Matematický časopis

Zdena Riečanová

On Regularity of a Measure on a σ-Algebra

Matematický časopis, Vol. 19 (1969), No. 2, 135--137

Persistent URL: http://dml.cz/dmlcz/127091

Matematický časopis 19 (1969), No. 2

ON REGULARITY OF A MEASURE ON A σ-ALGEBRA

ZDENA RIEČANOVÁ, Bratislava

In the paper we show that the question of regularity of a measure on a σ-algebra can be reduced to the question of regularity on a σ-ring.

If X is a nonempty set of elements and $\mathbf{C}$ is a nonempty family of substes of X, then by $\mathbf{S}(\mathbf{C})$ we denote the σ-ring and by $\mathbf{A}(\mathbf{C})$ the σ-algebra generated by $\mathbf{C}$. The family $[\mathbf{S}(\mathbf{C})]_\lambda = \{B \subset X : E \cap B \in \mathbf{S}(\mathbf{C})$ for all $E \in \mathbf{S}(\mathbf{C})\}$ is a σ-algebra of subsets of X.(1)

Definition.(2) *Let X be a nonempty set of elements, $\mathbf{C}$, $\mathbf{U}$, $\mathbf{S}$ be nonempty families of subsets of X such that $\mathbf{S}$ is a σ-ring, $\mathbf{C} \subset \mathbf{S}$, $\mathbf{U} \subset \mathbf{S}$. Then a measure μ defined on $\mathbf{S}$ will be called to be $(\mathbf{C}, \mathbf{U})$-regular on a σ-ring $\mathbf{S}_0 \subset \mathbf{S}$ if and only if*

$$\mu(E) = \sup\{\mu(C) : E \supset C \in \mathbf{C}\} = \inf\{\mu(U) : E \subset U \in \mathbf{U}\}$$

for each $E \in \mathbf{S}_0$.

Theorem 1. *Let X be a nonempty set of elements, $\mathbf{C}$ and $\mathbf{U}$ be nonempty families of subsets of X, $\mathbf{A}$ be a σ-algebra of subsets of X such that $\mathbf{C} \subset \mathbf{A} \subset [\mathbf{S}(\mathbf{C})]_\lambda$, μ be a σ-finite measure on $\mathbf{A}$. Let $\mathbf{U}$ and $\mathbf{C}$ satisfy the following conditions:*

a. *$\mathbf{U}$ is a finitely additive subfamily of $\mathbf{A}$.*

b. *$U - C \in \mathbf{U}$ for each $U \in \mathbf{U}$, $C \in \mathbf{C}$.*

Then μ is a $(\mathbf{C}, \mathbf{U})$-regular measure on $\mathbf{A}$ if and only if the following two conditions are simultaneously satisfied:

1. *μ is a $(\mathbf{C}, \mathbf{U})$-regular measure on $\mathbf{S}(\mathbf{C})$.*
2. *There are sets $Y \in \mathbf{S}(\mathbf{C})$ and $U \in \mathbf{U}$ such that $\mu(X - Y) = 0$, $X - Y \subset U$, $\mu(U) < \infty$.*(3)

First we prove two lemmas. We assume in both lemmas that $\mathbf{C}$ is a nonempty family of subsets of X and $\mathbf{A}$ is a σ-algebra of subsets of X such that $\mathbf{C} \subset \mathbf{A}$.

(1) We use the terminology according to [1].

(2) See also [2], p. 187 and [3].

(3) We can suppose that $Y = \bigcup_{n=1}^{\infty} C_n$, $C_n \in \mathbf{C}$.

Lemma 1. *Let μ be a measure on* $\mathbf{A}$. *If* $E \in \mathbf{A}$, $\mu(E) < \infty$ *and* $\mu(E) = \sup\{\mu(C) : E \supset C \in \mathbf{C}\}$, *then there exists* $Y \in \mathbf{S}(\mathbf{C})$, $Y \subset E$, $\mu(E - Y) = 0$.

Proof. By an assumption there are sets $C_n \in \mathbf{C}$ $(n = 1, 2, \ldots)$ such that

$$C_n \subset E,\ \mu(E) < \mu(C_n) + \frac{1}{n} \leqq \mu(\bigcup_{k=1}^{n} C_k) + \frac{1}{n}.$$

Hence

$$\bigcup_{n=1}^{\infty} C_n \subset E,\ \mu(E) \leqq \lim_{n\to\infty}\left[\mu(\bigcup_{k=1}^{n} C_k) + \frac{1}{n}\right] = \mu(\bigcup_{n=1}^{\infty} C_n)$$

and therefore

$$0 \leqq \mu(E - \bigcup_{n=1}^{\infty} C_n) = \mu(E) - \mu(\bigcup_{n=1}^{\infty} C_n) \leqq 0.$$

Lemma 2. *Let μ be a σ-finite measure on* $\mathbf{A}$. *Let*

$$\mu(E) = \sup\{\mu(C) : E \supset C \in \mathbf{C}\},$$

for each $E \in \mathbf{A}$. *Then there is a set* $Y \in \mathbf{S}(\mathbf{C})$ *such that* $\mu(X - Y) = 0$.

Proof. As μ is σ-finite there is a sequence $\{A_n\}_{n=1}^{\infty}$ of sets of $\mathbf{A}$, $\mu(A_n) < \infty$ $(n = 1, 2, \ldots)$ such that $X = \bigcup_{n=1}^{\infty} A_n$. By Lemma 1 there are sets $Y_n \in \mathbf{S}(\mathbf{C})$ $(n = 1, 2, \ldots)$ such that $\mu(A_n - Y_n) = 0$. Put $Y = \bigcup_{n=1}^{\infty} Y_n$. Then $Y \in \mathbf{S}(\mathbf{C})$ and

$$\mu(X - Y) = \mu[(\bigcup_{i=1}^{\infty} A_i) - (\bigcup_{n=1}^{\infty} Y_n)] \leqq \mu[\bigcup_{i=1}^{\infty}(A_i - Y_i)] \leqq \sum_{i=1}^{\infty} \mu(A_i - Y_i) = 0.$$

Proof of Theorem 1. A. If μ is a $(\mathbf{C}, \mathbf{U})$-regular measure on $\mathbf{A}$, then the condition 1 is evident and the condition 2 follows from Lemma 2 and the regularity of the set $X - Y$ with respect to μ.

B. Let the conditions 1 and 2 hold and $E \in \mathbf{A}$. Then $E \cap Y \in \mathbf{S}(\mathbf{C})$ and

$$\mu(E) = \mu(E \cap Y) = \sup\{\mu(C) : E \cap Y \supset C \in \mathbf{C}\} \leqq$$
$$\leqq \sup\{\mu(C) : E \supset C \in \mathbf{C}\} \leqq \mu(E).$$

If $\mu(E) = \infty$, then $\mu(E) = \inf\{\mu(U) : E \subset U \in \mathbf{U}\}$. Let $\mu(E) < \infty$. Choose an $\mathscr{E} > 0$. By assumptions we have $U \in \mathbf{U}$, $C \in \mathbf{C}$, $U \supset E - Y$, $\mu(U) < \infty$, $C \subset U \cap Y$, $\mu[(U \cap Y) - C] < \frac{\varepsilon}{2}$.

Hence we have $U - C \in \mathbf{U}$, $U - C \supset E - Y$, $\mu(U - C) \leqq \mu(U - Y) + \mu[(U \cap Y) - C] = \mu[(U \cap Y) - C] < \frac{\varepsilon}{2}$.

Since $E \cap Y \in \mathbf{S}(\mathbf{C})$, we have $V \in \mathbf{U}$, $V \supset E \cap Y$, $\mu[V - (E \cap Y)] < \frac{\varepsilon}{2}$.

Put $O = V \cup (U - C)$. Then

$$O \in \mathbf{U},\ O \supset E,\ \mu(O) - \mu(E) = \mu(O - E) \leqq \mu[(V - E) \cup (U - C)] \leqq$$
$$\leqq \mu(V - E) + \mu(U - C) < \varepsilon.$$

Hence the Theorem is proved.

Corollary. *Let X, $\mathbf{C}$ and $\mathbf{U}$ satisfy the assumptions of Theorem* 1, *μ be a σ-finite measure on $\mathbf{S}(\mathbf{C})$. Let μ_λ be the extension of μ on $[\mathbf{S}(\mathbf{C})]_\lambda$ defined in* [2], *example* 1, *p.* 53. *Then μ_λ is σ-finite and $(\mathbf{C}, \mathbf{U})$-regular on $[\mathbf{S}(\mathbf{C})]_\lambda$ if and only if the following conditions are satisfied:*

1. *μ_λ is a $(\mathbf{C}, \mathbf{U})$-regular measure on $\mathbf{S}(\mathbf{C})$.*
2. *There are sets $Y \in \mathbf{S}(\mathbf{C})$ and $U \in \mathbf{U}$ such that $\mu_\lambda(X - Y) = 0$, $X - Y \subset U$, $\mu_\lambda(U) < \infty$.*

Example 1. If in Theorem 1 X is a locally compact Hausdorff topological space and $\mathbf{C}$ is the family of all compact subsets of X, then we can put: 1. $\mathbf{A} = \mathbf{A}(\mathbf{C})$. 2. $\mathbf{A} = \mathbf{A}(\mathrm{D})$, where D are all closed subsets of X (we get weakly Borel sets). 3. $\mathbf{A} = [\mathbf{S}(\mathbf{C})]_\lambda$ (we get locally Borel sets). In these cases $\mathbf{U}$ is the family of all open sets belonging to $\mathbf{A}$.

Example 2. If in Theorem 1 X is a locally compact Hausdorff topological space and $\mathbf{C}$ is the family of all compact G_δ subsets of X, then we can choose: 1. $\mathbf{A} = \mathbf{A}(\mathbf{C})$. 2. $\mathbf{A} = \mathbf{A}(\mathrm{D}_0)$, where D_0 are all closed G_δ subsets of X (weakly Baire sets). 3. $\mathbf{A} = \mathbf{A}(\mathrm{Z})$, where Z is the family of all sets of the form $f^{-1}(\{0\})$, where f is a real — valued function continuous on X. 4. $\mathbf{A} = [\mathbf{S}(\mathbf{C}_0)]_\lambda$ (locally Baire sets). In the cases 1—4, $\mathbf{U}$ is the family of all open sets belonging to $\mathbf{A}$.

REFERENCES

[1] Halmos P. ., *Measure Theory*, New York 1950.
[2]Berberian S. K., *Measure and Integration*, New York 1965.
[3] Риечанова З., *О регулярности меры*, Mat .časop. *17* (1967), 38—47.

Received May 11, 1967.

Katedra matematiky a deskriptívnej geometrie
Elektrotechnickej fakulty
Slovenskej vysokej školy technickej,
Bratislava