A. K. Chongdar; N. K. Majumdar

Some novel generating functions of extended Jacobi polynomials by group theoretic method

Czechoslovak Mathematical Journal, Vol. 46 (1996), No. 1, 29–33

Persistent URL: http://dml.cz/dmlcz/127267

SOME NOVEL GENERATING FUNCTIONS OF EXTENDED JACOBI POLYNOMIALS BY GROUP THEORETIC METHOD

A.K. Chongdar and N.K. Majumdar, Calcutta

(Received April 15, 1993)

1. Introduction

The extended Jacobi polynomials defined by Patil and Thakare [1]

$$F_n(\alpha,\beta;x) = \frac{(-1)^n}{n!}(x-a)^{-\alpha}(b-x)^{-\beta}\Big(\frac{\lambda}{b-a}\Big)^n \times D^n[(x-a)^{n+\alpha}(b-x)^{n+\beta}], \tag{1.1}$$

where $D=\frac{\mathrm{d}}{\mathrm{d}x}$ and λ is a number such that $\frac{\lambda}{b-a}>0$, satisfy the ordinary differential equation [2]

$$[(x-a)(b-x)D^2+\{(\alpha+1)(b-x)-(\beta+1)(x-a)\}D+n(1+\alpha+\beta+n)]y=0. \tag{1.2}$$

Very recently, attempts have been made [2, 3] in connection with the derivation of generating functions of the extended Jacobi polynomials from the Lie-group viewpoint.

The aim of the present paper is to investigate some novel generating relations of the extended Jacobi polynomial $F_n(\alpha,\beta;x)$ by the application of L. Weisner's group-theoretic method [4] which is vividly presented in the monograph by E.B. McBride [5]. It may be of interest to remark that in course of constructing a Lie algebra we obtain a pair of linear partial differential operators which simultaneously raise (lower) and lower (raise) the parameters α and β of the polynomial under consideration. We would like to mention that our results differ from the traditional concept of a generating function for orthogonal polynomials.

2. Group-theoretic method

Replacing $\frac{\mathrm{d}}{\mathrm{d}x}$ by $\frac{\partial}{\partial x}$, α by $y\frac{\partial}{\partial y}$, β by $z\frac{\partial}{\partial z}$ and y by $u(x,y,z)$ in (1.2), we get the partial differential equation

$$\begin{aligned}(2.1)\qquad &\Big[(x-a)(b-x)\frac{\partial^2}{\partial x^2} + \Big\{\Big(y\frac{\partial}{\partial y}+1\Big)(b-x) - \Big(z\frac{\partial}{\partial z}+1\Big)(x-a)\Big\}\frac{\partial}{\partial x} \\ &+ n\Big(1 + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + n\Big)\Big]u(x,y,z) = 0.\end{aligned}$$

Thus we see that $u(x,y,z) = F_n(\alpha,\beta;x)y^\alpha z^\beta$ is a solution of (2.1), since $F_n(\alpha,\beta;x)$ is a solution of (1.2).

We now define linear partial differential operators

$$\begin{aligned}(2.2)\qquad A_1 &= y\frac{\partial}{\partial y}, \\ A_2 &= z\frac{\partial}{\partial z}, \\ A_3 &= (x-b)yz^{-1}\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \\ A_4 &= (x-a)y^{-1}z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}\end{aligned}$$

such that

$$\begin{aligned}(2.3)\qquad A_1[F_n(\alpha,\beta;x)y^\alpha z^\beta] &= \alpha F_n(\alpha,\beta;x)y^\alpha z^\beta, \\ A_2[F_n(\alpha,\beta;x)y^\alpha z^\beta] &= \beta F_n(\alpha,\beta;x)y^\alpha z^\beta, \\ A_3[F_n(\alpha,\beta;x)y^\alpha z^\beta] &= (\beta+n)F^n(\alpha+1,\beta-1;x)y^{\alpha+1}z^{\beta-1}, \\ A_4[F_n(\alpha,\beta;x)y^\alpha z^\beta] &= (n+\alpha)F_n(\alpha-1,\beta+1;x)y^{\alpha-1}z^{\beta+1}.\end{aligned}$$

The commutator relations satisfied by A_i (i = 1, 2, 3, 4) are

$$\begin{aligned}(2.4)\qquad [A_1,A_2] &= 0, & [A_2,A_3] &= -A_3, \\ [A_1,A_3] &= A_3, & [A_2,A_4] &= A_4, \\ [A_1,A_4] &= -A_4, & [A_3,A_4] &= A_1 - A_2\end{aligned}$$

where $[A,B]u = (AB - BA)u$.

Thus we arrive at the following theorem:

Theorem. *The set of operators* $\{1, A_i\ (i = 1,2,3,4)\}$ *where* 1 *stands for the identity operator, generates a Lie algebra* $\mathcal{L}$.

It can be shown that the partial differential operator L,

$$L = (x-a)(b-x)\frac{\partial^2}{\partial x^2} + \left[\left(y\frac{\partial}{\partial y}+1\right)(b-x) - \left(z\frac{\partial}{\partial z}+1\right)(x-a)\right]\frac{\partial}{\partial x} + n\left(1 + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + n\right) \tag{2.5}$$

can be related to A_i ($i = 1, 2, 3, 4$) as follows:

$$L = -A_3A_4 + A_1A_2 + A_1 + n(1 + A_1 + A_2 + n). \tag{2.6}$$

Now one can easily verify that L commutes with each A_i ($i = 1, 2, 3, 4$), i.e.,

$$[L, A_i] = 0. \tag{2.7}$$

The extended form of the groups generated by A_i ($i = 1, 2, 3, 4$) are

$$\begin{aligned} e^{a_1A_1}f(x,y,z) &= f(x, e^{a_1}y, z),\\ e^{a_2A_2}f(x,y,z) &= f(x, y, e^{a_2}z),\\ e^{a_3A_3}f(x,y,z) &= f\left(x + a_3\frac{(x-b)y}{z}, y, z + a_3y\right),\\ e^{a_4A_4}f(x,y,z) &= f\left(x + a_4\frac{(x-a)y}{z}, y + a_4z, z\right). \end{aligned} \tag{2.8}$$

From he above we get

$$\begin{aligned} &e^{a_4A_4}e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}f(x,y,z)\\ &\quad = f\left(\left(x + a_4\frac{(x-a)z}{y}\right)\left[1 + a_3\left(a_4 + \frac{y}{z}\right)\right] - a_3b\left(a_4 + \frac{y}{z}\right),\right.\\ &\qquad \left. e^{a_1}y\left(1 + \frac{a_4}{y}z\right), e^{a_2}z\left\{1 + a_3\left(\frac{y}{z} + a_4\right)\right\}\right). \end{aligned} \tag{2.9}$$

3. Generating functions

Now it follows from (2.1) that $u_1(x,y,z) = F_n(\alpha,\beta;x)y^\alpha z^\beta$ is a solution of the system

$$\begin{aligned} Lu &= 0,\\ (A_1 - \alpha)u &= 0;\\ Lu &= 0,\\ (A_2 - \beta)u &= 0;\\ Lu &= 0,\\ (A_1 + A_2 - \alpha - \beta)u &= 0. \end{aligned}$$

From (2.7) one can easily verify that

$$S\big(L\big(F_n(\alpha,\beta;x)y^\alpha z^\beta\big)\big)=L\big(S\big(F_n(\alpha,\beta;x)y^\alpha z^\beta\big)\big)=0,$$

where

$$S=\mathrm{e}^{a_4A_4}\mathrm{e}^{a_3A_3}\mathrm{e}^{a_1A_2}\mathrm{e}^{a_1A_1}.$$

Therefore $S\big(F_n(\alpha,\beta;x)y^\alpha z^\beta\big)$ is annihilated by L.

Putting $a_1=a_2=0$ and replacing $f(x,y,z)$ by $F_n(\alpha,\beta;x)y^\alpha z^\beta$ in (2.9), we get

$$\begin{aligned}(3.1)\quad &\mathrm{e}^{a_4A_4}\mathrm{e}^{e_3A_3}[F_n(\alpha,\beta;x)y^\alpha z^\beta]\\ &=F_n\Big(\alpha,\beta;\Big\{x+a_4\frac{(x-a)z}{y}\Big\}\Big\{1+a_3\Big(a_4+\frac{y}{z}\Big)\Big\}-a_3b\Big(a_4+\frac{y}{z}\Big)\Big)\\ &\quad\times y^\alpha\Big(1+\frac{a_4}{y}z\Big)^\alpha\times z^\beta\Big\{1+a_3\Big(a_4+\frac{y}{z}\Big)\Big\}^\beta.\end{aligned}$$

However,

$$\begin{aligned}(3.2)\quad &\mathrm{e}^{a_4A_4}\mathrm{e}^{e_3A_3}[F_n(\alpha,\beta;x)y^\alpha z^\beta]\\ &=\sum_{p=0}^{\infty}\frac{(-a_4)^p}{p!}(-n-\alpha-k)_p\sum_{k=0}^{\infty}\frac{(-a_3)^k}{k!}(-\beta-n)_k\\ &\quad\times F_n(\alpha+k-p,\beta-k+p;x)y^{\alpha+k-p}z^{\beta-k+p}.\end{aligned}$$

Equating (3.1) and (3.2), we get

$$\begin{aligned}(3.3)\quad &\Big(1+a_4\frac{y}{z}\Big)^\alpha\Big\{1+a_3\Big(a_4+\frac{y}{z}\Big)\Big\}^\beta F_n\Big(\alpha,\beta;\Big\{x+a_4\frac{(x-a)z}{y}\Big\}\\ &\quad\times\Big\{1+a_3\Big(a_4+\frac{y}{z}\Big)\Big\}-a_3b\Big(a_4+\frac{y}{z}\Big)\Big)\\ &=\sum_{p=0}^{\infty}\frac{(-a_4)^p}{p!}(-n-\alpha-k)_p\sum_{k=0}^{\infty}\frac{(-a_3)^k}{k!}(-\beta-n)_k\\ &\quad\times F_n(\alpha+k-p,\beta-k+p;x)y^{k-p}z^{-k+p}.\end{aligned}$$

We now consider the following cases:

Case 1. Putting $a_3=1$, $a_4=0$ and replacing $\frac{a_3y}{z}$ by $-t$, we get

$$(3.4)\qquad (1-t)^\beta F_n\big(\alpha,\beta;x-(x-b)t\big)=\sum_{k=0}^{\infty}\frac{(-\beta-n)_k}{k!}F_n(\alpha+k,\beta-k;x)t^k.$$

Case 2. Putting $a_3 = 0$, $a_4 = 1$ and substituting $\frac{a_4 z}{y} = t$, we get

$$(3.5)\quad (1+t)^{\alpha} F_n\big(\alpha, \beta; x + (x-a)t\big) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} (-n-\alpha)_p F_n(\alpha - p, \beta + p; x) t^p.$$

Case 3. Putting $a_3 = \frac{1}{w}$, $a_4 = 1$ and $\frac{z}{y} = t$, we get

$$(3.6)\quad (1+t)^{\alpha}\Big(1 + \frac{1}{w}\Big(1+\frac{1}{t}\Big)\Big)^{\beta}$$
$$\times F_n\Big(\alpha, \beta; \{x + (x-a)t\}\Big\{1 + \frac{1}{w}\Big(1+\frac{1}{t}\Big)\Big\} - \frac{b}{w}\Big(1+\frac{1}{t}\Big)\Big)$$
$$= \sum_{p=0}^{\infty} \frac{(-t)^p}{p!}(-n-\alpha-k)_p \sum_{k=0}^{\infty} \frac{(-\frac{1}{wt})^k}{k!}(-\beta-n)_k$$
$$\times F_n(\alpha + k - p, \beta - k + p; x).$$

References

[1] *Patil, K.R. and Thakare, N.K.*: Operational formulae and generating function in the united form for the classical orthogonal polynomials. Mathematics Student *46* (1977), no. 1, 41–51.

[2] *Shrivastava, P.N. and Dhillon, S.S.*: Lie operator and classical orthogonal polynomials. Pure Math. Manuscript *7* (1988), 129–136.

[3] *Chongdar, A.K. and Pan, S.K.*: Quelques fonctions génératrices de certains polynômes orthogonaux du point de vue du groupe Lie. Communicated.

[4] *Weisner, L.*: Group theoretic origin of certain generating functions. Pacific J. Math. *5* (1955), 1033–1039.

[5] *McBride, E.B.*: Obtaining Generating Functions. Springer Verlag, Berlin, 1972.

Authors' addresses: A.K. Chongdar, Department of Mathematics, Bangabasi Evening College, 19, R. K. Chakraborty Sarani, Calcutta – 700009, India; N. K. Majumdar, Department of Mathematics, Bagnan College, P.O. – Bagnan, Dist. – Howrah, India.