Wenxue Huang

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ON CLOSED 4-MANIFOLDS ADMITTING A MORSE FUNCTION WITH 4 CRITICAL POINTS

WENXUE HUANG, London

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1. INTRODUCTION

It is not hard to see that both $S^2 \times S^2$ and $S^1 \times S^3$ amdit a Morse function with exactly 4 critical points (see our Lemma 2's proof); we can also find a topological S^4 having this property (see §4). The main concern here is the inverse problem. Similar topics have been well investigated which can be found in [1], [2], [4], etc.

By $\mathbb{Z}$, E^k, S^i we denote the groups of integers, *k—Euclidean* space and *i—Euclidean* sphere, respectively.

Our main result is as follows

Theorem. *Let M with $\chi(M) \neq 0$ be a closed connected C^∞ 4-manifold which admits a Morse function with exactly 4 critical points, where $\chi(M)$ is the Euler characteristic of M. Then either M is a topological S^4 or M is simply connected and*

$$H_*(M; \mathbb{Z}) \cong H_*(S^2 \times S^2; \mathbb{Z}), \tag{1}$$

i.e., M has the integral homology groups of $S^2 \times S^2$. Furthermore,

a) *If (1) holds and such isomorphisms can be geometrically realized (i.e., if there exists a continuous mapping*

$$h\colon S^2 \times S^2 \longrightarrow M$$

such that its induced homomorphisms

$$h_\sharp\colon H_i(S^2 \times S^2; \mathbb{Z}) \longrightarrow H_i(M; \mathbb{Z}), \qquad i = 0, 2, 4$$

are isomorphisms), M has the homotopy type of $S^2 \times S^2$;

b) *if M is of a C^∞ product structure, M is diffeomorphic with $S^2 \times S^2$;*

c) *if M admits a Riemannian metric of positive curvature and with this metric M can be isometrically immersed into E^6, M is a topological S^4; if M admits a Riemannian structure of non-negative curvature and with this structure M can be isometrically embedded into E^6, either M is a topological S^4 or M is diffeomorphic with $S^2 \times S^2$.*

Remarks. 1. (See §3) We have two alternate versions for the hypothesis "$\chi(M) \neq 0$" in the theorem.

2. I wonder whether the condition in a) is superfluous or not. The realization for general simply connected spaces is not always possible. (See [15, p. 183])

The proof of the theorem will be given in §3. We shall present some preliminaries in §2 and discuss the case "$c_2(f) = 1$" of the theorem in §4.

2. Preliminaries

Unless otherwise specified, all manifolds involved in this paper are closed, connected, smooth and finite dimensional. M^n means manifold M is n-dimensional, e^n an n-cell, $\mathbb{F}$ an arbitrary field, $\chi(M)$ the Euler characteristic of M,

$$\beta_i(M^n; \mathbb{Z}) = \operatorname{rank} H_i(M^n; \mathbb{Z});$$
$$\beta_i(M^n; \mathbb{F}) = \dim H_i(M^n; \mathbb{F});$$
$$\beta(M^n; \mathbb{F}) = \sum_{i=0}^{n} \beta_i(M^n; \mathbb{F}).$$

Given a Morse function f defined on a smooth manifold M^n, by $c(f), c_i(f)$ we denote the number of critical points of f and that of index i, respectively. The Morse number of M^n is denoted by $\gamma(M^n)$, i.e.

$$\gamma(M^n) = \min\{c(\varphi) \mid \varphi\colon M^n \to \mathbb{R} \text{ is a Morse function}\}.$$

Similarly,

$$\gamma_i(M^n) = \min\{c_i(\varphi) \mid \varphi\colon M^n \to \mathbb{R} \text{ is a Morse function}\}.$$

Clearly, for any Morse function φ defined on M, we have

$$\beta_i(M^n; \mathbb{F}) \leqslant \gamma_i(M^n) \leqslant c_i(\varphi),$$
$$\beta(M^n; \mathbb{F}) \leqslant \sum_{i=0}^{n} \gamma_i(M^n) \leqslant c(\varphi).$$

In particular, if $\beta(M^n;\mathbb{F}) = c(\varphi)$, all inequalities above become equalities.

Using Kunneth's formula, we slightly modify the result in [5, p. 217–218] as follows

Lemma 1. *Given two Morse functions*

$$\varphi\colon N^n \to \mathbb{R},\ \psi\colon Q^q \to \mathbb{R},$$

then the function $\varphi + \psi\colon N^n \times Q^q \to \mathbb{R}$ *defined by*

$$(x, y) \in N^n \times Q^q \mapsto \varphi(x) + \psi(y) \in \mathbb{R}$$

is a Morse function and

$$c_i(\varphi + \psi) = \sum_{j+k=i} c_j(\varphi)c_k(\psi).$$

In particular, if both φ *and* ψ *are tight (i.e.,* $c(\varphi) = \gamma(N)$ *and* $c(\psi) = \gamma(Q)$*),*

$$c(\varphi + \psi) = \gamma(N^n)\gamma(Q^q) \geqslant \gamma(N^n \times Q^q).$$

If there exists a field $\mathbb{F}$ *such that*

$$\gamma(N^n) = \beta(N^n;\mathbb{F}) \text{ and } \gamma(Q^q) = \beta(Q^q;\mathbb{F}),$$

then $\gamma(N^n \times Q^q) = \gamma(N^n)\gamma(Q^q)$.

3. The Proof of The Theorem

To prove the theorem and study the general case, we establish first the following main lemma.

Lemma 2. *Let* f *be a Morse function defined on a closed connected smooth 4-manifold* M *and* $c(f) = 4$. *Then*

(a) *TFAE*

$(a)_1$ *M is a topological S^4;*

$(a)_2$ $c_0(f) + c_4(f) = 3$ or $c_2(f) = 1$;

$(a)_3$ $\chi(M) = 2$.

(b) *TFAE*

$(b)_1$ *M is simply connected and has the integral homology groups of $S^2 \times S^2$;*

$(b)_2$ $c_2(f) = 2$;

(b)$_3$ $\chi(\mathrm{M}) = 4$;

(b)$_4$ *there exist CW-complexes K and L with the same collection of cells such that M and $S^2 \times S^2$ have the homotopy type of K and L respectively.*

(c) *TFAE*

(c)$_1$ *M has the mod 2 homology groups of $S^1 \times S^3$ and the "mod 2" is replaced by "integral" when M is orientable;*

(c)$_2$ $c_1(f) = c_3(f) = 1$;

(c)$_3$ $\chi(M) = 0$;

(c)$_4$ *there exist CW-complexes S and T with the same collection of cells such that M and $S^1 \times S^3$ have the homotopy type of S and T respectively;*

(c)$_5$ *M is non-simply connected.*

(d) *TFAE*

(d)$_1$ $\chi(M) \neq 0$;

(d)$_2$ *M is simply connected;*

(d)$_3$ $c_1(f) + c_3(f) \leqslant 1$.

Proof. The condition $c(f) = 4$ and Theorem 12.1 in [10, p. 383] imply

$$2 \leqslant c_0(f) + c_4(f) \leqslant 3.$$

1) When $c_0(f) + c_4(f) = 3$, M is a topological S^4 by Theorem 12.1 in [10, p. 383] and Reeb theorem.

If $c_2(f) = 1$, then $c_0(f) = 1 = c_4(f)$, otherwise we can set $c_4(f) = 2$, then by Theorem 12.1 in [10, p. 383] we have $c_3(f) \geqslant 1$ that implies $c(f) \geqslant 5$, contradicting the hypothesis $c(f) = 4$. Therefore we can set

$$c_1(f) = 1, \qquad c_3(f) = 0.$$

By the improved Morse inequalities by Pitcher that

$$c_i(f) \geqslant \beta_i(M; \mathbb{Z}) + t_i(M; \mathbb{Z}) + t_{i-1}(M; \mathbb{Z}),$$

where $t_i(M; \mathbb{Z})$ is the torsion number of $H_i(M; \mathbb{Z})$, we have

$$\beta_i(M; \mathbb{Z}) = \beta_i(S^4; \mathbb{Z}), \quad i = 0, 1, 2, 3, 4,$$
$$H_*(S^4; \mathbb{Z}) \cong H_*(M; \mathbb{Z}).$$

Since $c_1(-f) = 0$, M is simply connected by Cor. 10.18 in [13, p. 225], it follows that M is a homotopy S^4, i.e. M is a topological S^4 by Freedman's theorem in [3, p. 371].

2) If $c_2(f) = 2$, then $c_0(f) = c_4(f) = 1$ and $c_1(f) = c_3(f) = 0$. It follows that M has the integral homology groups of $S^2 \times S^2$. Since $c_1(f) = 0$, M is simply connected by Cor. 10.18 in [13, p. 225].

3) If $c_1(f) = c_3(f) = 1, c_0(f) = c_4(f) = 1$ and $c_2(f) = 0$. By Morse inequalities, we have

$$\beta_2(M;\mathbb{F}) = 0,\ \beta_1(M;\mathbb{F}) = \beta_0(M;\mathbb{F}),\ \beta_4(M;\mathbb{F}) = \beta_3(M;\mathbb{F})$$

hold for any field $\mathbb{F}$. Therefore

$$H_*(M;\mathbb{Z}_2) \cong H_*(S^1 \times S^3;\mathbb{Z}_2);$$

and so

$$\beta_i(M;\mathbb{Z}_2) = \beta_i(S^1 \times S^3;\mathbb{Z}).$$

In particular, if M is orientable, then by the homology duality and the improved Morse inequalities, we have

$$H_*(M;\mathbb{Z}) \cong H_*(S^1 \times S^3;\mathbb{Z}).$$

4) We claim $c_1(f) \neq 2$(equivalently $c_3(f) \neq 2$). Otherwise $c_0(f) = c_4(f) = 1$ and $c_2(f) = c_3(f) = 0$, and then

$$\beta_i(M;\mathbb{F}) = 0,\ i = 1,2,3; \quad \beta_0(M;\mathbb{F}) = 1,$$

resulting in

$$-1 = c_2(f) - c_1(f) + c_0(f) \geqslant \beta_2(M;\mathbb{F}) - \beta_1(M;\mathbb{F}) + \beta_0(M;\mathbb{F}) = 1,$$

which is absurd.

We have exhibited all possible values of $c_i(f)$ and proved that $(a)_2 \Rightarrow (a)_1$, $(b)_2 \Rightarrow (b)_1$ and $(c)_2 \Rightarrow (c)_1$. That $(a)_1 \Rightarrow (a)_3, (a)_2 \Rightarrow (a)_3$, $(b)_1 \Rightarrow (b)_3$ and $(c)_1 \Rightarrow (c)_3$ are trivially true.

Our conclusion $(b)_3 \Rightarrow (b)_2$ follows from the facts that 1) implies $\chi(M) = 2$ and that 3) implies $\chi(M) = 0$.

The proof of $(b)_2 \Rightarrow (b)_4$: Given $c_2(f) = 2$, then $c_0(f) = c_4(f) = 1$ and $c_1(f) = c_3(f) = 0$. By Theorem 3.5 in [7, p. 20], M has the homotopy type of a *CW-complex* with a collection of one e^0, two e^2's and one e^4.

On the other hand, given a natural embedding $S^n \hookrightarrow E^{n+1}$, then for any unit vector $p \in E^{n+1}$, the linear height function $l_p\colon S^n \to \mathbb{R}$ defined by $x \in S^n \mapsto \langle p, x\rangle$ (where $\langle .,. \rangle$ denotes the usual inner product in E^{n+1}) is a Morse function with only

2 critical points, so l_p is tight and $\beta(S^n;\mathbb{F}) = 2 = \gamma(S^n)$. From Lemma 1, we know that

$$\varphi = l_p + l_q \colon S^2 \times S^2 \to \mathbb{R}$$

satisfies

$$c(\varphi) = \gamma(S^2)\gamma(S^2) = \gamma(S^2 \times S^2)$$

and

$$c_0(\varphi) = c_4(\varphi) = 1,\ c_2(\varphi) = 2,\ c_1(\varphi) = c_3(\varphi) = 0,$$

so $S^2 \times S^2$ like M has the homotopy type of a CW-complex with a collection of one e^0, two e^2's and one e^4.

The proof of $(b)_4 \Rightarrow (b)_3$: The Euler characteristic of a manifold is a homotopy type invariant and K and L have the same collection of cells, so

$$\chi(M) = \chi(K) = \chi(L) = \chi(S^2 \times S^2) = 4.$$

The proofs of $(c)_1 \Rightarrow (c)_3 \Rightarrow (c)_2 \Rightarrow (c)_4 \Rightarrow (c)_3$ are the analogues of that of (b).

The proof of $(c)_5 \Rightarrow (c)_2$: M is non-simply connected if and only if $c_1(f) = c_3(f) = 1$ holds according to 1) and 2).

$(c)_1 \Rightarrow (c)_5$ is trival.

The proof of $(a)_3 \Rightarrow (a)_2$: When $\chi(M) = 2$, only $c_0(f) + c_4(f) = 3$ or $c_2(f) = 1$ holds by 2) and 3).

Now the (d) follows immediately from (a), (b) and (c).

This concludes the proof of the lemma. □

Now we are in a position to prove the theorem.

P r o o f o f T h e o r e m. Since $\chi(M) \neq 0$, from the proof of Theorem 2 we know that M is simply connected and furthermore either M is a topological S^4 or M has the integral homology groups of $S^2 \times S^2$.

a) If M is the latter and there exists a continuous map

$$h \colon S^2 \times S^2 \to M$$

for which its induced homomorphisms

$$h_\sharp \colon H_i(S^2 \times S^2; \mathbb{Z}) \to H_i(M; \mathbb{Z}),\ i = 0, 2, 4$$

are isomorphisms, then since both M and $S^2 \times S^2$ are simply connected CW-complexes, using Theorem 25 in [14, p. 406], we conclude that M is homotopically equivalent to $S^2 \times S^2$.

b) Let M have a C^∞ product structure, i.e. $M = N \times Q$, then $\dim N = 2$ or 1.

If $\dim N = 1$, $N \approx S^1$ (here we denote "diffeomorphic to"by $\approx$), which contradicts the simply-connectedness of M. It follows that

$$\dim N = 2 = \dim Q.$$

But $\gamma(N) = 4 - \chi(N), \gamma(Q) = 4 - \chi(Q)$, thus

$$\beta(N;\mathbb{Z}_2) = \gamma(N),\ \beta(Q;\mathbb{Z}_2) = \gamma(Q).$$

Applying Lemma 1 to N and Q, we get

$$\gamma(M) = 4, \qquad \gamma(N) = 2 = \gamma(Q).$$

Therefore

$$M = N \times Q \approx S^2 \times S^2.$$

c) Let M be an orientable Riemannian manifold of positive curvature and let $I\colon M \to E^6$ be an isometrical immersing. By Moore's theorem (e.g. see [6, p. 116]) or [8, p. 72]), $I(M)$ is a topological S^4 and so I is an embedding, M is therefore a topological S^4.

Let M with a Riemannian structure of non-negative curvature be isometrically embedded into E^6 and $I\colon M \to E^6$ such an embedding. Since M is simply connected, by Baldin and Mercuri's result (see [6, p. 116]), we conclude that either M is a homotopy S^4 and hence a topological S^4 or $M \approx S^2 \times S^2$. This completes the proof of the theorem. □

Remark. Under the hypothesis of Lemma 2, if M is a non-simply connected product manifold, $M \approx S^1 \times Q^3$; if Q^3 satisfies $\gamma(Q^3) = \beta(Q^3;\mathbb{Z}_2)$, $M \approx S^1 \times S^3$. Because product manifold M is non-simply connected, $M = N^1 \times Q^3 \approx S^1 \times Q^3$ by the proof of the Theorem. Therefore

$$H_*(M;\mathbb{F}) \cong H_*(S^1 \times S^3;\mathbb{F})$$

holds for any field $\mathbb{F}$ and therefore

$$\beta(M;\mathbb{F}) = 4 = \gamma(M).$$

By Kunneth's formula, we know that

$$\beta(Q^3;\mathbb{F}) = 2$$

holds for any field $\mathbb{F}$, so

$$H_*(Q^3;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z}).$$

If, in addition, $\gamma(Q^3) = \beta(Q^3;\mathbb{Z}_2), Q^3 \approx S^3$. Hence

$$M \approx S^1 \times S^3.$$

4. On Case $c_2(f) = 1$

For (M, f) satisfing

$$c_0(f) + c_4(f) = 3 \tag{2}$$

or

$$c_2(f) = 2 \tag{3}$$

or

$$c_1(f) = c_3(f) = 1 \tag{4}$$

we have its corresponding models. In fact, models (M, f) satisfying (3) and (4) have been shown in the proof of Lemma 2; the model (M, f) for (2) can be realized by a hypersurface of E^5, which is similar to a U-shape tube with two smooth caps on its two ends, and a linear height function defined on the hypersurface. We show the model as follows:

A subset T of $E^5 = \{(x, y, z, u, v) \mid x, y, z, u, v \in E^1\}$ is defined by the equation

$$(\sqrt{u^2+v^2} - a)^2 + x^2 + y^2 + z^2 = b^2, \quad a > b > 0.$$

Obviously, T can be obtained by "revolving a 3-sphere in E^5

$$\begin{cases} (v-a)^2 + x^2 + y^2 + z^2 = b^2, \\ u = 0 \end{cases}$$

around subspace $Oxyz$"; so T is connected and closed. If hyperplane $v = 0$ is regarded as a "level surface" and v-axis as the "vertical" axis, then sublevel set T_- : $v < 0$ can be given by

$$v = -\sqrt{(a \pm \sqrt{b^2 - x^2 - y^2 - z^2})^2 - u^2}.$$

Similar to the case of a "vertical" torus in E^3, the linear height function on T_- can be expressed as

$$f(x,y,z,u,v) = -\sqrt{(a \pm \sqrt{b^2 - x^2 - y^2 - z^2})^2 - u^2} + a + b.$$

Then

$$df = 0 \Leftrightarrow x = y = z = u = 0, \quad v = \pm b - a,$$

i.e. f has just 2 critical points $(0,0,0,0,\pm b - a)$ on T_- and of which $(0,0,0,0,-b-a)$ is the minimum point of f. It is easily verified that the Hassian matrices of f at the 2 critical points are nondegenerate, so f is a Morse function on T_-.

Since level surface $v = 0$, which is the boundary of T_-, consists of two 3-spheres in E^5

$$\begin{cases} (u \pm a)^2 + x^2 + y^2 + z^2 = b^2, \\ \hfill v = 0, \end{cases}$$

T_- is a "U-shape tube" with two upward ends. We cover its each end with a "cap", i.e., a smooth $4-disc$ and then obtain the required hypersurface M of E^5. Meanwhile, we extend f naturally onto the two "caps". We still denote the extension of f, which is a linear height function defined on M, by f, then the two tops of the caps are critical points of f of index 4 and hence f has exactly 4 critical points on M, and

$$c_0(f) + c_4(f) = 3.$$

Then M is a topological S^4 by the Theorem. This concludes the construction of the required model.

Our main purpose of this section is to probe into (just!) the probability of the existence of (M, f) satisfying

$$(5) \qquad c_0(f) = c_2(f) = c_3(f) = c_4(f) = 1, \quad c_1(f) = 0.$$

To the end, we assume that (5) holds and under the assumption we determine the types of the 4 critical points of f and calculate the homology groups of the sublevels f_t and level manifolds f^t. We need some preliminaries.

Morse introduced the following notions and results in his [11, p. 257–258] and [12, p. 259–260]:

For a Morse function f defined on an orientable manifold M^n and a real number t, we denote the sublevel set $\{x \in M \mid f(x) \leqslant t\}$ by f_t, and $\beta_k(f_d; \mathbb{F})$ by $\beta_k(d)$. Suppose open interval (a, b) contains just one critical value c of f and f take its critical value c only at one critical point p_c of index k. Set

$$\Delta\beta_q(c) = \beta_q(b) - \beta_q(a), \quad q = 0, 1, \ldots, n.$$

Then $\Delta\beta_k(c) = 1$ or $\Delta\beta_{k-1} = -1$. If the former (resp. latter) holds, the critical point p_c is said to be of increasing (resp. decreasing) type or linking (resp. nonlinking) type.

Notice that

$$\beta_i(b) = \beta_i(c), \quad \beta_i(a) = \beta_i(f_c - \{p_c\}),$$
$$\Delta\beta_q(c) = \beta_q(c) - \beta_q(f_c - \{p_c\}).$$

By Theorem 29.2 in [12, p. 260],

$$\Delta\beta_q(c) = \begin{cases} 1, & \text{if } q = k \text{ and } p_c \text{ is of linking type,} \\ -1, & \text{if } q = k-1 \text{ and } p_c \text{ is of nonlinking type,} \\ 0, & \text{in other cases.} \end{cases}$$

Thus the notions of linking and nonlinking types are mutually exclusive and complementary.

We write

$$\Delta B_i(c) = \beta_i(f^{c+r}) - \beta_i(f^{c-\varepsilon})$$

for any regular value c of f and any sufficiently small real number ε.

Let (M, f) satisfy the conditions of Theorem 2 and (5), then M is a topological S^4. Applying Corollary 39.1 of [12, p. 361] to this (M, f), we choose f such that

$$f(p_i) = i, \qquad i = 0, 2, 3, 4$$

for which p_i is a critical point of f of index i. Take regular values a, b, c, d, e of f such that

$$a < 0 < b < 2 < c < 3 < d < 4 < e.$$

Then we have

Proposition 3. *For f chosen above, the critical points p_0, p_2 and p_4 are of linking type and p_3 nonlinking type. Moreover,*

$$\beta_q(i) = \begin{cases} 1, & \textit{if } (q,i) = (2,2), (4,4) \text{ or } (0,i), \textit{ where } i = 0,2,3,4. \\ 0, & \textit{in other cases.} \end{cases}$$

P r o o f. Clearly, p_0, p_4 are of linking type and as is p_2, since by applying Morse inequalities to f, f_c, we have

$$\beta_2(c) = 1,$$

thus

$$\Delta\beta_2(2) = 1.$$

It is easily checked that

$$\Delta\beta_3(3) = 0,$$

and so

$$\Delta\beta_2(3) = -1, \quad \beta_2(d) = 0.$$

Hence p_3 is of nonlinking type. □

Remarks. 1. An analogous argument shows that to $-f$, its critical points p_4, p_3, and p_0, with indeces 0, 1, 4, resp., are of linking type but p_2, with index 2, nonlinking type.

2. [1, p. 8–9] indicates: Let f be a Morse function defined on a closed C^∞ manifold M^n, then the following three conditions are equivalent

a) For any field $\mathbb{F}$, $c_k(f|f_t) = \beta_k(f_t;\mathbb{F})$ holds for any $t \in \mathbb{R}$ and $k = 0, 1, 2, \ldots, n$;

b) The homomorphisms between homology groups

$$H_i(f_t;\mathbb{F}) \to H_i(M^n;\mathbb{F}), \quad i = 0, 1, 2, \ldots, n$$

induced by inclusion $f_t \hookrightarrow M^n$ are injective;

c) Every critical point of f is of linking type.

Now our Proposition 3 implies that for (M, f) satisfying (5),

a)' $1 = c_k(f) > \beta_k(M;\mathbb{F}) = 0$, $k = 2, 3$;

b)' The induced homomorphism

$$(\mathbb{F} \cong) \qquad H_2(f_c;\mathbb{F}) \to H_2(M;\mathbb{F}) \qquad (= 0)$$

is not injective;

c)' the critical point p_3 of f is of nonlinking type.

It follows that our Proposition 3 does not contradict the results in [1, p. 8–9]. It can be verified that (5) is compatible with Morse inequalities, the theorem on a character of homology S^4 in [11, p. 259] and Corollary 1.2 as well as (7.11) in [9, p. 256–257]. Besides, by Lemma 1.1 in [10, p. 352], the critical point p_3 of $-f$ of index 1 is of linking type,which is continent with our Proposition 3. All these facts seem to be to a great extent in favor of the existence of (M, f) satisfying (5).

3. In his [16, p. 100], Willmore said that recent work by Cerf made it appear that (which has not been proved! cf. V. V. Sharko's work, say, $MR1989f, 57038$)

$$\gamma(M) = \sum_{i=0}^{n} \gamma_i(M)$$

holds for any closed C^∞ manifold. If the equality holds, our f satisfying (5) has two superfluous saddle points, i.e., there exists a Morse function $f^*: M \to \mathbb{R}$ with only 2 critical points.

As the end of this paper, we study the topology of the level hypersurfaces of M with respect to any regular value of f and obtain

Proposition 4. *Under the hypothesis of Proposition* 3, $f^b \approx S^3 \approx f^d$, f^c *has the integral homology groups of* $S^1 \times S^2$.

P r o o f. We denote set $\{x \in M \mid f(x) \geqslant t\}$ by f_t^+. Since the points p_0 in f_b and p_4 in f_d^+ are extreme points of f and f_b (resp. f_d^+) contains no critical points of f other than p_0 (resp. p_4), and f^b (resp. f^d) is the boundary of f_b (resp. f_d^+). Then by Morse lemma,

$$f^b \approx S^3 \approx f^d.$$

For f^c, since M is a homology S^4, p_0 and p_2 as critical points of f are of increasing type rel. f^0 and f^2, resp., but p_3 decreasing type rel. f^3 by Corollary 7.2 in [9, p. 256]. Thus by Theorem 5.1 in [9, p. 252],

$$\begin{aligned} 1 &= \Delta B_2(2) = \beta_2(f^c), \\ 1 &= \Delta B_1(2) = \beta_1(f^c), \\ 0 &= \Delta B_i(2) = \beta_i(f^c) - 1, \qquad i = 0, 3. \end{aligned}$$

that is,

$$\beta_i(f^c; \mathbb{F}) = \beta_i(S^1 \times S^2; \mathbb{F}) = \beta_i(S^1 \times S^2; \mathbb{Z}), \qquad i = 0, 1, 2, 3$$

hold for any field $\mathbb{F}$. Then by the improved Morse inequalities, we have

$$H_*(f^c; \mathbb{Z}) \cong H_*(S^1 \times S^2; \mathbb{Z}).$$

It follows, moreover, that f^b, f^c and f^d are connected closed orientable hypersurfaces of M. This proves the proposition. □

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Author's address: Department of Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada.