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ON THE NON-INVARIANCE OF SPAN AND IMMERSION CO-DIMENSION FOR MANIFOLDS

Diarmuid J. Crowley and Peter D. Zvengrowski

Abstract. In this note we give examples in every dimension $m \geq 9$ of piecewise linearly homeomorphic, closed, connected, smooth m-manifolds which admit two smoothness structures with differing spans, stable spans, and immersion co-dimensions. In dimension 15 the examples include the total spaces of certain 7-sphere bundles over S^8. The construction of such manifolds is based on the topological variance of the second Pontrjagin class: a fact which goes back to Milnor and which was used by Roitberg to give examples of span variation in dimensions $m \geq 18$.

We also show that span does not vary for piecewise linearly homeomorphic smooth manifolds in dimensions less than or equal to 8, or under connected sum with a smooth homotopy sphere in any dimension. Finally, we use results of Morita to show that in all dimensions $m \geq 19$ there are topological manifolds admitting two piecewise linear structures having different PL-spans.

1. Introduction

We shall use the notation M for a closed, connected, topological manifold, $M_A, M_B, \dots$ for M together with a given piecewise linear (henceforth PL) structure, and $M_\alpha, M_\beta, \dots$ for M together with a given smoothness structure. Recall that for a smooth m-dimensional manifold M_α, two basic and classical geometric invariants are its span and its immersion co-dimension. The span is the maximal number r such that M_α admits r pointwise linearly independent vector fields, while the immersion co-dimension is the least k such that M_α immerses in $\mathbb{R}^{m+k}$. Clearly $0 \leq r \leq m$, and from the Whitney Immersion Theorem (together with the fact that a closed m-manifold cannot immerse in dimension m), one has $1 \leq k \leq m-1$. A fundamental question is whether these two invariants can differ for distinct smooth structures, M_α and M_β, on the same PL-manifold M_A. An affirmative answer was first given by Roitberg [22] in 1969, in all dimensions $m \geq 18$. In this paper we use smoothing theory to settle this question in all dimensions: we give an affirmative answer for dimensions $m \geq 9$ and show that span and immersion co-dimension are PL invariants in dimensions less than or equal to 8.

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Let us first fix some definitions and notation. For a vector bundle ξ over a space X, we define

$$\operatorname{span}(\xi) := \max\{r : \xi \approx r\varepsilon \oplus \eta\}$$

where $\approx$ denotes isomorphism of vector bundles, $r\varepsilon$ denotes the trivial bundle of rank r and η is some other vector bundle over X. This is the same as the maximal number of pointwise linearly independent sections of ξ, and if ξ is of rank m, then clearly $0 \leq \operatorname{span}(\xi) \leq m$. We also write $m-\operatorname{span}(\xi) = \operatorname{gd}(\xi)$, the geometric dimension of ξ, and this clearly equals $\operatorname{rank}(\eta)$. Replacing isomorphism $\approx$ by stable isomorphism $\sim$ in the above definitions gives the corresponding notions of stable span and stable geometric dimension, written respectively span^0, gd^0. Writing ξ^0 for the stable vector bundle represented by ξ we also define $\operatorname{span}(\xi^0) := \operatorname{span}^0(\xi)$ and similarly for geometric dimension. Evidently

$$0 \leq \operatorname{span}(\xi) \leq \operatorname{span}^0(\xi) = \operatorname{span}(\xi^0) \leq m\,, \quad m \geq \operatorname{gd}(\xi) \geq \operatorname{gd}^0(\xi) = \operatorname{gd}(\xi^0) \geq 0\,.$$

We remark that in the literature "geometric dimension" is often used to denote what we are calling "stable geometric dimension". Let M_α be a smooth m-dimensional manifold with underlying topological manifold M. With the above definitions, the span (resp. stable span) of M_α is simply the span (resp. stable span) of its tangent bundle $\tau_\alpha = \tau(M_\alpha)$, i.e.

$$\operatorname{span}(M_\alpha) := \operatorname{span}(\tau_\alpha)\,, \quad \operatorname{span}^0(M_\alpha) := \operatorname{span}^0(\tau_\alpha)\,.$$

The manifold M is also a CW-complex of dimension $m = \operatorname{rank}(\tau)$, it is then useful to note that by standard stability properties of vector bundles (cf. [8, Ch. 9]), $\operatorname{span}^0(M_\alpha) = \max\{r : \tau_\alpha \oplus \varepsilon \approx (r+1)\varepsilon \oplus \eta\}$. The notation $M^{(k)}$ will be used, as usual, to denote the k-skeleton of M.

Turning to the normal bundle $\nu^0_\alpha = \nu^0(M_\alpha)$ (which is a stable bundle), the Hirsch immersion theorem states that the immersion co-dimension k of M_α is given by the formula $k = \max\{1, \operatorname{gd}(\nu^0_\alpha)\}$. The stable isomorphism $\tau^0_\alpha \oplus \nu^0_\alpha \sim 0$ suggests a possible relation between the stable span and the immersion co-dimension. For interesting inequalities relating these with the Lyusternik-Schnirel'man category of M we refer the reader to Korbaš and Szűcs, [12].

Now let M_A be the PL-manifold underlying M_α and let $\mathcal{C}(M_A)$ denote the finite set of concordance classes of smooth structures on M_A (see Section 2). We define the *smooth span variation* of M_A to be to be the maximal difference of spans over all the smooth structures on M_A and similarly define the *smooth stable span variation of* M_A:

$$\mathrm{ssv(M_A)} := \max\{\operatorname{span}(M_\alpha) \mid [M_\alpha] \in \mathcal{C}(M_A)\} - \min\left\{\operatorname{span}(M_\alpha) \mid [M_\alpha] \in \mathcal{C}(M_A))\right\},$$

$$\mathrm{ss^0v}(M_A) := \max\{\operatorname{span}^0(M_\alpha) \mid [M_\alpha] \in \mathcal{C}(M_A)\} - \min\{\operatorname{span}^0(M_\alpha) \mid [M_\alpha] \in \mathcal{C}(M_A)\}\,.$$

Evidently $\mathrm{ssv}(M_A)$ and $\mathrm{ss^0v(M_A)}$ are invariants of the PL-homeomorphism type of M_A. We also note that both span variations can be defined to give topological invariants of M by replacing $\mathcal{C}(M_A)$ with $\mathcal{C}(M)$, the finite set of concordance classes of smooth structures on M: we write $\mathrm{ssv}(M)$ and $\mathrm{ss^0v}(M)$. Of course $\mathrm{ssv}(M) \geq \mathrm{ssv}(M_A)$ and $\mathrm{ss^0v}(M) \geq \mathrm{ss^0v}(M_A)$. As an example, if M is a manifold with non-zero Euler characteristic (whence $\dim(M)$ is necessarily even), then the tangent bundle of every smooth structure on M admits no nowhere zero sections so $\mathrm{ssv}(M) = \mathrm{ssv}(M_A) = 0$. If also the Euler characteristic of M is odd then by [13, Theorem 2.2] we even have that $\mathrm{ss^0v}(M) = \mathrm{ss^0v}(M) = 0$.

We mention one of the reasons why span variation is surprising: by definition the span of a smooth manifold M_α depends upon its tangent bundle τ_α and a result of Atiyah [1] says that the stable spherical fibration associated to the tangent bundle of a smooth manifold is in fact a homotopy invariant. This was later strengthened by Dupont [6], and by Benlian-Wagoner [2], so that the word “stable” may be omitted. Thus the examples of Theorem 1.1 below and of Roitberg entail span variation amongst vector bundles in the kernel of the J-homomorphism.

We now state our main theorems for span, where we use $\sharp$ to denote the connected sum of locally oriented, smooth manifolds and S^m_0 to denote the standard smooth m-sphere. Analogous results hold for immersion co-dimension.

Theorem 1.1. *In every dimension $m \geq 9$ there are PL-manifolds M_A for which* $\mathrm{ssv}(M_A) \geq 4$ *and* $\mathrm{ss^0v}(M_A) \geq 4$.

Theorem 1.2.

(a) *Let M be a topological manifold with* $\dim(M) \leq 8$ *which admits a PL-structure M_A. Then* $\mathrm{ssv}(M_A) = \mathrm{ss^0v}(M_A) = 0$. *If also $H^3(M;\mathbb{Z}/2) = 0$ then* $\mathrm{ssv}(M) = \mathrm{ss^0v}(M) = 0$.

(b) *For every oriented homotopy sphere S^m_σ, and every locally oriented smooth manifold M_α,* $\mathrm{span}(M_\alpha) = \mathrm{span}(M_\alpha \# S^m_\sigma)$. *In particular for every homotopy sphere* $\mathrm{span}(S^m_\sigma) = \mathrm{span}(S^m_0)$.

Remark 1.3. All of the manifolds we find for Theorem 1.1 admit a smooth structure M_α which is parallelisable and another smooth structure M_β with non-vanishing second Pontrjagin class, $p_2(M_\beta) \neq 0$. This explains the 4, since $p_2(\xi) = 0$ for any vector bundle with stable geometric dimension less than 4. It was also stated in [19] that the second Pontrjagin class is not a topological invariant for closed manifolds, and a recent proof appears in [15].

One can also define the span and stable span of CAT-manifolds for $CAT = PL$ or Top as well as for smooth manifolds where $CAT = O$ (see [25] for the topological case and also [21]). Let $CAT(k)$ be the group of CAT-isomorphisms of $\mathbb{R}^k$ fixing zero. An m-dimensional CAT manifold $M_{\mathcal{A}}$ has a CAT-tangent bundle $\tau(M_{\mathcal{A}})$ and a stable CAT-bundle $\tau^0(M_{\mathcal{A}})$. The span of $M_{\mathcal{A}}$ equals j if the principal $CAT(m)$-bundle associated to $\tau(M_{\mathcal{A}})$ has a $CAT(m-j)$ reduction but no $CAT(m-j-1)$-reduction. The stable span of $M_{\mathcal{A}}$ is j if the same is true of the principal CAT-bundle associated to $\tau^0(M_{\mathcal{A}})$. Analogously to the case of smooth span variations, we obtain the PL-span variations of a topological manifold M by setting $\mathcal{C}_{PL}(M)$ to be the finite set of concordance classes of PL-structures on M

and defining

$$\begin{aligned}&\mathrm{plsv}(M) :=\\&\qquad\max\{\mathrm{span}(M_C) \mid [M_C] \in \mathcal{C}_{PL}(M)\} - \min\{\mathrm{span}(M_C) \mid [M_C] \in \mathcal{C}_{PL}(M)\}\,,\end{aligned}$$

$$\begin{aligned}&\mathrm{pls}^0\mathrm{v}(M) :=\\&\qquad\max\{\mathrm{span}^0(M_C) \mid [M_C] \in \mathcal{C}_{PL}(M)\} - \min\{\mathrm{span}^0(M_C) \mid [M_C] \in \mathcal{C}_{PL}(M)\}\,.\end{aligned}$$

In [18] Morita discovered topological manifolds M in each dimension $m \geq 22$ which admit PL structures M_A and M_B which cannot both be smoothed. It is a relatively simple matter to combine Morita's resuls with a theorem of Wall [26] to prove

Theorem 1.4. *In all dimensions $m \geq 19$ there are topological manifolds M such that* $\mathrm{plsv}(M) > 0$ *and* $\mathrm{pls}^0\mathrm{v}(M) > 0$.

The remainder of the paper is organised as follows. In Section 2 we review the smoothing theory we need and prove Theorem 1.2. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.4. We now conclude the introduction with a list of open problems concerning span variation.

Problem 1.5 (Problems about span variation and span)**.** Let M be a closed topological manifold. We state these problems for $\mathrm{ssv}(M)$ and $\mathrm{plsv}(M)$ for brevity but the analogous problems are open and interesting for $\mathrm{ss}^0\mathrm{v}(M)$ and $\mathrm{pls}^0\mathrm{v}(M)$, as well as for immersion co-dimension.

(1) Relate $\mathrm{ssv}(M)$ to other topological invariants of M.
(2) For a dimension m, determine the largest $\mathrm{ssv}(M)$ for an m-dimensional manifold.
(3) If possible, find families of manifolds M_i such that $\lim_{i\to\infty}\mathrm{ssv}(M_i) = \infty$.
(4) Find a manifold M where the spherical fibration associated to $\tau(M)$ is non-trivial and $\mathrm{ssv}(M) > 0$.
(5) Determine the dimensions m for which $\mathrm{plsv}(M^m) = 0$ is always zero. This relates to the next problem.
(6) Determine whether the assumption that $H^3(M;\mathbb{Z}/2) = 0$ can be removed from the second part of Theorem 1.2 (a).
(7) Compute $\mathrm{ssv}(M)$ for well known manifolds. In particular, for the total spaces of 7-bundles over S^8. This relates to the next problem.
(8) Determine the span of stably parallelisable topological 15-manifolds. (Bredon and Kosinski calculated the span of stably parallelisable smooth manifolds in [3]. In [25] Varadarajan extended their result to stably parallelisable topological manifolds except in dimension 15.)

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2. A rapid review of smoothing theory

Recall the notation established in the introduction: M_α is a closed, connected smooth manifold with underlying PL-manifold M_A and underlying topological manifold M. In this section we review the implications of Cairns-Hirsch smoothing theory for the question of whether the smooth span of M_α depends upon the choice of smooth structure α. We use [16] as our reference for smoothing theory and for further details relating to this brief review.

A concordance between smooth structures M_α and M_β is a smooth structure on $M_A \times [0,1]$, compatibile with the PL structure of $M_A \times [0,1]$, which restricts to M_α on $M_A \times \{0\}$ and to M_β on $M_A \times \{1\}$. The set of concordance classes of smooth structures on M_A is denoted by $\mathcal{C}(M_A)$, and $[M_\alpha] \in \mathcal{C}(M_A)$ will denote the equivalence class of M_α, i.e. the set of all M_β refining M_A that are concordant to M_α. We are interested in the difference a choice of smooth structure can make to the smooth tangent bundle considered as an abstract vector bundle up to isomorphism. Notice that if M_α and M_β are concordant, then their tangent bundles are stably equivalent. The following lemma implies that this remains true unstably.

Lemma 2.1. *Let M_α and M_β be smooth structures on the topological manifold M. Then $\tau(M_\alpha) \sim \tau(M_\beta)$ if and only if $\tau(M_\alpha) \approx \tau(M_\beta)$.*

Proof. One implication is trivial, so let $\tau(M_\alpha)$ and $\tau(M_\beta)$ be classified by $f_\alpha \colon M \to BO(m)$ and $f_\beta \colon M \to BO(m)$, and suppose these bundles are stably equivalent. Then they agree over $M^{(m-1)}$. Now let $O_{\alpha,\beta} \in H^m(M;K)$ be the obstruction to a homotopy $f_\alpha \simeq f_\beta$, where $K = \mathrm{Ker}\big(\pi_{m-1}(O(m)) \to \pi_{m-1}(O)\big) \cong 0,\ \mathbb{Z}/2,\ \mathbb{Z}$, corresponding to $m \in \{1,3,7\}$, or m odd and $m \notin \{1,3,7\}$, or m even, respectively. We now show this obstruction vanishes in turn for the cases: m is odd, m is even with M orientable, and m is even with M non-orientable.

If $m = 2r+1$ is odd, it follows from [9] that there are either one or two isomorphism classes of rank m vector bundles over M, stably equivalent to $\tau(M_\alpha)$, this number being called the James-Thomas number. If the James-Thomas number is one then automatically $\tau(M_\alpha) \approx \tau(M_\beta)$. On the other hand, if this number is two, then the two isomorphism classes are distinguished by the Browder-Dupont invariant b_B, cf. [24]. But according to [24], $b_B\big(\tau(M_\alpha)\big)$ and $b_B\big(\tau(M_\beta)\big)$ must both equal the mod-2 Kervaire semi-characteristic $\chi_2(M) := \Sigma_{i=0}^{r}\mathrm{rank}\big(H^i(M;\mathbb{Z}/2)\big) \pmod 2$, so $O_{\alpha,\beta} = 0$.

If m is even and M is orientable then $O_{\alpha,\beta}$ lies in $H^m(M;\mathbb{Z})$, where the coefficients are untwisted. In this case $O_{\alpha,\beta}$ measures the difference in the Euler classes of the bundles $\tau(M_\alpha)$ and $\tau(M_\beta)$, but these are both determined by the Euler characteristic of M and hence the same. Thus $O_{\alpha,\beta}$ vanishes.

If m is even and non-orientable let $\omega \colon \pi_1(M) \twoheadrightarrow \mathbb{Z}/2 = \{1,-1\}$ be the first Stiefel-Whitney class. In this case $O_{\alpha,\beta} \in H^m(M;\widetilde{\mathbb{Z}})$ where the coefficients are twisted and $\widetilde{\mathbb{Z}}$ denotes the $\mathbb{Z}[\pi_1(M)]$-module with $g \in \pi_1(M)$ acting via multiplication by $\omega(g)$. By twisted Poincaré duality (see, for example, [5, §5]), $H^m(M;\widetilde{\mathbb{Z}}) \cong H_0(M;\mathbb{Z}) \cong \mathbb{Z}$. Now let $p \colon \widetilde{M} \twoheadrightarrow M$ denote the orientation double cover of M and $\widetilde{M}_{\widetilde{\alpha}}$, $\widetilde{M}_{\widetilde{\beta}}$ the corresponding smooth structures on $\widetilde{M}$ induced via p. Of course the classifying map for $\tau(\widetilde{M}_{\widetilde{\alpha}})$ is $f_\alpha \circ p$ and similarly for the classifying map of $\tau(\widetilde{M}_{\widetilde{\beta}})$. We write $O_{\widetilde{\alpha},\widetilde{\beta}}$ for the obstruction to a homotopy of the classifying map for $\tau(\widetilde{M}_{\widetilde{\alpha}})$ to that of $\tau(\widetilde{M}_{\widetilde{\beta}})$, which is zero by the oriented case. The covering map p induces $p^* : H^m(M;\widetilde{\mathbb{Z}}) \to H^m(\widetilde{M};\mathbb{Z})$ where the latter coefficients are untwisted and we have that $p^*(O_{\alpha,\beta}) = O_{\widetilde{\alpha},\widetilde{\beta}}$. Since p^* is induced by a double covering it is isomorphic to $\times 2\colon \mathbb{Z} \to \mathbb{Z}$ and we conclude that $O_{\alpha,\beta} = 0$. □

Let us now define the following sets of isomorphism classes of vector bundles and stable vector bundles:

$$Tv(M_A) := \big\{[\tau(M_\alpha)] \mid [M_\alpha] \in \mathcal{C}(M_A)\big\}$$

and

$$T^0v(M_A) := \big\{[\tau^0(M_\alpha)] \mid [M_\alpha] \in \mathcal{C}(M_A)\big\}\,.$$

Observe that Lemma 2.1 shows that there is a bijection $T^0v(M_A) \equiv Tv(M_A)$. We first show that $Tv(M_A)$ is a singleton in dimensions $m \leq 4$.

Lemma 2.2. *Let $h\colon M_\alpha \to N_\beta$ be a homotopy equivalence between smooth m-manifolds with $m \leq 4$. Then h preserves the tangent bundles; i.e. $h^*\big(\tau(N_\beta)\big) \approx \tau(M_\alpha)$.*

Proof. By Lemma 2.1 it is enough to show that $h^*\big(\tau^0(N_\beta)\big) \sim \tau^0(M_\alpha)$. Let $f_\alpha\colon M \to BO$ and $g_\beta\colon N \to BO$ classify the stable tangent bundles of M_α and N_β, let $p\colon BO \to BG$ be the canonical fibration, and let $i\colon G/O \to BO$ be the inclusion of a fibre. By [1], h preserves the stable spherical fibrations underlying $\tau^0(M_\alpha)$ and $\tau^0(N_\beta)$ and so $p \circ f_\alpha$ is homotopic to $p \circ g_\beta \circ h$. As p is an isomorphism on π_1 and π_2 and as $\pi_3(BO) = 0$, f_α and $g_\beta \circ h$ agree on $M^{(3)}$. Hence the lemma holds in dimensions $m \leq 3$.

Now assume that $\dim(M) = 4$. There is a cohomology class $O_{\alpha,\beta} \in H^4\big(M;\pi_4(BO)\big)$ which is the obstruction to a homotopy from f_α to $g_\beta \circ h$. The coefficients are untwisted since $\pi_1(BO)$ acts trivially on $\pi_4(BO)$. Moreover we see that $O_{\alpha,\beta}$ lies in the image of the map from $H^4\big(M;\pi_4(G/O)\big)$. If M is not orientable then $H^4\big(M;\pi_4(G/O)\big)$ and $H^4\big(M;\pi_4(BO)\big)$ are both isomorphic to $\mathbb{Z}/2$ but the map $\pi_4(G/O) \to \pi_4(BO)$ is multiplication by 24, and since $O_{\alpha,\beta}$ lifts to $H^4\big(M;\pi_4(G/O)\big)$ it must vanish. If M and N are orientable then orient them so that h is orientation preserving and repeat the above argument replacing BO and BG respectively by BSO and BSG, and using the classifying maps of the oriented tangent bundles. The class $O_{\alpha,\beta}$ is now detected by the difference of the Pontrjagin classes $p_1\big(\tau^0(M_\alpha)\big) - h^*\big(p_1(\tau^0(N_\beta))\big)$ but by the signature theorem these classes agree since h is an orientation preserving homotopy equivalence from M to N. Hence $\tau^0(M_\alpha)$ and $h^*\big(\tau^0(M_\beta)\big)$ may be oriented so that they become isomorphic oriented stable vector bundles and so, in particular, they are isomorphic. □

We now recall how smoothing theory calculates $T^0v(M_A)$ and hence $Tv(M_A)$ in dimensions $m \geq 5$. Fixing a smooth structure, M_α, makes $\mathcal{C}(M_A)$ into a pointed set denoted $\mathcal{C}(M_\alpha)$. A fundamental result of smoothing theory is the following

Theorem 2.3 (Cairns-Hirsch, see [16, Theorem 7.2])**.** *Let M_α be a smooth manifold of dimension at least* 5*, then there is a bijection*

$$\Psi_\alpha \colon \mathcal{C}(M_A) \equiv [M, PL/O]$$

which takes the base point $[M_\alpha]$ to the homotopy class of the constant map.

Recall that PL/O has a commutative H-space structure which makes the fibration $PL/O \to BO \to BPL$ into a sequence of H-space maps where BO and BPL have compatible commutative H-space structures coming from the Whitney sum of bundles [16][p 92]. Associated to this fibration we have the long exact Puppe sequence of abelian groups, for any space X,

$$\ldots \longrightarrow [X, PL] \longrightarrow [X, PL/O] \xrightarrow{\partial_X} [X, BO] \longrightarrow [X, BPL].$$

When $X = M$ is homeomorphic to a smooth manifold M_α, ∂_M computes the difference a smooth structure makes to the isomorphism class of the stable tangent bundle. That is, for the appropriate choice of Ψ_α,

$$\partial_M\big(\Psi_\alpha(M_\beta)\big) = [\tau^0(M_\alpha)] - [\tau^0(M_\beta)] \in \widetilde{KO}(M) = [M, BO]\,.$$

Combining Lemma 2.2, the fact that PL/O is 6-connected and the above identity we deduce

Lemma 2.4. *The group* $\mathrm{Im}(\partial_M)$ *acts freely and transitively on $T^0v(M_A)$.*

Applying Lemma 2.1 we immediately obtain

Corollary 2.5. If $\partial_M = 0$ then $Tv(M_A)$ and $T^0v(M_A)$ are singletons and so $\mathrm{ssv}(M_A) = \mathrm{ss}^0\mathrm{v}(M_A) = 0$.

Proof of Theorem 1.2. Lemma 2.2 implies both parts in dimensions $m \leq 4$. So we now assume that $m \geq 5$ and start with part (b). If $M = S^m$, then it is known [**?**] that $\pi_m(PL) \to \pi_m(PL/O)$ is surjective and so $\partial_{S^m} = 0$. It follows that every exotic sphere gives rise to the same tangent bundle as the usual one (a fact already observed in [20]). Now for any smooth locally oriented manifold M_α and any homotopy m-sphere S^m_σ we have $M_{\alpha+\sigma} := M_\alpha \sharp S^m_\sigma$. Using smoothing theory we identify the smooth structure $\alpha + \sigma$ as follows. Identify $\mathcal{C}(S^m) = \pi_m(PL/O)$ using the standard smooth structure S^m_0 on the sphere so that $\sigma \in \pi_m(PL/O)$ corresponds to the exotic sphere S^m_σ under the bijection Ψ_0, and let $c : M \to S^m$ be the collapse map taking an open m-disc in M homeomorphically onto $S^m \setminus \{\mathrm{pt}\}$ and all points outside the open m-disc to pt. By definition we have that $\Psi_\alpha^{-1}(c^*\sigma) = M_{\alpha+\sigma}$. Now the induced maps $c^* : \pi_m(PL/O) \to [M, PL/O]$ and $c^* : \pi_m(BO) \to [M, BO]$ give rise to the following commutative diagram:

$$\begin{tikzcd}
\pi_m(PL/O) \arrow[r, "\partial_{S^m}"] \arrow[d, "c^*"] & \pi_m(BO) \arrow[d, "c^*"] \\
{[M, PL/O]} \arrow[r, "\partial_M"] & {[M, BO]}\,.
\end{tikzcd}$$

It follows that

$$\partial_M\big(\Psi_\alpha(M_{\alpha+\sigma})\big) = \partial_M\big(c^*(\sigma)\big) = c^*\big(\partial_{S^m}(\sigma)\big) = c^*(0) = 0\,.$$

Thus $\tau^0(M_\alpha) \sim \tau^0(M_{\alpha+\sigma})$. By Lemma 2.1 we have that $\tau(M_\alpha) \approx \tau(M_{\alpha+\sigma})$ and so $\mathrm{span}(M_\alpha) = \mathrm{span}(M_{\alpha+\sigma})$. This concludes the proof of part (*b*).

We now prove part (*a*). For the PL-statement, since $m \geq 5$ we apply Theorem 2.3. As PL/O is 6-connected, if M_A is 5 or 6 dimensional then M_A admits a unique smooth structure. If M_A is of dimension 7 then Theorem 2.3 implies that all smooth structures are obtained from a fixed one by connected sum with a homotopy 7-sphere and so by part (b) don't alter the span. If M is 8-dimensional it suffices, by Corollary 2.5, to show that $\partial_M = 0$. As usual, let M be the topological manifold underlying M_A and let $M^{(6)}$ be the 6-skeleton of a CW-decomposition for M containing just one 8-cell. Such a decomposition exists by [27]. As PL/O is 6-connected, $[M/M^{(6)}, PL/O] \twoheadrightarrow [M, BO]$ is surjective and thus the image of ∂_M lies in $\mathrm{Im}([M/M^{(6)}, BO] \to [M, BO])$. If M is orientable then $M/M^{(6)} \simeq (\vee S^7) \vee S^8$ is homotopy equivalent to a wedge of 7-spheres and an 8-sphere, then ∂_M splits as the sum of ∂_{S^7}'s and ∂_{S^8} but these are zero. If M is not orientable then $M/M^{(6)} \simeq M(\mathbb{Z}/2, 7) \vee (\vee S^7)$ is homotopy equivalent to a degree 7 Moore space wedged with a wedge of 7-spheres. Since the short exact sequence $\pi_7(O) \to \pi_7(PL) \to \pi_7(PL/O)$ (see Section 2) splits at the prime 2 it again follows that $\partial_M = 0$.

It remains to prove that $\mathrm{ssv}(M) = 0$ if $H^3(M; \mathbb{Z}/2) = 0$, in dimensions $5 \leq m \leq 8$. In dimensions $m \geq 5$ there is a smoothing theory for PL-structures on topological manifolds which is analogous to the smoothing theory for smooth structures on PL-manifolds we sketched above. In particular the set of concordance classes of PL-structures on M, $\mathcal{C}_{PL}(M)$, corresponds bijectively with $[M, TOP/PL]$. Moreover, the fundamental work of [11] shows that TOP/PL is homotopy equivalent to the Eilenberg-MacLane space $K(\mathbb{Z}/2, 3)$. Hence the assumption that $H^3(M; \mathbb{Z}/2) = 0$ ensures that there is a unique concordance class $[M_A]$ of PL structures on M. Thus the span variations for M and the span variations for M_A are zero by the PL case. □

We remark that our proof in fact shows

Corollary 2.6. Let M_A be a PL-manifold of dimension $m \leq 8$. Then $|Tv(M_A)| = 1$.

Turning our attention now to higher dimensions, if there is a PL-manifold M_A with $\partial_M \neq 0$ and which admits a parallelisable smooth structure M_α, i.e. $\tau(M_\alpha) \approx m\varepsilon$, then there will be a smooth structure M_β such that $\tau^0(M_\beta)$ is non-trivial and so $\mathrm{span}(M_\beta) \leq \mathrm{span}^0(M_\beta) < m$. However, $\mathrm{span}(M_\alpha) = \mathrm{span}^0(M_\alpha) = m$, so in such a case both $\mathrm{ssv}(M_A) > 0$ and $\mathrm{ss^0v}(M_A) > 0$. In the next section we produce examples of this sort.

3. PL-Manifolds with varying smooth spans

In this section we give examples of PL-manifolds M_A in dimensions 9 and higher with $\mathrm{ssv}(M_A) \geq 4$ and $\mathrm{ss^0v}(M_A) \geq 4$. Let $M(C_k, 1) = S^1 \cup_k e^2$ be the degree 1 Moore space with first homology group cyclic of order k. As $M(C_k, 1)$ is a 2-dimensional complex it can be embedded into $\mathbb{R}^5$; we take an embedding into $\mathbb{R}^{10}$ and then take a regular neighbourhood of $M(C_k, 1)$, $T^{10}_\alpha(k)$, which is a compact, smooth, parallelisable 10-manifold with boundary. Here α is the induced smoothness structure coming from the standard one on $\mathbb{R}^{10}$. Let $N^9_\alpha(k)$ be the boundary of $T^{10}_\alpha(k)$. We see that $N^9_\alpha(k)$ is a closed, connected, smooth stably parallelisable 9-manifold and we write $N^9_A(k)$ for the underlying PL-manifold.

Before starting the next theorem, we recall (following [3]) the definitions of the semi-characteristic $\chi^*(M)$ and the reduced semi-characteristic $\widehat{\chi}(M)$ of a manifold M. If $\dim(M)$ is even then $\chi^*(M)$ is the half-integer $\chi(M)/2$ where $\chi(M)$ is as usual the Euler characteristic of M. If $\dim(M)$ is odd then $\chi^*(M) \in \mathbb{Z}/2$ is equal to $\chi_2(M)$, the mod-2 Kervaire semi-characteristic (defined in the proof of Lemma 2.1). The reduced semi-characteristic is defined to be $\widehat{\chi}(M) = 1 - \chi^*(M)$ and satisfies $\widehat{\chi}(M_0 \sharp M_1) = \widehat{\chi}(M_0) + \widehat{\chi}(M_1)$. For example: $\widehat{\chi}(S^1 \times S^m) = 1$ if $m \geq 1$ and $\widehat{\chi}\big(N^9_\alpha(k)\big) = 0$. We also orient the manifolds $N^9_A(k)$ and use the notation $M\#_j T = M\#T\# \underset{j}{\cdots} \#T$ for the connected sum of M with j copies of an oriented manifold T, for any choice of $CAT = O, PL, Top$.

Theorem 3.1.

(1) *Let $n \geq 0$ and W^n_B be any closed, oriented PL-n-manifold admitting a stably parallelisable smooth structure. Assume that 7 divides k and set $l = \chi^*\big(N^9_A(k) \times W^n_B\big)$. Then for all $j \geq 0$*

$$\mathrm{ss}^0\mathrm{v}\big((N^9_A(k) \times W^n_B)\sharp_j(S^1 \times S^{n+8})\big) \geq 4 \quad \text{and} \quad \mathrm{ssv}\big((N^9_A(k) \times W^n_B)\sharp_l(S^1 \times S^{n+8})\big) \geq 4\,,$$

where we regard $S^1 \times S^{n+8}$ as a PL manifold.

(2) *Let ξ be a linear 7-sphere bundle over S^8 and let P^{15}_A be the PL-manifold underlying the total space of ξ. If the total space of ξ is stably parallelisable and 14 divides the Euler class of ξ, $e(\xi) \in H^8(S^8; \mathbb{Z}) \cong \mathbb{Z}$, then $\mathrm{ssv}(P^{15}_A) \geq 4$ and $\mathrm{ss}^0\mathrm{v}(P^{15}_A) \geq 4$.*

Remark 3.2. Of course in part (1) above one may take W^0_B to be a point, and $W^n_B = S^n$, $n > 0$. Furthermore, $l \in \mathbb{Z}$ because $\mathrm{span}^0(N^9_A(k) \times W^n_B) = 9 + n > 0$ implies $\chi(N^9_A(k) \times W^n_B)$ is even. The idea of taking neighbourhoods of appropriate Moore spaces to find examples of homeomorphic smooth manifolds with differing tangent bundles goes back to Milnor [17]. Roitberg [22] doubled compact neighbourhoods of Moore spaces of degree at least 7 to exhibit smooth span variation for closed manifolds in dimensions 18 and higher. We are able to get examples down to dimension 9 by using a degree 1 Moore space so that a "dual" Moore space appears in dimension 7. In (2), note that $E(\xi)$ has a standard smoothness structure because it is a linear 7-sphere bundle.

Remark 3.3. Total spaces as in Theorem 3.1 (2) exist: in the notation of [23, §2] take any 7-sphere bundle $\xi_{h,j} \in \pi_7\big(SO(8)\big) \cong \mathbb{Z} \oplus \mathbb{Z}$ with $(h, j) = (7k, 7k)$ and $k \neq 0$. By [23] the corresponding total spaces are almost parallelisable and hence stably parallelisable since $\pi_{14}(O) = 0$ (or cf. [14, Ch. 9 (8.5)]). We do not resolve whether the non-stably parallelisable smooth structures in this case are also realised as the total spaces of 7-sphere bundles over S^8.

Proof of Theorem 3.1. Let M_A^m be any manifold satisfying the hypotheses of the theorem. By assumption M_A admits a stably parallelisable smooth structure M_α, so $\mathrm{span}^0(M_\alpha) = m$. If, in addition, the semi-characteristic $\chi^*(M)$ vanishes then [3] asserts that $\mathrm{span}(M_\alpha) = m$ and it is a simple matter (using the addition formula for the reduced semicharacteristic $\widehat{\chi}$ under connected sums, as well as $\widehat{\chi}(S^1 \times S^{n+8}) = 1$) to check that the additional hypotheses in the theorem ensure that the semi-characteristic vanishes. We will show that each M_A admits a smooth structure M_β with non-zero second Pontrjagin class, $p_2(M_\beta) \neq 0$. The theorem then follows since any smooth m-manifold with stable span greater than $m-4$ has vanishing second Pontrjagin class, which shows

$$\mathrm{span}(M_\beta) \leq \mathrm{span}^0(M_\beta) \leq m-4\,.$$

It remains to show the existence of a smooth structure β with $p_2(M_\beta) \neq 0$. We may therefore specialize to the case where M_A^m is one of $N_A^9(k)$ or P_A^{15} using the product formula for the Pontrjagin classes of the manifolds in Theorem 3.1 (1). First recall [4, 7] that the homotopy exact sequence

$$0 \to \pi_7(O) \longrightarrow \pi_7(PL) \longrightarrow \pi_7(PL/O) \to 0$$

is isomorphic to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(7,1)} \mathbb{Z} \oplus \mathbb{Z}/4 \xrightarrow{\binom{-1}{7}} \mathbb{Z}/28 \longrightarrow 0\,.$$

We denote the Bockstein homomorphism associated to the first short exact sequence by Bk. We shall relate Bk to $\partial_M : [M, PL/O] \to [M, BO]$.

Since M_α is stably parallelisable and PL/O is 6-connected it follows for any smooth structure, M_γ, that $\tau^0(M_\gamma)$ is trivial when restricted to $M^{(6)}$. Further, since $\pi_7(BO) = 0$, we can extend this statement to $M^{(7)}$. Thus the primary obstruction to the triviality of $\tau^0(M_\gamma)$, $\mathrm{Ob}_O\big(\tau^0(M_\gamma)\big)$, lies in $H^8(M; \pi_7(O))$ and there is a commutative diagram

$$\begin{tikzcd}
{[M, PL/O]} \arrow[r, "\partial_M"] \arrow[d, "\mathrm{Ob}_{PL/O}"] & \mathrm{Im}(\partial_M) \arrow[d, "\mathrm{Ob}_O"] \\
H^7(M; \pi_7(PL/O)) \arrow[r, "\mathrm{Bk}"] & H^8(M; \pi_7(O))
\end{tikzcd}$$

where we have used Ψ_α to identify $\mathcal{C}(M_A) \equiv [M, PL/O]$ and $\mathrm{Ob}_{PL/O} \colon [M, PL/O] \to H^7\big(M; \pi_7(PL/O)\big)$ as the primary obstruction to a null-homotopy. Now for all the M to which we have specialized, $H^8\big(M; \pi_7(O)\big) \cong H^8(M; \mathbb{Z})$ contains a cyclic summand of order 7^a with $a \geq 1$. Let y be a generator for this summand. We claim that there is an element $x \in [M, PL/O]$ such that $\mathrm{Bk} \circ \mathrm{Ob}_{PL/O}(x) = 7^{a-1}y$. Firstly we observe that $\mathrm{Ob}_{PL/O}$ is onto the 7-torsion in $H^7\big(M; \pi_7(PL/O)\big)$ since the Atiyah-Hirzeburch spectral sequence to compute $[M, PL/O]$ gives an exact sequence

$$\cdots \longrightarrow [M, PL/O] \xrightarrow{\mathrm{Ob}_{PL/O}} H^7\big(M; \pi_7(PL/O)\big) \longrightarrow H^m(M; \pi_{m-1}PL/O) \longrightarrow \ldots$$

and $H^m(M;\pi_{m-1}PL/O) \cong \pi_{m-1}(PL/O)$ is prime to 7 ($m = 9$ or 15, and $\pi_8(PL/O) \cong \pi_{14}(PL/O) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$). Secondly, from the coefficient sequence above, we see that when restricted to the summand generated by y, the map $H^8\big(M;\pi_7(O)\big) \to H^8\big(M;\pi_7(PL/O)\big)$ is isomorphic to multiplication by 7. It follows that $7^{a-1}y \neq 0$ lies in the image of Bk and since it is 7-torsion it also lies in the image of $\mathrm{Bk} \circ \mathrm{Ob}_{PL/O}$.

From the claim and the commutativity of the above diagram we have an $x \in [M, PL/O]$ such that $\mathrm{Ob}_O \circ \partial_M(x) = 7^{a-1}y$. Setting $\beta = \Psi_\alpha^{-1}(x)$ we obtain a smooth structure β on M_A with $\mathrm{Ob}_O\big(\tau^0(M_\beta)\big) = 7^{a-1}y$. Finally, Kervaire [10] has shown that $p_2 = 6 \cdot \mathrm{Ob}_O$ for vector bundles which are trivial over $M^{(7)}$ and hence

$$p_2(M_\beta) = 6 \cdot \mathrm{Ob}_O\big(\tau^0(M_\beta)\big) = 6 \cdot 7^{a-1}y \neq 0\,.$$

□

4. Topological manifolds with varying PL spans

In this section we prove Theorem 1.4. We assume that the reader is familiar with the simply connected surgery exact sequences for smooth and PL-manifolds.

In every dimension $m \geq 22$, Morita [18, Theorem 6.1] defines a simply connected topological manifold $M = M^m(K)$ by embedding a 10-skeleton K of $PL/O \simeq K(\mathbb{Z}/2, 3)$ in $\mathbb{R}^m$, $m \geq 22$, taking a regular neighbourhood $T = T^m(K)$ of K and letting M be the trivial double of T: $M = T \cup_{\mathrm{Id}} T$. The manifold M admits two PL structures, M_A and M_B, such that M_A admits a stably parallelisable smooth structure and M_B is not smoothable (we explain this below). We first explain how to find examples of this type in dimensions 19 and higher. We observe that $M^m(K)$ is the boundary $T^m(K) \times [0,1]$ and hence is a closed, stably parallelisable, topological manifold which contains K as a retract. We observe also that these properties along with $K \to M$ being an 8-equivalence are all that is required in Morita's arguments to show that PL-structures A and B exist as above. Now by [26] K embedds into $\mathbb{R}^{19}$. Let $T^{19}(K)$ be a regular neighbourhood of such an embedding and let $M^{19}(K)$ be the boundary of $T^{19}(K) \times [0,1]$. Then $M^{19}(K)$ is a closed, stably parallelisable, topological manifold containing K as an 8-connected retract and hence admits PL structures A and B as above. We first prove the following

Lemma 4.1. *For all the manifolds $M = M^m(K)$, $m \geq 19$, M_A is stably parallelisable and M_B is not smoothable. Hence* $\mathrm{pls}^0\mathrm{v}(M) > 0$.

Proof. Morita's arugments show the following. Consider the PL-structure, in the sense of surgery theory, $f : M_B \to M$, f the identity map. This gives an element $[f]$ in the PL-structure set of M. As M is simply connected, the PL-structure set injects into the normal invariant set and so we obtain an element $[f] \in [M, G/PL]$ (where we use $\mathrm{Id_M} : M_A \to M$ as the base point to identify the normal invariants of M with $[M, G/PL]$). Morita showed that $[f]$ does not belong to the image of the canonical map $q : [M, G/O] \to [M, G/PL]$.

Similarly to Section 2, the map $\delta_M^{PL} : [M, G/PL] \to [M, BPL]$ maps $[f]$ to the difference of the stable PL-tangent bundles $\tau^0(M_A) - \tau^0(M_B) \in \widetilde{KPL}(M) = [M, BPL]$ and a similar statment holds for $\delta_M^O : [M, G/O] \to [M, BO]$ and the smooth normal invariant set. There is a commuting diagram of long exact sequences

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & [M,G] & \longrightarrow & [M,G/O] & \xrightarrow{\delta_M^O} & [M,BO] & \xrightarrow{BJ} & [M,BG] \quad \dots \\
 & & \downarrow = & & \downarrow q & & \downarrow & & \downarrow = \\
\dots & \longrightarrow & [M,G] & \longrightarrow & [M,G/PL] & \xrightarrow{\delta_M^{PL}} & [M,BPL] & \longrightarrow & [M,BG] \quad \dots
\end{array}$$

where BJ denotes the map induced on classifying spaces by the J-homomorphism $J : O \to G$. Suppose that $\tau^0(M_B)$ has a smooth reduction. Since $\tau^0(M_A)$ is trivial this means that $\delta_M([f])$ lifts to $x \in [M,BO]$. As $BJ(x)$ is defined by the stable spherical fibration of M and this is trivial we conclude that $x \in \mathrm{Im}(\delta_M^O)$. Now a simple diagram chase ensures that $y \in [M,G/O]$ can be chosen such that $q(y) = [f]$, contradicting Morita's results. Hence $\tau^0(M_B)$ cannot be smoothed, so it must be non-trivial and $\mathrm{span}^0(M_B) < m$. But $\mathrm{span}^0(M_A) = m$, so $\mathrm{pls}^0\mathrm{v}(M) > 0$. □

Proof of Theorem 1.4. Let $M = M^{19}(K)$ and let M_α be a stably parallelisable smooth structure refining M_A. By the Bredon-Kosinski theorem we know that $\tau(M_\alpha)$ is trivial if and only if $\chi_2(M) = 0$. However, we do not know $\chi_2(M)$ so similarly to Theorem 3.1 we let $N_\alpha = M_\alpha \sharp_l (S^1 \times S^{18})$ where $l = \chi_2(M)$ is 1 or 0. It follows that N_α is stably parallelisable and that $\chi_2(N) = 0$. Thus N_α is parallelisable and so $N_A = M_A \sharp_l (S^1 \times S^{18})$ is too. The manifold N also admits the PL-structure $N_B = M_B \sharp_l (S^1 \times S^{18})$ which is not smoothable. Hence $\mathrm{plsv}(N) > 0$ and $\mathrm{pls}^0\mathrm{v}(N) > 0$. In dimensions $m > 19$ we take $Q = N \times S^n$ for $n > 0$, for then Q admits a PL-structure $Q_A = N_A \times S^n$ which is parallelisable and another PL-structure $Q_B = N_B \times S^n$ which is not smoothable. Hence $\mathrm{plsv}(Q) > 0$ and $\mathrm{pls}^0\mathrm{v}(Q) > 0$. □

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