Matematický časopis 19 (1969), No. 2

COUPLED SYSTEMS OF TYPE T

VÁCLAV HAVEL, Brno

In § 1.3 of [1] there is explained the standart theory of free planar extensions of incidence structures. The incidence structures can be also interpreted as some special „coupled systems" ([1], Proposition 3 on p. 6). We have applied this viewpoint already in [2] for the study of partition properties of incidence structures. It is further possible to extend the mentioned free planar extension theory of incidence structures to the free complete extension theory of general coupled systems. This is indicated in [3]. Note that in general coupled systems the points (lines) are, essentially, the sets of some pairs of lines (points). On the other hand, the free complete extension theory of m-tuple systems with at most n-tuple intersections ($m - 1 > n \geqq 1$) can be built up.

In the present Note we wish to deduce first the properties belonging to the free complete extension theory of „coupled systems of type $\mathbf{T}$" (here the points (lines) are, essentially, the sets of certain lines (points) with some prescribed cardinalities).

Let $\mathbf{T}$ be a fixed class of cardinal numbers. By a *coupled system (of type* $\mathbf{T}$*)* we mean a collection $\mathbf{C} = (S_i, f_i; i = 1,2)$ where S_1, S_2 are nonempty sets and f_i is a map of a set $\operatorname{dom} f_i \subseteq \{X \subseteq S_i | \operatorname{card} X \in T\}$ into S_j; $(i, j) = (1,2), (2,1)$. If especially $\operatorname{dom} f_i = \{X \subseteq S_i | \operatorname{card} X \in T\}$ for $i = 1,2$ then $\mathbf{C}$ is said to be *complete.*

Let $\mathbf{C} = (S_i, f_i; i = 1,2)$, $\mathbf{C}' = (S'_i, f'_i; i = 1,2)$ be coupled systems(1) such that $S_i \subseteq S'_i$ and $f_i = f'_i|_{S_i}$, for $i = 1,2$. Then $\mathbf{C}$ is called a coupled *subsystem* of $\mathbf{C}'$.

Let $\mathbf{C}$, $\mathbf{C}'$ be given coupled systems. By a *surjection* $\sigma : \mathbf{C} \to \mathbf{C}'$ we shall mean a pair (σ_1, σ_2) where $\sigma_i : S_i \to S'_i$ is a surjection for $i = 1,2$. A surjection $\sigma : \mathbf{C} \to \mathbf{C}'$ is called an *epimorphism* if (i) $\{X^{[\sigma]} | X \in \operatorname{dom} f_i, \operatorname{card} X^{[\sigma]} \in \mathbf{T}\} = \operatorname{dom} f'_i$ for $X^{[\sigma]}$ defined as $\{\sigma_i x | x \in X\}$ where $X \in \operatorname{dom} f_i$; $i = 1,2$ and (ii) $f'_i X^{[\sigma]} = \sigma_j(f_i X)$ for all $X \in \operatorname{dom} f_i$ with $\operatorname{card} X^{[\sigma]} \in \mathbf{T}$; $(i, j) = (1,2), (2,1)$. A bijective epimorhism is called an *isomorphism.*(2)

(1) This denotation will be preserved throughout the whole paper.

(2) Note that if σ is a bijective epimorphism then $\sigma^{-1} = (\sigma_1^{-1}, \sigma_2^{-1})$ is also an epimorphism.

Construction 1. Let $\mathbf{C}$ be a coupled subsystem of a complete coupled system $\mathbf{C}'$. We put $\mathbf{C}^\circ = \mathbf{C}$ and suppose that $\mathbf{C}^n$ is already given for some n.(3) Construct $\mathbf{C}^{n+1}$ in such a way that $\operatorname{dom} f_i^{n+1} = \{X \subseteq S_i^n | \text{ card } X \in \mathbf{T}\}$, $S_j^{n+1} = S_j^n \cup T_j^n$, $T_j^n = \{f_i X | X \in \operatorname{dom} f_i^{n+1} \setminus \operatorname{dom} f_i^n\}$ for $(i, j) = (1,2), (2,1)$. By induction we get a sequence $(\mathbf{C}^n)_{n=0}^\infty$. Now there is precisely one minimal coupled system $\mathbf{C}_{\mathbf{C}'}$ containing all $\mathbf{C}^0, \mathbf{C}^1, \mathbf{C}^2, \ldots$ as coupled subsystems.

Construction 2. Let $\mathbf{C}$ be a coupled system. First put $\mathbf{C}^{(0)} = \mathbf{C}$. Secondly suppose that $\mathbf{C}^{(n)}$ is already determined for some n. Then construct $\mathbf{C}^{(n+1)}$ in such a way that $\operatorname{dom} f_i^{n+1} = \{X \subseteq S_i^{(n)} | \text{ card } X \in \mathbf{T}\}$, $S_j^{(n+1)} = S_j^{(n)} \cup T_j^{(n)}$ where $S_j^{(n)} \cap T_j^{(n)} = \emptyset$ with $\text{card } T_j^{(n)} = \text{card } (\operatorname{dom} f_i^{(n+1)} \setminus \operatorname{dom} f_j^{(n)})$ and $f_i^{(n+1)}|_{\operatorname{dom} f_i^{(n)}} = f_i^{(n)}$, $f_i^{(n+1)}|_{\operatorname{dom} f_i^{(n+1)} \setminus \operatorname{dom} f_i^{(n)}} : (\operatorname{dom} f_i^{(n+1)} \setminus \operatorname{dom} f_i^{(n)}) \to T_j^{(n)}$ is a bijection for $(i, j) = (1,2), (2,1)$. Inductively we obtain a sequence $(\mathbf{C}^{(n)})_{n=0}^\infty$. Now there is precisely one minimal coupled system $\overline{\mathbf{C}}$ containing all $\mathbf{C}^{(0)}, \mathbf{C}^{(1)}, \ldots$. $\overline{\mathbf{C}}$ is necessarily complete and is determined up to isomorphisms.

Proposition 1. *Let $\mathbf{C}$ be a coupled system and $\mathbf{C}'$ a coupled system such that $\mathbf{C}_{\mathbf{C}'} = \mathbf{C}'$. Then there is an epimorphism $\varphi : \overline{\mathbf{C}} \to \mathbf{C}$ with $\varphi|_{\mathbf{C}} = \mathrm{id}_{\mathbf{C}}$.*

Proof. Let $\varphi^{(0)} = \mathrm{id}_{\mathbf{C}}$. Suppose further that there is an epimorphism $\varphi^{(n)} : \mathbf{C}^{(n)} \to \mathbf{C}^n$ with $\varphi^{(n)}|_{\mathbf{C}} = \mathrm{id}_{\mathbf{C}}$ for a certain n. Then construct $\varphi^{(n+1)} : \mathbf{C}^{(n+1)} \to \mathbf{C}^{n+1}$ in such a way that $\varphi_i^{(n+1)} X = f^{(n+1)} X^{[\varphi^{(n)}]}$ for $X \in \operatorname{dom} f_i^{(n+1)}$ with card $X^{[\varphi^{(n)}]} \in \mathbf{T}$, $X^{[\varphi^{(n)}]} = \{\varphi_i^{(n)} x | x \in X\}$ whereas $\varphi_i^{n+1} X$ will be defined as an arbitrary element of S_i^{n+1} if $X \in \operatorname{dom} f_i^{(n+1)}$, card $X^{[\varphi^n]} \notin \mathbf{T}$; $(i, j) = (1,2), (2,1)$. It is easy to see that $\varphi^{(n+1)}$ is an epimorphism with $\varphi^{(n+1)}|_{\mathbf{C}} = \mathrm{id}_{\mathbf{C}}$. By induction we get a sequence $(\varphi^{(n)})_{n=0}^\infty$. Now there exists precisely one epimorphism $\varphi : \overline{\mathbf{C}} \to \mathbf{C}'$ prolonging simultaneously all $\varphi^{(0)}, \varphi^{(1)}, \varphi^{(2)}, \ldots$. Q.E.D.

Proposition 2. *Let $\mathbf{C}$ be a coupled system and $\mathbf{C}'$ some coupled system with $\mathbf{C}' = \mathbf{C}_{\mathbf{C}'}$ such that there is an epimorphism $\psi : \mathbf{C}' \to \overline{\mathbf{C}}$ with $\psi|_{\mathbf{C}} = \mathrm{id}_{\mathbf{C}}$. Then ψ is an isomorphism.*

Proof. Let $\varphi : \overline{\mathbf{C}} \to \mathbf{C}$ be the epimorphism from Proposition 1. Putting $\varphi^{(n)} = \varphi|_{\mathbf{C}^{(n)}}$, $\psi^n = \psi|_{\mathbf{C}^n}$ $(n = 0,1,2, \ldots)$ we shall show by induction that $\Theta^n = \varphi^{(n)} \circ \psi^n$ is equal to $\mathrm{id}_{\mathbf{C}^n}$ for each $n = 0,1,2, \ldots$. First $\Theta^0 = \mathrm{id}_{\mathbf{C}}$ because of $\varphi^{(0)} = \psi_0 = \mathrm{id}_{\mathbf{C}}$. Suppose further that $\Theta^n = \mathrm{id}_{\mathbf{C}^n}$ for some n. Then, to every $z \in T_j^n$ there is $X \in \operatorname{dom} f_i^{n+1}$ such that $f_i^{n+1} X = z$; because of card $X \in \mathbf{T}$, card $X^{[\psi]} \in \mathbf{T}$(4) it follows that $\psi_j z = f_i^{n+1} X^{[\psi]}$, $\varphi_j(\psi_j z) = f_i^{n+1} (X^{[\psi]})^{[\varphi]} = f_i^{n+1} X = z$, as $\Theta^n = \mathrm{id}_{\mathbf{C}^n}$ by assumption $((i, j) = (1,2), (2,1))$. Thus $(\Theta^n)_n^{\infty=0}$ is a sequence of identity mappings and consequently $\psi \circ \varphi = \mathrm{id}_{\mathbf{C}'}$ so that ψ must be an isomorphism. Q.E.D.

(3) More detailed: $\mathbf{C}^n = (S_i^n, f_i^n; i = 1, 2)$. Similarly for analogous cases.

(4) $X^{[\psi]} = \{\psi_i x | x \in X\}$ and similarly further for $Y^{[\varphi]}$.

(5) $X^{[\sigma^n]} = \{\sigma_i^n x | x \in X\}$.

Let $\mathbf{C}$ be a coupled system with $\operatorname{dom} f_1 = \operatorname{dom} f_2 = \emptyset$. Then $\overline{\mathbf{C}}$ is called a *free* coupled system.

Proposition 3. *To each complete coupled system* $\mathbf{C}'$ *there is an epimorphism* $\sigma : \overline{\mathbf{C}} \to \mathbf{C}'$ *where* $\mathbf{C}$ *is such that* $S_1 = S_1'$, $S_2 = S_2'$, $\operatorname{dom} f_1 = \operatorname{dom} f_1 = \emptyset$.

Proof. Inductively we shall construct a sequence $(\sigma^n)_{n=0}^{\infty}$ where each σ^n is a surjection of $\mathbf{C}^{(n)}$ onto some coupled subsystem of $\mathbf{C}'$. If $z \in T_j^{(n)}$, $z \in f_i^{(n+1)}X$, card $X^{[\sigma^n]} \in \mathbf{T}$ (respectively, $\notin \mathbf{T}$), define $\sigma_j^{n+1}z = f_i' X^{[\sigma^n]}$ (respectively, $\sigma_j^{n+1} z =$ an arbitrary element of $S_j^{(n+1)}$) for $(i, j) = (1,2), (2,1)$. The common prolongation $\sigma : \overline{\mathbf{C}} \to \mathbf{C}'$ of all $\sigma^0, \sigma^2, \sigma^2, \ldots$ presents the required epimorphism. Q.E.D.

Proposition 4. *Every complete coupled subsystem of a free coupled system is free.*

Proof. Let $\mathbf{C}$ be a coupled system such that $\operatorname{dom} f_1 = \operatorname{dom} f_2 = \emptyset$ and let $\mathbf{C}'$ be a complete coupled subsystem of $\overline{\mathbf{C}}$. Construct the set $S_j^{\Diamond}$ consisting of all elements of $S_j \cap S_j'$ and of all elements in S_j' having (as elements of $\bigcup_{n=0}^{\infty} S_j^{(n)}$) the form $f_i^{(n+1)} X$, $X \in \operatorname{dom} f_i^{(n+1)} \setminus \operatorname{dom} f_i^{(n)}$ where X contains an element of $S_i^{(n)} \setminus S_i'$; $(i, j) = (1,2), (2,1)$. Then $\overline{\mathbf{C}^{\Diamond}} = \mathbf{C}$ for $\mathbf{C}^{\Diamond} = (S_i^{\Diamond}, f_i^{\Diamond}; i = 1,2)$, $\operatorname{dom} f_1^{\Diamond} = \operatorname{dom} f_2^{\Diamond} = \emptyset$. Q.E.D.

REFERENCES

[1] Pickert G., *Projektive Ebenen*, Berlin—Göttingen—Heidelberg, 1955.

[2] Havel V., *Zerlegungen von Inzidenzstrukturen*, I, Publ. Math. *13* (1966), 99—102.

[3] Havel V., *Free extensions of coupled systems*, Arch. Math. Brno *3* (1967), 65—68.

Received May 4, 1967.

Katedra matematiky elektrotechnické fakulty
Vysokého učení technického,
Brno