to \mathrm{ca}(\sigma(\tilde{\mathscr{A}}))$$

be the standard isomorphisms (see [2], Theorem 7, page 30). Set $\overline{\mu} = T\tilde{\mu}$ and $\overline{\lambda} = S\tilde{\lambda}$. Then $\overline{\mu}$ is $\overline{\lambda}$-continuous. Denote by $L_1(\sigma(\tilde{\mathscr{A}}), \overline{\lambda})$ the space of all X-valued $\overline{\lambda}$-integrable functions. By assumption, there exists $f \in L_1(\sigma(\tilde{\mathscr{A}}), \overline{\lambda})$ such that

$$\overline{\mu}(A) = \int_A f \, \mathrm{d}\overline{\lambda} \quad \text{for every } A \in \sigma(\tilde{\mathscr{A}}).$$

From [3] (Theorem III.8.3), there exists a sequence $\{f_n\} \subseteq S(\tilde{\mathscr{A}})$ such that

$$\tilde{\mu}(A) = \lim_n \int_A f_n \, \mathrm{d}\tilde{\lambda} \quad \text{in } X$$

uniformly with respect to $A \in \tilde{\mathscr{A}}$ and

$$\lim_{n,m} \int \|f_n - f_m\| \, \mathrm{d}\tilde{\lambda} = 0.$$

For each $n \in \mathbb{N}$, set

$$\tilde{\mu}_n(A) = \int_A f_n \, \mathrm{d}\tilde{\lambda} \quad \text{and} \quad \mu_n(a) = \tilde{\mu}_n(h(a))$$

for each $A \in \tilde{\mathscr{A}}$ and $a \in \mathscr{A}$. Then $\mu_n \in S_\lambda(\mathscr{A})$ and

$$\lim_n \mu_n(a) = \lim_n \int_{h(a)} f_n \, \mathrm{d}\tilde{\lambda} = \tilde{\mu}(h(a)) = \mu(a)$$

for every $a \in \mathscr{A}$. Moreover (see [3, Lemma III.2.15])

$$\lim_{n,m} \|\mu_n - \mu_m\| = \lim_{n,m} \|\tilde{\mu}_n - \tilde{\mu}_m\| = \lim_{n,m} \int \|f_n - f_m\|_X \, \mathrm{d}\tilde{\lambda} = 0.$$

□

Theorem 2.5. *Suppose that X has the Radon-Nikodym property and let μ_0: $C \to X$ be a $\nu_{|C}$-continuous measure of bounded variation. Then there exists a bounded measure μ: $L \to X$, which extends μ_0.*

Proof. From the proposition (2.4), there exists a sequence $\{\mu_n\} \subseteq S_\nu(C)$ such that $\mu_0(x) = \lim_n \mu_n(x)$ uniformly for $x \in C$ and $\lim_{n,m} \|\mu_n - \mu_m\| = 0$. Moreover μ_n are measures on L and, from Lemma 2.3,

$$\|\mu_n(x) - \mu_m(x)\|_X \leqslant \|\mu_n - \mu_m\|$$

for every $x \in L$.

Set $\mu(x) = \lim_n \mu_n(x)$ $(x \in L)$. Then μ: $L \to X$ is a bounded measure and $\mu(x) = \mu_0(x)$ for every $x \in C$. The proof is complete. □

3. Extending measures to Boolean algebras and concrete logics

In this section, we prove some extension theorems for measures on set representable logics.

In the sequel, we will denote by X and X' a normed linear space and its topological dual, respectively.

We will use the following deep result (see a result of Lipecki in [16] and its reformulation in [20]).

Theorem 3.1 [16, 20]. *Let G be a commutative Hausdorff topological group and let $M \subseteq G$ be a complete set in G. Moreover let $\mathscr{A}_0$ be a Boolean subalgebra of a Boolean algebra $\mathscr{A}$. Then every s-bounded measure μ: $\mathscr{A}_0 \to M$ has a finitely additive s-bounded extension $\overline{\mu}$: $\mathscr{A} \to M$ such that $\mathscr{A}_0$ is dense in $\mathscr{A}$ with respect to the $\overline{\mu}$-topology.*

Moreover we will need the following (known) result.

Lemma 3.2. *Let X be a normed space and $\mathscr{A}$ a Boolean algebra. Then a measure μ: $\mathscr{A} \to X$ is bounded if and only if it is s-bounded with respect to the weak topology of X.*

Proof. A measure μ is bounded (resp. weakly s-bounded) if and only if, for every $x' \in X'$, the measure $x' \circ \mu$: $\mathscr{A} \to R$ is bounded (resp. s-bounded). Then the result follows from the equivalence between boundedness and s-boundedness of real-valued measures. □

Lemma 3.3. *Let X be a Hilbert space, $\mathscr{A}$ a Boolean algebra and $\mathscr{A}_0$ its Boolean subalgebra. Moreover let $\mu\colon \mathscr{A} \to X$ be a measure such that $\mathscr{A}_0$ is dense in $\mathscr{A}$ with respect to the μ-topology. Then, μ is orthogonal if and only if $\mu_{|\mathscr{A}_0}$ is orthogonal.*

Proof. Assume that $\mu_{|\mathscr{A}_0}$ is orthogonal. Take $a, b \in \mathscr{A}$ with $a \wedge b = 0$. Let $a_n, b_n \in \mathscr{A}_0$ be such that $a_n \to a$ and $b_n \to b$ with respect to the μ-topology. Replacing b_n by $a_n \wedge b_n$, we can suppose that $a_n \wedge b_n = 0$.

Then, from $0 = (\mu(a_n), \mu(b_n)) \to (\mu(a), \mu(b))$, we obtain $(\mu(a), \mu(b)) = 0$ and the proof is complete. □

The following result generalizes Theorems 9 and 10 of [11].

Theorem 3.4. *Let X be a normed space with predual, $\mathscr{A}$ a Boolean algebra and $\mathscr{A}_0$ a Boolean subalgebra of $\mathscr{A}$. Then every bounded measure $\mu\colon \mathscr{A}_0 \to X$ has a bounded finitely additive extension $\overline{\mu}\colon \mathscr{A} \to X$, such that $\mathscr{A}_0$ is dense in $\mathscr{A}$ with respect to the $\overline{\mu}$-topology. In particular, if X is a reflexive Banach space, then $\overline{\mu}$ is s-bounded. Moreover, if X is a Hilbert space and μ is orthogonal, then $\overline{\mu}$ is orthogonal too.*

Proof. Let E be a predual of X, i.e. $X = E'$. By assumption, there exists a ball $D \subseteq X$ such that $\mu(\mathscr{A}) \subseteq D$ and D is compact (and therefore complete) with respect to the weak-$*$ topology $\sigma(E', E)$ of E'.

Moreover, μ is s-bounded with respect to the weak topology $\sigma(X, X') = \sigma(E', E'')$ of X (Lemma 3.2) and therefore μ is s-bounded also with respect to the topology $\sigma(E', E) \leqslant \sigma(E', E'')$.

The result now follows from Theorem 3.1 applied for $(G, \tau) = (X, \sigma(E', E))$ and $M = D$.

If X is a reflexive Banach space, then $\overline{\mu}$ is s-bounded, because every bounded measure on a Boolean algebra with values in a reflexive Banach space is s-bounded (see [2, Theorem 2, page 20]).

The final part of the result follows from Lemma 3.3. □

The following result generalizes Theorem 3.5 of [12].

Theorem 3.5. *Let G be a commutative complete Hausdorff topological group, L a concrete logic and $\mathscr{B}$ a Boolean subalgebra of L. Then every s-bounded measure $\mu\colon \mathscr{B} \to G$ has a finitely additive extension $\overline{\mu}\colon L \to G$ such that $\overline{\mu}(L) \subseteq \overline{\mu(\mathscr{B})}$. Moreover, if G is a Hilbert space and μ is orthogonal, then $\overline{\mu}$ can be taken orthogonal too.*

Proof. Let Δ be a set such that $L \subseteq P(\Delta)$, where $P(\Delta)$ denotes the set of all subsets of Δ. By Theorem 3.1, there exists a finitely additive extension $\tilde{\mu}$:

$P(\Delta) \to G$ of μ to $P(\Delta)$ such that $\tilde{\mu}(P(\Delta)) \subseteq \overline{\mu(\mathscr{B})}$ and $\mathscr{B}$ is a dense subset of $P(\Delta)$ with respect to the $\tilde{\mu}$-topology. Then $\overline{\mu} = \tilde{\mu}_{|L}$ is the required extension.

Moreover, let G be a Hilbert space and $\mu\colon B \to G$ be an orthogonal measure. Since μ is bounded we can immediately apply Theorem 3.4. □

In some cases we can remove the assumption of s-boundedness in Theorem 3.5.

Theorem 3.6. *Let G be a commutative Hausdorff topological group, L a concrete logic and $\mathscr{B}$ a Boolean subalgebra of L. Then every measure $\mu\colon \mathscr{B} \to G$ such that $\mu(\mathscr{B})$ is relatively compact in G has a finitely additive extension $\overline{\mu}\colon L \to G$ such that $\overline{\mu}(L) \subseteq \overline{\mu(\mathscr{B})}$. Moreover, if G is a Hilbert space and μ is orthogonal, then $\overline{\mu}$ can be taken orthogonal too.*

Proof. Again let Δ be such that $L \subseteq P(\Delta)$. Without loss of generality (see the proof of Theorem 3.5) we can suppose that $L = P(\Omega)$ and that $\mathscr{B}$ is a Boolean subalgebra of L.

Let $\mathscr{P}$ be the set of all partitions of Δ in $\mathscr{B}$ (i.e. $\mathscr{P}$ consists of all finite orthogonal subsets of $\mathscr{B}$ with join Δ). Put $K = \overline{\mu(\mathscr{B})}$ and

$$\mathscr{F}_P = \{\nu\colon P(\Delta) \to G\colon\ \nu \text{ is finitely additive, } \nu(P(\Delta)) \subseteq K \text{ and } \nu_{|P} = \mu_{|P}\},$$

$(P \in \mathscr{P})$.

We prove that $\mathscr{F}_P$ is non-void for every $P \in \mathscr{P}$. Indeed, let $P \in \mathscr{P}$, with $P = \{A_1, \dots, A_n\}$. Choose $x_i \in A_i$, and set $y_i = \mu(A_i)$ ($i \leqslant n$). Let δ_{x_i} be the point measure concentrated at x_i. Put $\overline{\nu} = \sum_{i=1}^{n} y_i \delta_{x_i}$. Then $\overline{\nu}\colon P(\Omega) \to G$ is finitely additive. For any $M \subseteq \Omega$, we have

$$\overline{\nu}(M) = \sum_{x_i \in M} y_i = \mu\Big(\bigcup_{x_i \in M} A_i\Big).$$

Therefore $\overline{\nu}(M) \in \mu(\mathscr{B}) \subseteq K$. Moreover, if $M = A_j$, then $\overline{\nu}(M) = \mu(A_j)$. Thus $\overline{\nu} \in \mathscr{F}_P$.

Since K is a compact set, the set $M(P(\Delta), K)$ of all finitely additive K-valued measures on $P(\Delta)$ considered with the topology of pointwise convergence is compact (Tychonoff theorem). Let $P_1, P_2, \dots, P_n \in \mathscr{P}$ and let $R \in \mathscr{P}$ be a common refinement of the partitions $P_1, P_2, \dots, P_n$. Then $\bigcap \mathscr{F}_{P_i} \supseteq \mathscr{F}_R \neq \emptyset$. Moreover every set $\mathscr{F}_p$ is a closed subset of $M(P(\Omega), K)$. Thus, $\bigcap_{P \in \mathscr{P}} \mathscr{F}_P \neq \emptyset$ and this intersection consists of required extensions.

If G is a Hilbert space and μ is orthogonal, since μ is bounded, μ extends to an orthogonal measure on L by Theorem 3.4. □

We have shown that every orthogonal measure on a Boolean algebra extends to an orthogonal measure on an arbitrary larger concrete logic with values in the same Hilbert space. As Example 1.3, shows this result is no longer valid for general orthomodular structures.

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