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ON CYCLES IN TOURNAMENTS[1]

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§ 1. SUMMARY

If R denotes a set of r points chosen from a tournament (complete oriented graph) T_n with n points, then for which integers h do there exists cycles in T_n of length h that contain every point of R? Kotzig [2] answered this question when n is odd and there are an equal number of arcs oriented towards and away from each point of T_n. Our object here is to show that Kotzig's argument may be extended to yield analogous results for irreducible tournaments in general.

§ 2. DEFINITIONS

A *tournament* T_n consists of n points $p_1, p_2, \ldots, p_n$ such that each pair of distinct points p_i and p_j is joined by one and only one of the oriented arcs $\overrightarrow{p_ip_j}$ or $\overrightarrow{p_jp_i}$. If the arc $\overrightarrow{p_ip_j}$ is in T_n, then we say that p_i beats p_j and that p_j loses to p_i. The *score* of a point p is the number $s(p)$ of points that p beats. A tournament T_n is *regular* if the scores of its points are as nearly equal as possible, that is, if $s(p) = m$ for every point p when $n = 2m + 1$ and $s(p) = m - 1$ or m for every point p when $n = 2m$.

A sequence of the type $(a, \overrightarrow{ab}, b, \overrightarrow{bc}, \ldots, l, \overrightarrow{lm}, m)$ is called a *path* from a to m; if the arc $\overrightarrow{ma}$ is also included in the sequence then it is called a *cycle* (we assume that the points $a, b, \ldots, m$ are all distinct). The *length* of a path or cycle is the number of arcs it contains; we adapt the convention that a single point constitutes a path of length zero and a cycle of length one. Cycles and paths of length k will be denoted by C_k and P_k.

If it is possible to partition the points of a tournament T_n into two non-

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empty sets B and A such that every point of B beats every point of A, then T_n is *reducible*; if not, then T_n is *irreducible*. Every reducible tournament T_n has a unique decomposition into irreducible subtournaments $T^{(1)}, T^{(2)}, \ldots, T^{(l)}$ such that every point of $T^{(j)}$ beats every point of $T^{(i)}$ if $1 \leqslant i < j \leqslant l$; the subtournaments $T^{(1)}$ and $T^{(l)}$ are the *bottom* and *top components* of T_n and the remaining irreducible subtournaments are the *intermediate components* of T_n. A tournament T_n is irreducible if and only if there exists a path from p to q for every ordered pair of points p and q of T_n (see Roy [4]).

If C_k is a cycle in the tournament T_n, let $B(C_k)$ ($L(C_k)$) denote the set of points of $T_n - C_k$ that beat (lose to) every point of C_k and let $M(C_k)$ denote the remaining points of $T_n - C_k$. It is easy to see that a point p of $T_n - C_k$ belongs to $M(C_k)$ if and only if there exist two consecutive points of the cycle C_k, e and f say, such that e beats p and p beats f. We shall frequently use the same symbol to denote the set of points of a tournament or cycle as we use to denote the tournament or cycle itself.

§ 3. RESULTS ON CYCLES

The following two lemmas are direct consequences of the hypotheses and the definition of irreducibility (see lemmas 1 and 2 of [2]).

Lemma 1. *If C_k is a cycle of the irreducible tournament T_n and if p is any point of $T_n - C_k$, then there exists a cycle C_{k+1} in T_n such that $C_k \cup p \subset C_{k+1}$ if and only if $p \in M(C_k)$.*

Lemma 2. *If C_k is a cycle of the irreducible tournament T_n such that $k < n$ and $M(C_k) = \emptyset$, then there exists at least one point l in $L(C_k)$ and at least one point b in $B(C_k)$ such that l beats b. Furthermore,*

(a) *there exists a cycle C_{k+2} in T_n such that $C_k \cup l \cup b \subset C_{k+2}$, and*

(b) *if w is any point of the cycle C_k and $k \neq 1$, then there exists a cycle C_{k+1} in T_n such that $(C_k - w) \cup l \cup b \subset C_{k+1}$.*

Theorem 1. *If C_k is a cycle of the irreducible tournament T_n and if $1 \leqslant k < h \leqslant n$, then there exists a cycle C_h in T_n such that $C_k \subset C_h$ except when $h = k + 1$ and $M(C_k) = \emptyset$.*

This follows by induction on h, using Lemmas 1 and 2; whenever we apply Lemma 2b we take w to be one of the points added to C_k at an earlier stage. (See Lemmas 3, 4, and 5 of [2].)

The following result, obtained by letting $k = 1$, is stated in [3].

Corollary 1. *If p is any point of the irreducible tournament T_n and if $3 \leqslant h \leqslant n$, then there exists a cycle C_h in T_n such that $p \in C_h$.*

A tournament is *Hamiltonian* if it contains a cycle passing through every point once and only once; if a tournament is reducible then it obviously is not Hamiltonian. Hence, corollary 1 implies the following result due to Camion [1].

Corollary 2. *A tournament is Hamiltonian if and only if it is irreducible.*

§ 4. RESULTS ON REDUCIBLE SUBTOURNAMENTS

If T_u and T_v denote the bottom and top components of a reducible subtournament T_r of an irreducible tournament T_n, let $x(y)$ be one of the points of $T_u(T_v)$ that beats the smallest (greatest) number of other points of $T_u(T_v)$. (These numbers need not be the same as the scores $s(x)$ and $s(y)$ of x and y in the tournament T_n). Let $m(T_r)$ denote the length of any shortest path P_m in T_n of the form $(t_0, \overrightarrow{t_0 t_1}, t_1, \ldots, \overrightarrow{t_{m-1} t_m}, t_m)$ where t_0 is in T_u and t_m is in T_v. It is clear that P_m exists (since T_n is irreducible) and that none of the points $t_1, \ldots, t_{m-1}$ belong to T_u or T_v. Let z denote the number of points in the intermediate components of T_r that do not belong to the path P_m.

Lemma 3. *If T_r is a reducible subtournament of an irreducible tournament T_n, then*

$$2 \leqslant m(T_r) \leqslant \max\{2, 3 + s(y) - s(x) - z - \tfrac{1}{2}(u + v)\}.$$

Proof. Every point of T_v beats every point of T_u, so it must be that $m = m(T_r) \geqslant 2$; let us suppose that $m > 2$. Since P_m is a shortest path from T_u to T_v it follows that every point of T_u loses to the $m - 2$ points $t_2, t_3, \ldots, t_{m-1}$ and that every point of T_v beats the $m - 2$ points $t_1, t_2, \ldots, t_{m-2}$. Furthermore, every point of T_u loses to the z points in the intermediate components of T_r that do not belong to P_m and to the v points of T_v; similarly, every point of T_v beats these same z points and the u points of T_u.

Let e and f denote the number points of T_n belonging neither to T_r nor to P_m that the point x beats and loses to; the point y must beat all the e points that lose to x (since $m > 2$) and

$$e + f + u + v + z + (m - 1) = n. \tag{1}$$

Finally, x must lose to at least $\frac{1}{2}(u - 1)$ points of T_u and y must beat at least $\frac{1}{2}(v - 1)$ points of T_v.

If we combine all these statements we obtain the inequalities

$$(n - 1) - s(x) \geqslant (m - 2) + z + v + f + \tfrac{1}{2}(u - 1) \tag{2}$$

and

$$s(y) \geqslant (m - 2) + z + u + e + \tfrac{1}{2}(v - 1). \tag{3}$$

It follows from (1), (2), and (3) that

$$m \leqslant 3 + s(y) - s(x) - z - \tfrac{1}{2}(u + v), \tag{4}$$

and the lemma is proved.

Lemma 4. *If T_r is a reducible subtournament of an irreducible tournament T_n and if $k(T_r)$ denotes the length of any shortest cycle C_k in T_n such that $T_r \subset C_k$, then*

$$r + 1 \leqslant k(T_r) \leqslant \max\{r + 1, 2 + s(y) - s(x) + \tfrac{1}{2}(u + v)\}.$$

Proof. Let P_m denote, as before, a shortest path from the bottom component T_u of T_r to the top component T_v. It follows from corollary 2, that the points of T_u and T_v can be labelled $u_1, u_2, \ldots, u_u$ and $v_1, v_2, \ldots, v_v$ so that $u_u = t_0$, $v_1 = t_m$, and u_i beats u_{i+1} for $i = 1,2, \ldots, u - 1$ and v_j beats v_{j+1} for $j = 1,2, \ldots, v - 1$. The cycle

$$C = P_m \cup (\overrightarrow{v_1 v_2}, v_2, \overrightarrow{v_2 v_3}, \ldots, v_v, \overrightarrow{v_v u_1}, u_1, \overrightarrow{u_1 u_2}, \ldots, \overrightarrow{u_{u-1} u_u})$$

has the length $m(T_r) - 1 + u + v$ and it contains every point of T_r except the z points in the intermediate components of T_r that do not belong to the path P_m. We may apply Lemma 1 to these z points and conclude that there exists a cycle of length $m(T_r) - 1 + u + v + z$ that contains every point of T_r. The required result now follows from Lemma 3.

§ 5. MAIN THEOREM

The preceding results may be combined to yield the following theorem. (If T_r is a subtournament of T_n we let $M(T_r)$ denote the set of points p of $T_n - T_r$ such that p beats at least one and loses to at least one point of T_r).

Theorem 2. *Let T_r denote a subtournament of an irreducible tournament T_n.*

(a) *If T_r is irreducible and $r \leqslant h \leqslant n$, then there exists a cycle C_h in T_n such that $T_r \subset C_h$ except when $h = r + 1$ and $M(T_r) = \emptyset$.*

(b) *If T_r is reducible and $k(T_r) \leqslant h \leqslant n$, then there exists a cycle C_h in T_n such that $T_r \subset C_h$.*

Corollary 3. *Let T_r denote a subtournament of a regular tournament T_n. If $1 \leqslant r \leqslant h \leqslant n$, then there exists a cycle C_h in T_n such that $T_r \subset C_h$ except when*

(a) *T_r is irreducible, $h = r + 1$, and $M(T_r) = \emptyset$,*

(b) *T_r is reducible and $h = r$, or*

(c) *$r = 2$, $h = 3$, n is even, and $M(T_r) = \emptyset$.*

Proof. (This corollary is essentially the same as Theorem 5 of [2] when n

is odd and $r > 2$). It is easy to show that a regular tournament T_n is irreducible when $n \neq 2$ (recall that a tournament T_n is reducible if and only if the sum of the k smallest scores of T_n equals $\binom{k}{2}$ for some integer k where $k < n$). Hence, we can apply Theorem 2 when $n \neq 2$. If T_r is irreducible there is nothing more to prove. If T_r is reducible and $r \neq 2$ then it follows from Lemma 4 that $k(T_r) = r + 1$ when T_n is regular since $2 + s(y) - s(x) + \frac{1}{2}(u + v) \leqslant \leqslant 3 + \frac{1}{2}r < r + 2$ if $r > 2$; similarly, if $r = 2$ then $k(T_r) = r + 1$ when n is odd, and $k(T_r) = r + 2$ or $r + 1$ according as $M(T_r)$ is or is not empty when n is even. This suffices to complete the proof of the corollary since it is clearly true when $n = 2$.

In closing we remark that in view of the unique decomposition every reducible tournament has into irreducible components there is no serious loss of generality in assuming T_n is irreducile in Theorem 2.

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