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CONSTRUCTION OF ALL HOMOMORPHISMS OF GROUPOIDS

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To Otakar Borůvka on the occasion of his 95th birthday

1. INTRODUCTION

We prove that the category of ternary structures with homomorphisms as morphisms is isomorphic to a particular category of binary structures. Objects of the latter category are binary structures whose carriers are Cartesian squares and binary relations on squares are "binding" in a very natural sense; morphisms of this category are decomposable homomorphisms of binary structures. The category of groupoids with homomorphisms as morphisms is a full subcategory of the category of ternary structures. The above mentioned isomorphism puts the category of groupoids in correspondence with a particular category of mono-unary algebras on Cartesian squares with decomposable homomorphisms as morphisms. This allows us to construct all homomorphisms of a groupoid into another one: The corresponding mono-unary algebras are constructed, all homomorphisms of the first mono-unary algebra into the other are found basing on a known construction. These homomorphisms are tested to find all decomposable ones among them. The decomposable homomorphisms of mono-unary algebras define exactly all homomorphisms of the given groupoids.

This paper is dedicated to Otakar Borůvka on the occasion of his 95th birthday. Borůvka belongs to the founders of groupoid theory. He inspired the author with a problem leading to the investigation of mono-unary algebras. The present paper may be regarded as a result of Borůvka's fruitful activity and of author's gratitude.

We now present the details of our constructions. The instruments of the theory of categories used here can be easily found in [3] or in [11].

First, we construct a category **TER** (category of TERnary structures) as follows (cf. [9], [10]): Any object of **TER** is a set A with a ternary relation $t \subseteq A \times A \times A$; it will be written as (A, t) and called a *ternary structure.* A morphism h of (A, t) into a ternary structure (A', t') in **TER** is a mapping of A into A' such that for any x, y, z in A the condition $(x, y, z) \in t$ implies that $(h(x), h(y), h(z)) \in t'$. Such a mapping will be called a *homomorphism* of (A, t) into (A', t').

It is easy to see that $1_{(A,t)}$ is a homomorphism of (A, t) into itself and that a composition of a homomorphism of (A, t) into (A', t') and a homomorphism of (A', t') into (A'', t'') is a homomorphism of (A, t) into (A'', t'') supposing that (A, t), (A', t'), (A'', t'') are ternary structures. It follows that **TER** is a category.

Before defining the category **BIP** we need some auxiliary results.

Let A, A' be sets, h a mapping of A into A'. We define a mapping f of $A \times A$ into $A' \times A'$ by putting $f(x, y) = (h(x), h(y))$ for any $(x, y) \in A \times A$. We denote the mapping f by $h \times h$. A mapping f of $A \times A$ into $A' \times A'$ will be called *decomposable* if there exists a mapping h of A into A' such that $f = h \times h$.

Lemma 1. *Let A, A' be sets, f a decomposable mapping of $A \times A$ into $A' \times A'$. If $h \times h = f = g \times g$, then $h = g$.*

P r o o f. Indeed, for any $x \in A$ we have $(h(x), h(x)) = f(x, x) = (g(x), g(x))$, which implies $h(x) = g(x)$. □

Thus, any decomposable mapping f of $A \times A$ into $A' \times A'$ defines exactly one mapping h of A into A' such that $f = h \times h$.

We shall need a characterization of decomposable mappings.

Theorem 1. *Let A, A' be sets, f a mapping of $A \times A$ into $A' \times A'$. Then the following assertions are equivalent.*

(i) *The mapping f is decomposable.*

(ii) *For any x_0, x_1, x_2 in A the conditions $f(x_0, x_1) = (y_0, y_1)$, $f(x_0, x_2) = (y_0', y_2)$ imply $f(x_1, x_2) = (y_1, y_2)$.*

P r o o f. If (i) holds, there exists a mapping h of A into A' such that $f = h \times h$. Suppose x_0, x_1, x_2 in A, $f(x_0, x_1) = (y_0, y_1)$, $f(x_0, x_2) = (y_0', y_2)$. Then $y_1 = h(x_1)$, $y_2 = h(x_2)$ and, therefore, $f(x_1, x_2) = (h(x_1), h(x_2)) = (y_1, y_2)$. Thus, (ii) holds.

Suppose that (ii) holds. First, we prove

(iii) If $f(x_0, x_1) = (y_0, y_1)$, $f(x_0, x_2) = (y_0', y_2)$, then $y_0' = y_0$.

Indeed, put $f(x_0, x_0) = (z_0, z_1)$. This condition together with $f(x_0, x_1) = (y_0, y_1)$ implies $f(x_0, x_1) = (z_1, y_1)$ by (ii) and hence, $z_1 = y_0$. Thus $f(x_0, x_0) = (z_0, y_0)$.

This condition together with $f(x_0, x_2) = (y'_0, y_2)$ implies that $f(x_0, x_2) = (y_0, y_2)$ by (ii) and thus, $y'_0 = y_0$. Hence (iii) holds.

Thus, we choose a fixed element $x_0 \in A$. There exists $y_0 \in A'$ such that $f(x_0, x) = (y_0, y)$ for any $x \in A$ where $y \in A'$; this is a consequence of (iii). We put $h(x) = y$. Since f is a mapping, h is a correctly defined mapping of A into A'.

Let x, x' in A be arbitrary. Then $f(x_0, x) = (y_0, h(x))$, $f(x_0, x') = (y_0, h(x'))$ by (iii). By (ii), we obtain $f(x, x') = (h(x), h(x'))$. Thus, $f = h \times h$ and (i) holds. □

Example 1. Let A be a set having at least two elements, $x_0 \in A$ an element, f a mapping of $A \times A$ into itself such that $f(x_1, x_2) = (x_0, x_2)$ for any $(x_1, x_2) \in A \times A$. Then f is not decomposable. Indeed, choose $x \in A$ such that $x \neq x_0$. Then $f(x_0, x) = (x_0, x)$, $f(x, x_0) = (x_0, x_0)$, which implies—by (ii) of Theorem 1—that $f(x, x_0) = (x, x_0)$; this is a contradiction.

Let A be a set, $((x, y), (u, v))$ an ordered pair of elements in $A \times A$; this pair is said to be *bound* if $y = u$. A binary relation r on $A \times A$ is said to be *binding* if the condition $((x, y), (u, v)) \in r$ implies that the ordered pair $((x, y), (u, v))$ is bound.

Lemma 2. *Let A, A' be sets, $((x, y), (u, v))$ a bound ordered pair in $A \times A$. If f is a decomposable mapping of $A \times A$ into $A' \times A'$, then $(f(x, y), f(u, v))$ is a bound ordered pair in $A' \times A'$.*

Proof. Indeed, $f = h \times h$ holds for some mapping h of A into A'. Then $(f(x, y), f(u, v)) = ((h(x), h(y)), (h(u), h(v)))$ where $h(y) = h(u)$, which completes the proof. □

Lemma 3. *Let A, A', A'' be sets, h a mapping of A into A', h' a mapping of A' into A''. Then $(h' \times h') \circ (h \times h) = (h' \circ h) \times (h' \circ h)$.*

Proof. If x, y are in A, then $((h' \times h') \circ (h \times h))(x, y) = (h' \times h')(h(x), h(y)) = (h'(h(x)), h'(h(y))) = ((h' \circ h)(x), (h' \circ h)(y)) = ((h' \circ h) \times (h' \circ h))(x, y)$, which implies the assertion of Lemma. □

We now define the category **BIP** (category of BInary structures with Particular carriers and relations) as follows. Any object of **BIP** is a Cartesian square $A \times A$ with a binding binary relation r; it will be denoted by $(A \times A, r)$. Let $(A \times A, r)$, $(A' \times A', r')$ be objects in **BIP**. A morphism f of $(A \times A, r)$ into $(A' \times A', r')$ in **BIP** is a decomposable mapping of $A \times A$ into $A' \times A'$ such that, for any x, y, u, v in A with $((x, y), (u, v)) \in r$, the condition $(f(x, y), f(u, v)) \in r'$ holds. Such a mapping f will be called a *decomposable homomorphism* of $(A \times A, r)$ into $(A' \times A', r')$.

It is easy to see that $1_{(A \times A, r)}$ is a decomposable homomorphism of $(A \times A, r)$ into itself. Let $(A \times A, r)$, $(A' \times A', r')$, $(A'' \times A'', r'')$ be objects in **BIP**, f a decomposable homomorphism of $(A \times A, r)$ into $(A' \times A', r')$, f' a decomposable homomorphism of $(A' \times A', r')$ into $(A'' \times A'', r'')$. By Lemma 3, $f' \circ f$ is a decomposable mapping of $A \times A$ into $A'' \times A''$. If $((x,y),(u,v)) \in r$, then it is a bound ordered pair and $(f(x,y), f(u,v))$ is a bound ordered pair by Lemma 2; it is in r' because f is a decomposable homomorphism. Similarly, $(f'(f(x,y)), f'(f(u,v)))$ is a bound ordered pair in r''. Thus, $f' \circ f$ is a decomposable homomorphism of $(A \times A, r)$ into $(A'' \times A'', r'')$. Thus, a composition of two morphisms in **BIP** is a morphism of **BIP**. It follows that **BIP** is a category.

3. Isomorphism of categories TER and BIP

We now define two functors: F is a functor of the category **TER** into the category **BIP** and G is a functor of **BIP** into **TER**. These functors will be defined by presenting the object mappings Fo, Go and the morphism mappings Fm, Gm.

Let (A, t) be an object in **TER**. We define a binary relation $\mathbf{b}[t]$ on $A \times A$ as follows:

$$\mathbf{b}[t] = \{((x,y),(y,z));\ (x,y,z) \in t\}.$$

Clearly, $\mathbf{b}[t]$ is a binding binary relation on $A \times A$. Cf also [4]. Thus, we put

$$Fo(A,t) = (A \times A, \mathbf{b}[t]).$$

If h is a homomorphism of (A,t) into (A',t') where (A,t), (A',t') are objects in **TER**, we put

$$Fm(h) = h \times h.$$

Lemma 4. *F is a functor of the category* **TER** *into* **BIP**.

P r o o f. If (A,t) is an object in **TER**, then $Fo(A,t) = (A \times A, \mathbf{b}[t])$ is an object in **BIP**, by the definition of **BIP**. Let (A,t), (A',t') be objects in **TER** and h a homomorphism of (A,t) into (A',t'). We prove that $h \times h$ is a homomorphism of $(A \times A, \mathbf{b}[t])$ into $(A' \times A', \mathbf{b}[t'])$. Indeed, if $((x,y),(u,v)) \in \mathbf{b}[t]$, then $y = u$ and $(x,y,v) \in t$, which implies that $(h(x), h(y), h(v)) \in t'$ and, therefore, $((h \times h)(x,y), (h \times h)(u,v)) = ((h(x), h(y)), (h(y), h(v))) \in \mathbf{b}[t']$.

Furthermore, $Fm(1_{(A,t)}) = 1_{(A,t)} \times 1_{(A,t)} = 1_{(A \times A, \mathbf{b}[t])} = 1_{Fo(A,t)}$.

Finally, if (A,t), (A',t'), (A'',t'') are objects in **TER** and if h is a morphism of (A,t) into (A',t') and if h' is a morphism of (A',t') into (A'',t''), then for any $(x,y) \in A \times A$ we have the following: $(Fm(h') \circ Fm(h))(x,y) = ((h' \times h') \circ (h \times h))(x,y) = ((h' \circ h) \times (h' \circ h))(x,y) = (Fm(h' \circ h))(x,y)$ by Lemma 3. This completes the proof. □

Let $(A \times A, r)$ be an object in **BIP**. We define a ternary relation $\mathbf{t}[r]$ as follows:

$$\mathbf{t}[r] = \{(x, y, v);\ ((x, y), (u, v)) \in r, y = u\}.$$

Clearly, $\mathbf{t}[r]$ is a ternary relation on A. Thus, we put

$$Go(A \times A, r) = (A, \mathbf{t}[r]).$$

Let $(A \times A, r)$, $(A' \times A', r')$ be objects in **BIP**, f a morphism of $(A \times A, r)$ into $(A' \times A', r')$. There exists a mapping h defined in a unique way such that $f = h \times h$. We put

$$Gm(f) = Gm(h \times h) = h.$$

Lemma 5. *G is a functor of the category* **BIP** *into* **TER**.

P r o o f. By definition, $Go(A \times A, r) = (A, \mathbf{t}[r])$ is an object of **TER** for any object $(A \times A, r)$ in **BIP**. Let $(A \times A, r)$, $(A' \times A', r')$ be objects in **BIP**, f a morphism of $(A \times A, r)$ into $(A' \times A', r')$. Thus, there exists a uniquely defined mapping h of A into A' such that $f = h \times h$. Suppose $(x, y, z) \in \mathbf{t}[r]$. Then $((x, y), (y, z)) \in r$ which implies that $(((h(x), h(y)), (h(y), h(z))) = ((h \times h)(x, y), (h \times h)(y, z)) = (f(x, y), f(y, z)) \in r'$, which entails $(h(x), h(y), h(z)) \in \mathbf{t}[r']$. We have proved that $h = Gm(f)$ is a morphism of $Go(A \times A, r) = (A, \mathbf{t}[r])$ into $Go(A' \times A', r') = (A', \mathbf{t}[r'])$.

Furthermore, $Gm(1_{(A \times A, r)}) = Gm(1_{(A, \mathbf{t}[r])} \times 1_{(A, \mathbf{t}[r])}) = 1_{(A, \mathbf{t}[r])} = 1_{Go(A \times A, r)}$.

Let $(A \times A, r)$, $(A' \times A', r')$, $(A'' \times A'', r'')$ be objects in **BIP**, f a morphism of $(A \times A, r)$ into $(A' \times A', r')$ and f' a morphism of $(A' \times A', r')$ into $(A'' \times A'', r'')$. Then there exist mappings h, h' such that $f = h \times h$, $f' = h' \times h'$ where h is a homomorphism of $(A, \mathbf{t}[r])$ into $(A', \mathbf{t}[r'])$, h' a homomorphism of $(A', \mathbf{t}[r'])$ into $(A'', \mathbf{t}[r''])$ as we have seen. Then $(Gm(f') \circ Gm(f))(x) = (Gm(h' \times h') \circ Gm(h \times h))(x) = (h' \circ h)(x) = Gm((h' \circ h) \times (h' \circ h))(x) = Gm((h' \times h') \circ (h \times h))(x) = (Gm(f' \circ f))(x)$ for any $x \in A$ by virtue of Lemma 3. This completes the proof. □

Lemma 6. *$G \circ F$ is the identity functor on the category* **TER** *and $F \circ G$ is the identity functor on the category* **BIP**.

P r o o f. If (A, t) is an object in **TER**, then $Fo(A, t) = (A \times A, \mathbf{b}[t])$ and $Go(Fo(A, t)) = (A, \mathbf{t}[\mathbf{b}[t]])$. Let x, y, z be in A. Then $(x, y, z) \in t$ is equivalent to $((x, y), (y, z)) \in \mathbf{b}[t]$, which means $(x, y, z) \in \mathbf{t}[\mathbf{b}[t]]$. Hence $\mathbf{t}[\mathbf{b}[t]] = t$ and $Go(Fo(A, t)) = (A, t)$, i.e., $Go \circ Fo$ is the identity on the class of all objects in **TER**.

Similarly, if $(A \times A, r)$ is an object in **BIP**, then $Go(A \times A, r) = (A, \mathbf{t}[r])$ and $Fo(Go(A \times A, r)) = (A \times A, \mathbf{b}[\mathbf{t}[r]])$. But $((x, y), (u, v)) \in r$ holds if and only if $y = u$ and $(x, y, v) \in \mathbf{t}[r]$, which means $((x, y), (u, v)) \in \mathbf{b}[\mathbf{t}[r]]$. Thus $r = \mathbf{b}[\mathbf{t}[r]]$ and

$Fo(Go(A \times A, r)) = (A \times A, r)$. We have proved that $Fo \circ Go$ is the identity on the class of all objects of **BIP**.

If h is a morphism in **TER**, then $Fm(h) = h \times h$ and $Gm(Fm(h)) = Gm(h \times h) = h$. Similarly, if f is a morphism in **BIP**, then $f = h \times h$ and $Gm(f) = h$, which implies that $Fm(Gm(f)) = h \times h = f$. Thus, $Gm \circ Fm$ is the identity on the class of all morphisms of **TER** and $Fm \circ Gm$ is the identity on the class of all morphisms of **BIP**. □

By Lemmas 4, 5, 6 we obtain

Theorem 2. *F is a functor of the category* **TER** *onto* **BIP** *and G is a functor of the category* **BIP** *onto* **TER** *such that $F \circ G$ and $G \circ F$ are identity functors.*

Corollary 1. *The following assertions hold.*

(i) *The functor F is an isomorphism of the category* **TER** *onto the category* **BIP**.

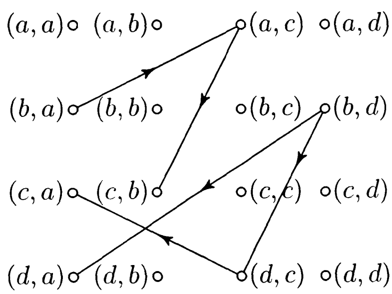
(ii) *The functor G is an isomorphism of the category* **BIP** *onto the category* **TER**.

Corollary 2. *Let (A, t), (A', t') be ternary structures.*

(i) *For any homomorphism h of (A, t) into (A', t') there exists a decomposable homomorphism f of $(A \times A, \mathbf{b}[t])$ into $(A' \times A', \mathbf{b}[t'])$ such that $f = h \times h$.*

(ii) *If f is a decomposable homomorphism of $(A \times A, \mathbf{b}[t])$ into $(A' \times A', \mathbf{b}[t'])$, then $f = h \times h$ and h is a homomorphism of (A, t) into (A', t').*

Example 2. Put $A = \{a, b, c, d\}$, $t = \{(a, c, b), (b, a, c), (b, d, a), (b, d, c), (d, c, a)\}$. Then $\mathbf{b}[t] = \{((a, c), (c, b)), ((b, a), (a, c)), ((b, d), (d, a)), ((b, d), (d, c)), ((d, c), (c, a))\}$. The oriented graph representing $(A \times A, \mathbf{b}[t])$ may be regarded as a representation of the ternary structure (A, t) (cf. [4]). See the following figure.



4. Categories GRD and MAP

We now define a category **GRD** (category of GRoupoiDs) as follows. Any object of **GRD** is a groupoid (cf. [1], [2]), i.e. a set A with a binary operation o. It will be denoted by (A, o). This means that o is a mapping of $A \times A$ into A. It will be convenient to regard o as a ternary relation $\mathbf{tr}[o]$: For arbitrary x, y, z in A we put $(x, y, z) \in \mathbf{tr}[o]$ if and only if $o(x, y) = z$. Hence, an object of **GRD** can be characterized as a ternary structure (A, t) where for any $(x, y) \in A \times A$ there exists exactly one $z \in A$ such that $(x, y, z) \in t$.

Let (A, o), (A', o') be groupoids. A morphism h of (A, o) into (A', o') is a mapping of A into A' such that $o'(h(x), h(y)) = h(o(x, y))$ holds for any $(x, y) \in A \times A$; hence, a morphism is a homomorphism of groupoids. It is easy to see that $1_{(A,o)}$ is a morphism of (A, o) into itself and that the composite of two morphisms is a morphism as well. Thus, **GRD** is a category.

As we have seen, objects in **GRD** are objects of **TER** with a particular property. Let (A, o), (A', o') be groupoids, h a morphism of (A, o) into (A', o') in **GRD**. Suppose that x, y, z in A are such that $(x, y, z) \in \mathbf{tr}[o]$. Then $z = o(x, y)$ and $h(z) = h(o(x, y)) = o'(h(x), h(y))$, which implies that $(h(x), h(y), h(z)) \in \mathbf{tr}[o']$. Thus, h is a morphism of $(A, \mathbf{tr}[o])$ into $(A', \mathbf{tr}[o'])$ in **TER**.

If (A, o), (A', o') are groupoids and h is a morphism of $(A, \mathbf{tr}[o])$ into $(A', \mathbf{tr}[o'])$ in **TER**, then for any $(x, y) \in A \times A$ we obtain $(x, y, o(x, y)) \in \mathbf{tr}[o]$, which implies that $\big(h(x), h(y), h(o(x, y))\big) \in \mathbf{tr}[o']$; this means $o'(h(x), h(y)) = h(o(x, y))$. Hence, h is a morphism of (A, o) into (A', o') in **GRD**.

In what follows we do not distinguish between (A, o) and $(A, \mathbf{tr}[o])$; indeed, o is a subset of $(A \times A) \times A$ and $\mathbf{tr}[o]$ is a subset of $A \times A \times A$ where $(x, y, z) \in \mathbf{tr}[o]$ holds if and only if $((x, y), z) \in o$. Basing on the above presented considerations, we obtain after this identification:

Theorem 3. *The category* **GRD** *is a full subcategory of the category* **TER**.

We now present the definition of a category **MAP** (category of Mono-unary Algebras with Particular properties). Any object of **MAP** is a Cartesian square $A \times A$ with a unary operation w satisfying the following condition: If $(x, y) \in A \times A$, $(u, v) \in A \times A$ are such that $w(x, y) = (u, v)$, then $u = y$. This means that the ordered pair $((x, y), w(x, y))$ in $A \times A$ is bound; for this reason, the operation w will be called *binding*. The resulting mono-unary algebra will be denoted by $(A \times A, w)$. The binary relation on $A \times A$ corresponding to w will be denoted by $\mathbf{br}[w]$; hence, for any $(x, y) \in A \times A$, $(u, v) \in A \times A$ the condition $((x, y), (u, v)) \in \mathbf{br}[w]$ is satisfied if and only if $y = u$ and $w(x, y) = (u, v)$; as we have seen, $\mathbf{br}[w]$ is a binding relation.

Let $(A \times A, w)$, $(A' \times A', w')$ be objects of **MAP**. A morphism f of $(A \times A, w)$ into $(A' \times A', w')$ in **MAP** is a decomposable mapping of $A \times A$ into $A' \times A'$ such that $f(w(x,y)) = w'(f(x,y))$ for any $(x,y) \in A \times A$, i.e. a decomposable mapping of $A \times A$ into $A' \times A'$ that is a homomorphism of the mono-unary algebra $(A \times A, w)$ into $(A' \times A', w')$. It is easy to see that $1_{(A\times A, w)}$ is a morphism of $(A \times A, w)$ into itself. By Lemma 3, the composite of two decomposable mappings is decomposable; furthermore, the composite of two homomorphisms of mono-unary algebras is a homomorphism. It follows that the composite of two morphisms of **MAP** is a morphism as well. Thus, **MAP** is a category.

As we have seen, objects in **MAP** are objects in **BIP** with a particular property. Let $(A \times A, w)$, $(A' \times A', w')$ be objects in **MAP**, f a morphism of $(A \times A, w)$ into $(A' \times A', w')$ in **MAP**. Suppose that x, y, u, v in A are such that $((x,y),(u,v)) \in \mathbf{br}[w]$. Then $w(x,y) = (u,v)$ and $y = u$. It follows that $w'(f(x,y)) = f(w(x,y)) = f(u,v)$. Since $f = h \times h$ for some mapping h of A into A', we have $w'(h(x),h(y)) = (h(u),h(v))$ and $(f(x,y),f(u,v)) = ((h(x),h(y)),(h(u),(h(v))) \in \mathbf{br}[w']$. Thus, f is a morphism of $(A \times A, \mathbf{br}[w])$ into $(A' \times A', \mathbf{br}[w'])$ in **BIP**.

If $(A \times A, w)$, $(A' \times A', w')$ are objects in **MAP** and f is a morphism of $(A \times A, \mathbf{br}[w])$ into $(A' \times A', \mathbf{br}[w'])$ in **BIP**, we take an arbitrary $(x,y) \in A \times A$. Then $w(x,y) = (y,v)$ for some $v \in A$. This means $((x,y),(y,v)) \in \mathbf{br}[w]$. It follows that $(f(x,y),f(y,v)) \in \mathbf{br}[w']$. Putting $f = h \times h$, we obtain $((h(x),h(y)),(h(y),h(v))) \in \mathbf{br}[w']$ which implies that $w'(h(x),h(y)) = (h(y),h(v))$, i.e. $w'(f(x,y)) = f(y,v) = f(w(x,y))$. Thus, f is a morphism of $(A \times A, w)$ into $(A' \times A', w')$ in **MAP**.

In what follows, we do not distinguish between $(A \times A, w)$ and $(A \times A, \mathbf{br}[w])$; indeed, w, $\mathbf{br}[w]$ are subsets of $(A \times A) \times (A \times A)$ and $((x,y),(u,v)) \in \mathbf{br}[w]$ holds if and only if $u = y$ and $(u,v) = w(x,y)$. After this identification, the above presented results may be summarized as follows.

Theorem 4. *The category* **MAP** *is a full subcategory of the category* **BIP**.

We now prove that the functor F transforms the category **GRD** onto **MAP** and the functor G transforms the category **MAP** onto **GRD**. First, we prove some simple results facilitating the proof.

Let (A, o) be a groupoid. We define a unary operation w on $A \times A$ putting $w(x,y) = (y, o(x,y))$ for any $(x,y) \in A \times A$; since it depends on o, we shall denote it by $\mathbf{un}[o]$. Thus, $(\mathbf{un}[o])(x,y) = (u,v)$ if and only if $u = y$, $v = o(x,y)$.

Lemma 7. *Let* (A, o) *be a groupoid. Then* $\mathbf{b}[\mathbf{tr}[o]] = \mathbf{br}[\mathbf{un}[o]]$.

P r o o f. Suppose that x, y, u, v are in A. Then any two consecutive conditions in the following sequence are equivalent.

(a) $((x,y),(u,v)) \in \mathbf{b}[\mathbf{tr}[o]]$;
(b) $y = u$, $(x,y,v) \in \mathbf{tr}[o]$;
(c) $y = u$, $v = o(x,y)$;
(d) $y = u$, $(u,v) = (\mathbf{un}[o])(x,y)$;
(e) $((x,y),(u,v)) \in \mathbf{br}[\mathbf{un}[o]]$.
The equivalence of (a) and (e) implies the assertion of Lemma. □

Let $(A \times A, w)$ be a mono-unary algebra with a binding operation. We define a binary operation o on A: For any $(x,y) \in A \times A$ we put $o(x,y) = z$ if and only if $w(x,y) = (y,z)$. Since o depends on w, it will be denoted by $\mathbf{bo}[w]$. Thus, for any x, y, z in A the condition $(\mathbf{bo}[w])(x,y) = z$ is satisfied if and only if $w(x,y) = (y,z)$.

Lemma 8. *Let $(A \times A, w)$ be a mono-unary algebra with a binding operation. Then $\mathbf{t}[\mathbf{br}[w]] = \mathbf{tr}[\mathbf{bo}[w]]$.*

Proof. Suppose that x, y, z are arbitrary elements in A. Then any two consecutive conditions in the following sequence are equivalent.
(a) $(x,y,z) \in \mathbf{t}[\mathbf{br}[w]]$;
(b) $((x,y),(y,z)) \in \mathbf{br}[w]$;
(c) $w(x,y) = (y,z)$;
(d) $(\mathbf{bo}[w])(x,y) = z$;
(e) $(x,y,z) \in \mathbf{tr}[\mathbf{bo}[w]]$.
The equivalence of (a), (e) implies the assertion of Lemma. □

Theorem 5. *The following assertions hold.*

(i) *The restriction of the functor F to the category* **GRD** *is an isomorphism of* **GRD** *onto* **MAP**.

(ii) *The restriction of the functor G to the category* **MAP** *is an isomorphism of* **MAP** *onto* **GRD**.

Proof. If (A,o) is an object in **GRD**, then $Fo(A, \mathbf{tr}[o]) = (A \times A, \mathbf{b}[\mathbf{tr}[o]]) = (A \times A, \mathbf{br}[\mathbf{un}[o]])$ by Lemma 7; the last structure is an object in **MAP**.

If $(A \times A, w)$ is an object in **MAP**, then $Go(A \times A, \mathbf{br}[w]) = (A, \mathbf{t}[\mathbf{br}[w]]) = (A, \mathbf{tr}[\mathbf{bo}[w]])$ by Lemma 8. The last structure is an object in **GRD**.

Hence, the restriction of Fo to the class of objects in **GRD** maps the class of these objects into the class of objects in **MAP** while the restriction of Go to the class of all objects in **MAP** maps the class of these objects into the class of objects in **GRD**. Taking into account Theorem 2, we obtain that these restrictions are bijections between the classes of objects in **GRD** and **MAP**.

The mapping Fm is a bijection of the class of all morphisms in **TER** onto the class of all morphisms in **BIP** by Corollary 1, **GRD** is a full subcategory of **TER** by Theorem 3, and **MAP** is a full subcategory of **BIP**. It follows that the restriction of Fm to the class of all morphisms in **GRD** is a bijection of this class onto the class of all morphisms in **MAP**. Similarly, the restriction of Gm to the class of all morphisms in **MAP** is a bijection onto the class of all morphisms in **GRD**. This completes the proof. □

In particular, by Corollary 2 we obtain

Corollary 3. *Let (A, o), (A', o') be groupoids.*

(i) *For any homomorphism h of (A, o) into (A', o') there exists a decomposable homomorphism f of $(A \times A, \mathbf{b}[\mathbf{tr}[o]])$ into $(A' \times A', \mathbf{b}[\mathbf{tr}[o']])$ such that $f = h \times h$.*

(ii) *If f is a decomposable homomorphism of $(A \times A, \mathbf{b}[\mathbf{tr}[o]])$ into $(A' \times A', \mathbf{b}[\mathbf{tr}[o']])$, then $f = h \times h$ and h is a homomorphism of (A, o) into (A', o').*

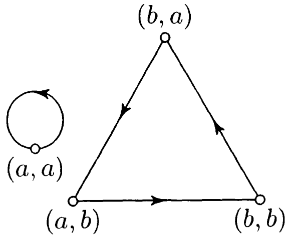
Example 3. Let us have a groupoid (A, o) where $A = \{a, b\}$ and o is given by the following table.

	a	b
a	a	b
b	b	a

Then the corresponding mono-unary algebra is $Fo(A, o) = (A \times A, w)$ where $w(x, y) = (y, o(x, y))$ for any $(x, y) \in A \times A$. Then w is represented by the following table.

(x, y)	(a, a)	(a, b)	(b, a)	(b, b)
$w(x, y)$	(a, a)	(b, b)	(a, b)	(b, a)

This mono-unary algebra is represented by the following graph.



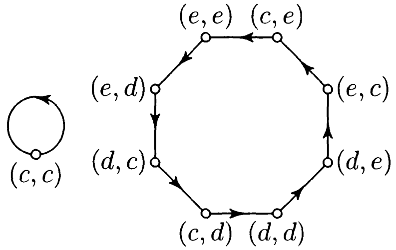
Example 4. Let a groupoid (A', o') be given where $A' = \{c, d, e\}$ and o' is represented by the following table.

	c	d	e
c	c	d	e
d	d	e	c
e	e	c	d

Then the corresponding mono-unary algebra is $Fo(A', o') = (A' \times A', w')$ where $w'(x, y) = (y, o'(x, y))$ for any $(x, y) \in A' \times A'$. Then w' is given by the following table.

(x,y)	(c,c)	(c,d)	(c,e)	(d,c)	(d,d)	(d,e)	(e,c)	(e,d)	(e,e)
$w'(x,y)$	(c,c)	(d,d)	(e,e)	(c,d)	(d,e)	(e,c)	(c,e)	(d,c)	(e,d)

This mono-unary algebra is represented by the following graph.



5. Construction of all homomorphisms of a groupoid into another one

In the papers [5], [6], [7] a construction of all homomorphisms of a mono-unary algebra into another one is presented. We now describe a similar construction for groupoids.

Construction. Let (A, o), (A', o') be groupoids.

Construct $Fo(A, o) = (A \times A, w)$ where $w(x, y) = (y, o(x, y))$ for any $(x, y) \in A \times A$.

Construct $Fo(A', o') = (A' \times A', w')$ where $w'(x, y) = (y, o'(x, y))$ for any $(x, y) \in A' \times A'$.

Construct all homomorphisms of the mono-unary algebra $(A \times A, w)$ into $(A' \times A', w')$ using the known construction from [5], [6], [7].

Test all the constructed homomorphisms and reject all that are not decomposable.

For any decomposable homomorphism f of $(A \times A, w)$ into $(A' \times A', w')$ construct the mapping h such that $f = h \times h$. Then h is a homomorphism of (A, o) into (A', o') and any homomorphism of (A, o) into (A', o') can be constructed in this way.

The assertions contained in the text of Construction paraphrase Corollary 3.

Example 5. Let us find all homomorphisms of (A, o) into (A', o') for the groupoids defined in Examples 3 and 4. Put $w = \mathbf{un}[o]$, $w' = \mathbf{un}[o']$. The mono-unary algebra $(A \times A, w)$ has two components: C_1 with the node (a, a) and C_2 with the nodes (a,b), (b,b), (b,a). Similarly $(A' \times A', w')$ has two components C_1' with the node (c,c) and C_2' with the nodes (d,d), (d,e), (e,c), (c,e), (e,e), (e,d), (d,c), (c,d). By [7], any homomorphism of $(A \times A, w)$ into $(A' \times A', w')$ maps a component C of the first algebra into an admissible component C' of the other. The admissibility of C' to C means that the number of elements in the cycle of C' divides the number of elements in the cycle of C if both algebras are finite. Hence, in our case, C_1' is the only component of $(A' \times A', w')$ admissible to C_1 and also the only component of $(A' \times A', w')$ admissible to C_2. It follows that the only homomorphism of $(A \times A, w)$ into $(A' \times A', w')$ is $f(x,y) = (c,c)$ for any $(x,y) \in A \times A$. It is easy to see that $f = h \times h$ where $h(x) = c$ for any $x \in A$. Thus, this constant mapping is the only homomorphism of (A, o) into (A', o').

Example 6. We construct all endomorphisms of the groupoid (A', o') from Example 4. Similarly as in Example 5, we state that $h(x) = c$ is a constant endomorphism of (A', o'). Furthermore, any endomorphism f of $(A' \times A', w')$ that is not constant is given by the conditions $f(c,c) = (c,c)$ and $f(d,d) = (x,y)$ where $(x,y) \in A \times A$, $(x,y) \neq (c,c)$. This is a consequence of the construction presented in [7] because C_1' is the only component admissible to C_1' and C_2' is the only component that is different from C_1' and admissible to C_2'. If f is decomposable, then $f = h \times h$ for some h. Then $f(d,d) = (h(d), h(d))$ and, therefore, $x = y$. Thus either $f(d,d) = (d,d)$ or $f(d,d) = (e,e)$. In the first case we have $f = 1_{A\times A} = 1_A \times 1_A$ and $h = 1_A$. In the other case we obtain—by definition of w'—the following values for f.

(x,y)	(c,c)	(c,d)	(c,e)	(d,c)	(d,d)	(d,e)	(e,c)	(e,d)	(e,e)
$f(x,y)$	(c,c)	(c,e)	(c,d)	(e,c)	(e,e)	(e,d)	(d,c)	(d,e)	(d,d)

We see that $f = h \times h$ where $h(c) = c$, $h(d) = e$, $h(e) = d$. Thus, our groupoid (A', o') has three endomorphisms with the following tables.

x	c	d	e
$h_1(x)$	c	c	c

x	c	d	e
$h_2(x)$	c	d	e

x	c	d	e
$h_3(x)$	c	e	d

6. Concluding remarks

This article demonstrates an application of the construction of all homomorphisms of a mono-unary algebra. Another application of this construction can be found in [8]. The idea of solving algebraic problems by using isofunctors appears also in [9], [10] and [12]. These problems are near to those considered in [11].

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