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1. INTRODUCTION

By a semilattice decomposition (SL decomposition) of a semigroup S we mean a partition of S which belongs to a factor semigroup $\bar{S}$ on S such that $\bar{S}$ is a semilattice [1].

Every element A of an SL decomposition of S is a subsemigroup of S. If A and B are two elements of an SL decomposition of S, then there exists an element C of this SL decomposition such that $AB \subseteq C$ and $BA \subseteq C$. $A \leqslant B$ holds if and only if $AB \subseteq A$. This relation is a partial ordering of $\bar{S}$.

A congruence relation on S that belongs to an SL decomposition of S will be called a semilattice congruence (SL congruence).

The minimal semilattice congruence (MSL congruence) is the intersection of all SL congruences on S.

By a maximal semilattice decomposition (MSL decomposition) of S we mean an SL decomposition of S belonging to an MSL congruence on S.

A semigroup S is called indecomposable if the only SL congruence on S is the universal relation on S.

T. Tamura [9], [10] and M. Petrich [4] proved the indecomposability of the classes of the MSL decomposition of a semigroup S. At the Semigroup Symposium held in Smolenice in 1968 T. Tamura suggested to me to try to prove this statement by means of the notions of my paper [6]. In the present paper a new proof of the indecomposability of the classes of the MSL decomposition of a semigroup S is given. We obtain it as a simple consequence of Theorem 3, which in itself is also of great interest. O. Steinfeld [5] proved this Theorem for rings. Moreover some results of papers [4] and [6] are proved again.

A K-semigroup S is a semigroup in which $xyzyx = yxzxy$ holds for all $x, y, z \in S$ [2]. For a K-semigroup a generalization of the construction of classes of the MSL decomposition of a commutative semigroup is given.

By an ideal we mean a two-sided ideal.

Let J be an ideal of S. Let x be an element of S such that for some positive integer n, $x^n \in J$ holds. Then x will be called J-potent. The set of all J-potent elements of S will be denoted by $\tilde{N}(J)$.

An ideal (subsemigroup) I of S, for which there exists a positive integer n such that $I^n \subseteq J$, is called a J-potent ideal (subsemigroup) of S. The union $R(J)$ of all J-potent ideals of S will be called the Schwarz J-radical [7], [8].

Finally we give an example of a K-semigroup S, the Schwarz J-radical of S being distinct from the set $\tilde{N}(J)$ of all J-potent elements of S. This completes the results of the paper [2].

An ideal P of S is called completely prime if $ab \in P$ implies either $a \in P$ or $b \in P$. (In [1] $P \neq S$ is called a prime ideal.)

The face of a semigroup S is a (eventually empty) subset $M \subseteq S$ for which $ab \in M$ holds if and only if $a \in M$ and $b \in M$.

A subset P of a semigroup S is a completely prime ideal of S if and only if $S \setminus P$ is a face of S distinct from S.

A subset M of S is a face of S if and only if $S \setminus M$ is either a completely prime ideal of S or the empty set $\square$.

If M' is a face of M and M is a face of S, then M' is a face of S [6].

Let φ be a homomorphism of the semigroup S onto the semigroup S'. If P' is a completely prime ideal of S', then its inverse $\varphi^{-1}(P')$ is a completely prime ideal of S. If M' is a face of S', then $\varphi^{-1}(M')$ is a face of S.

2. THE MSL DECOMPOSITION OF A SEMIGROUP

The existence and the construction of the classes of the MSL decomposition of a semigroup have been studied in papers [4], [6], [9], [10], [11] and [12]. From paper [4] it is evident that by the construction of the MSL decomposition we can use the restriction to the SL decompositions having two classes (see also [6]).

We shall now deal with SL decompositions having two classes. By an SL congruence having two classes we mean an SL congruence that belongs to an SL decomposition having two classes.

By a class of a congruence relation we mean a class of the partition belonging to this congruence relation.

Lemma 1. *An SL decomposition having two classes is of the form $S = P \cup N$, $P \cap N = \square$, where P is a completely prime ideal and N the corresponding face. Conversely such a partition of S is an SL decomposition of S.*

Proof. Let A and B be two distinct classes of an SL decomposition having two classes. We have $A^2 \subseteq A$, $B^2 \subseteq B$. Without loss of generality suppose that $AB \subseteq B$ and $BA \subseteq B$. Then $B^2 \cup AB \subseteq B$, hence $(B \cup A)B \subseteq B$

and $B^2 \cup BA \subseteq B$, hence $B(B \cup A) \subseteq B$, so that B is an ideal of S. If $ab \in B$, then $a \in A$ and $b \in A$ cannot hold. Therefore either $a \in B$ or $b \in B$. Hence B is a completely prime ideal and A is a face. The converse statement is evident.

Theorem 1 ([4], [6]). *The MSL congruence on a semigroup S is the intersection of all SL congruences having two classes and of the universal relation on S.*

Proof. Let Φ be the intersection of all SL congruences having two classes and of the universal relation on S. Clearly Φ is a SL congruence on S. Moreover every nonempty face of S either contains a class of the congruence Φ or it is disjoint with this class.

Let η be the MSL congruence on S. Evidently $\eta \subseteq \Phi$. Suppose for an indirect proof that $\eta \neq \Phi$. Then at least one class T of the congruence Φ contains at least two classes A and B of the congruence η. These classes A and B are elements of a factor semigroup $\bar{S}$ belonging to the congruence η. A and B are either incomparable in the partial ordering of $\bar{S}$ or one of them is greater than the other. In the second case we can suppose without loss of generality that $A \geqslant B$. In both cases the set $\bar{M}$ of all classes $X \geqslant A$ is a face of $\bar{S}$. The inverse $\varphi^{-1}(\bar{M}) = M$ in the natural homomorphism φ of the semigroup S onto $\bar{S}$ is a face of S. However, M contains the set A, but it does not contain the set B. Hence M does not contain the whole class T of the SL congruence Φ and at the same time it has a nonempty intersection with T. This is a contradiction.

Let us denote by N_x the class of the MSL decomposition that contains the element $x \in S$, by $N(x)$ the intersection of all faces containing x and by $C(x)$ the intersection of all completely prime ideals containing x. Theorem 1 implies:

Corollary ([6]). $N_x = N(x) \cap C(x)$.

3. ARBITRARY SL DECOMPOSITION OF A SEMIGROUP

The Corollary of Theorem 1 enables us to prove.

Theorem 2 ([4]). *Every SL congruence on a semigroup S is an intersection of a system of SL congruences having two classes and of the universal relation on S.*

Proof. Let Φ be the given SL congruence on S. Denote by $\bar{S}$ the factor semigroup belonging to Φ. Let $\bar{\mathfrak{P}}$ be the system of all completely prime ideals of $\bar{S}$ and $\bar{\mathfrak{M}}$ the system of all nonempty faces of $\bar{S}$ that are distinct from $\bar{S}$. Let φ be the natural homomorphism of S onto $\bar{S}$. Let us denote $\mathfrak{P} = \{\varphi^{-1}(\bar{P}) \mid \bar{P} \in \bar{\mathfrak{P}}\}$, $\mathfrak{M} = \{\varphi^{-1}(\bar{M}) \mid \bar{M} \in \bar{\mathfrak{M}}\}$. Clearly $\varphi^{-1}(\bar{P})$ is a completely prime ideal and $\varphi^{-1}(\bar{M})$ a face of S. We shall show that the SL congruence Φ

is the intersection of all SL congruences having two classes whose classes are elements of $\mathfrak{P}$ and $\mathfrak{M}$ and of the universal relation on S.

Let T_x be the class of the SL congruence Φ that contains the element $x \in S$. We have to show that T_x is equal to the class (containing x) of the SL decomposition that belongs to the intersection of the SL congruences whose classes are elements of $\mathfrak{P}$ and $\mathfrak{M}$.

Let $x \in S$. Then $x \in P \in \mathfrak{P}$ if and only if $\varphi(x) \in \bar{P} = \varphi(P) \in \bar{\mathfrak{P}}$ and $x \in \in M \in \mathfrak{M}$ if and only if $\varphi(x) \in \bar{M} = \varphi(M) \in \bar{\mathfrak{M}}$.

Let $\mathfrak{P}_x = \{P \mid P \in \mathfrak{P},\ x \in P\}$, $\mathfrak{M}_x = \{M \mid M \in \mathfrak{M}, x \in M\}$, $\bar{\mathfrak{P}}_x = \{\bar{P} \mid \bar{P} \in \bar{\mathfrak{P}}, \varphi(x) \in \bar{P}\}$ and $\bar{\mathfrak{M}}_x = \{\bar{M} \mid \bar{M} \in \bar{\mathfrak{M}},\ \varphi(x) \in \bar{M}\}$.

Then $\mathfrak{P}_x = \{\varphi^{-1}(\bar{P}) \mid \bar{P} \in \bar{\mathfrak{P}}_x\}$ and $\mathfrak{M}_x = \{\varphi^{-1}(\bar{M}) \mid \bar{M} \in \bar{\mathfrak{M}}_x\}$. $\bar{S}$ is a semilattice. Hence $\{\varphi(x)\} = (\bigcap_{\bar{P} \in \bar{\mathfrak{P}}_x} \bar{P}) \cap (\bigcap_{\bar{M} \in \bar{\mathfrak{M}}_x} \bar{M})$. This implies

$$T_x = \varphi^{-1}(\varphi(x)) = (\bigcap_{\bar{P} \in \bar{\mathfrak{P}}_x} \varphi^{-1}(\bar{P})) \cap (\bigcap_{\bar{M} \in \bar{\mathfrak{M}}_x} \varphi^{-1}(\bar{M})) = (\bigcap_{P \in \mathfrak{P}_x} P) \cap (\bigcap_{M \in \mathfrak{M}_x} M).$$ This proves our statement.

4. THE INDECOMPOSABILITY OF CLASSES OF THE MSL DECOMPOSITION

We first prove a Theorem from which the indecomposability of classes of the MSL decomposition follows as a simple consequence. O. Steinfeld [5] proved this Theorem for completely prime ideals in rings. The same proof can be carried out for semigroups. However, we want to give a proof of this Theorem.

Theorem 3 ([5]). *Let S be a semigroup, I an ideal of S. Let be $I = P \cup M$, $P \cap M = \square$, $P \neq I$, where P is a completely prime ideal of I and M the corresponding face. Then there is a completely prime ideal A of S such that $P = I \cap A$ and $M = I \cap (S \setminus A)$.*

Proof. Let $b \in M$ be an arbitrary but fixed element. Let $A = \{x \mid x \in S, xb \in P\}$ and $C = \{x \mid x \in S, xb \in M\}$. Clearly $P \subseteq A$ and $M \subseteq C$ hold. Hence A and C are nonempty sets. Evidently every element $x \in S$ belongs to just one of the sets A and C. Hence $\{A, C\}$ is a partition of S and $C = S \setminus A$. Moreover $A \cap I = P$ and $C \cap I = M$.

Next we shall prove that $xb \in P$ if and only if $bx \in P$, for every $x \in S, b \in M$. Let $xb \in P$. Then $b(xb) = (bx)b \in P$, but $bx \in I, b \notin P$. Hence $bx \in P$, because P is a completely prime ideal of I. If $bx \in P$, then $(bx)b = b(xb) \in P$, $b \notin P$, $xb \in I$. Hence $xb \in P$.

Finally we show that A is an ideal and C a subsemigroup of S. This implies that A is a completely prime ideal and C a face of S.

a) Let $a \in A$, $s \in S$. Then $ab \in P$, $sb \in I$. From this it follows that $ba \in P$, $sb \in I$. Further $(ba)(sb) = b(asb) \in P$, $b \notin P$, $asb \in I$. Hence $(as)b \in P$, i. e. $as \in A$. We proved that $AS \subseteq A$.

Analogously we obtain

b) $SA \subseteq A$

c) $CC \subseteq C$

and the proof is completed.

Corollary 1 ([4], [9], [10]). *Every class of the MSL decomposition of a semigroup S is indecomposable.*

Proof. The Corollary of Theorem 1 states that $N_x = N(x) \cap C(x)$. Hence N_x is an ideal of $N(x)$. If N_x can be decomposed (i. e. there exists a SL decomposition $\{P, M\}$ of N_x) then according to Theorem 3 $N(x)$ can be also decomposed. Hence there exists an SL decomposition $\{A, C\}$ of $N(x)$, where A is a completely prime ideal of $N(x)$ and C is a face of $N(x)$. Since C is a face of $N(x)$ and $N(x)$ is a face of S, C is a face of S. C contains some elements of N_x, but it does not contain all elements of N_x. This is impossible since $\{S \setminus C, C\}$ is an SL decomposition and every class of the MSL decomposition is contained either in $S \setminus C$ or in C.

Corollary 2 ([4]). *Let J be any ideal of N_x. Then J is indecomposable.*

Proof. The existence of an SL decomposition $\{P, M\}$ of the ideal J would imply again according to Theorem 3 the existence of an SL decomposition $\{A, C\}$ of N_x. But this is a contradiction.

5. THE CONSTRUCTION OF THE CLASSES OF THE MSL DECOMPOSITION

From the Corollary of Theorem 1 and from Theorem 3 there follow two simple constructions of the MSL decomposition of a semigroup.

Theorem 4. (See [6]). *Let N_x be the class of the MSL decomposition of a semigroup S containing x. Then*

a) *N_x is the intersection of all completely prime ideals of $N(x)$ containing x.*

b) *N_x is the intersection of all faces of $C(x)$ containing x.*

Proof. According to the Corollary of Theorem 1 $N_x = C(x) \cap N(x)$.

a) The intersection of $N(x)$ and of a completely prime ideal of S is a completely prime ideal of $N(x)$. Therefore it is sufficient to show that every completely prime ideal of $N(x)$ is the intersection of $N(x)$ and of a completely prime ideal of S. Let P' be a completely prime ideal of $N(x)$. Since $N(x) \setminus P'$ is a face of $N(x)$ and $N(x)$ is a face of S, $N(x) \setminus P'$ is a face of S. Hence $P' \cup (S \setminus N(x)) = S \setminus (N(x) \setminus P') = P$ is a completely prime ideal of S.

b) The intersection of the ideal $C(x)$ and of a face of S is a face of $C(x)$. Therefore it is sufficient to show that every face of $C(x)$ is the intersection of $C(x)$ and of any face of S. Let M' be a face of the ideal $C(x)$. Then Theorem 3 implies the existence of a face M of S such that $M' = C(x) \cap M$. This completes the proof.

6. THE CLASSES OF THE MSL DECOMPOSITION IN K-SEMIGROUPS

T. Tamura and N. Kimura [11] gave a construction of the classes of the *MSL* decomposition of a commutative semigroup. This construction can be generalized to the class of *K*-semigroups.

We introduce some notions and notations.

Let J be an ideal of the semigroup S and $J(x) = x \cup xS \cup Sx \cup SxS$.

a) An ideal I of S, each element of which is J-potent, will be called a J-ideal. The union $R^*(J)$ of all J-ideals of S is called the Clifford J-radical.

b) An ideal I of S with the property that each subsemigroup of I generated by a finite number of elements is J-potent, is called a locally J-potent ideal of S. The union $L(J)$ of all locally J-potent ideals of S will be called the Ševrin J-radical.

c) An ideal P of S is called a prime ideal of S if for any two ideals A and B of S, $AB \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$. The intersection $M(J)$ of all prime ideals of S which contain J is called the McCoy J-radical.

d) The intersection $C(J)$ of all completely prime ideals of S which contain J will be called the Jiang Luh J-radical.

Lemma 2. *Let S be a K-semigroup. Then*

$$(gh)^{3^k} = h^{\frac{3^k-1}{2}} g^{3^k} h^{\frac{3^k+1}{2}} = g^{\frac{3^k+1}{2}} h^{3^k} g^{\frac{3^k-1}{2}}$$

for all $g, h \in S$ and for every positive integer k.

Proof (by induction).

We shall prove only the first part of this statement. The proof of the rest is evident.

Our statement is true for $k = 1$, since $(gh)^3 = (gh)(gh)(gh) = (ghghg)h = hg^3h^2$.

Suppose that our statement holds for k. Then

$$(gh)^{3^{k+1}} = (gh)^{3 \cdot 3^k} = (gh)^{3^k} (gh)^{3^k} (gh)^{3^k} =$$
$$= \left(h^{\frac{3^k-1}{2}} g^{3^k} h^{\frac{3^k+1}{2}}\right) \left(h^{\frac{3^k-1}{2}} g^{3^k} h^{\frac{3^k+1}{2}}\right) \left(h^{\frac{3^k-1}{2}} g^{3^k} h^{\frac{3^k+1}{2}}\right) =$$

$$= h^{\frac{3^k-1}{2}} (g^{3^k}h^{3^k}g^{3^k}h^{3^k}g^{3^k})\, h^{\frac{3^k+1}{2}} = h^{\frac{3^k-1}{2}+3^k}\, g^{3^k.3}\, h^{3^k+\frac{3^k+1}{2}} = h^{\frac{3^{k+1}-1}{2}}\, g^{3^{k+1}}\, h^{\frac{3^{k+1}+1}{2}}.$$

Hence our statement holds for $k + 1$ and the proof is completed.

Lemma 3. *Let S be a K-semigroup. Then*
$(exf)^{3^k} = \alpha x^{3^k}\beta$, for all $e, x, f \in S$ and for all positive integers k, where α and β are any elements of S.

Proof. Using Lemma 2 twice we have

$$(exf)^{3^k} = [(ex)f]^{3^k} = f^{\frac{3^k-1}{2}} (ex)^{3^k} f^{\frac{3^k+1}{2}} = f^{\frac{3^k-1}{2}} e^{\frac{3^k+1}{2}} x^{3^k} e^{\frac{3^k-1}{2}} f^{\frac{3^k+1}{2}} = \alpha x^{3^k}\beta.$$

According to Theorem 1 (see also [6]) the elements $x, y \in S$ are contained in the same class of the MSL decomposition of S if and only if $C(x) = C(y)$, i. e. $C(J(x)) = C(J(y))$. Let S be a K-semigroup. J. E. Kuczkowski ([2]) showed that in such a semigroup $C(J) = \tilde{N}(J)$ if J is an ideal of S. Hence two elements x, y of such a semigroup S are contained in the same class of the MSL decomposition of S if and only if $\tilde{N}(J(x)) = \tilde{N}(J(y))$.

This enables us to prove:

Theorem 5. *Let S be a K-semigroup. Then x and y are contained in the same class of the MSL decomposition of S if and only if there exist positive integers m, n and elements $a, b, c, d \in S$ such that $x^m = ayb$ and $y^n = cxd$.*

Proof. *a*) Let $\tilde{N}(J(x)) = \tilde{N}(J(y))$. Then $x \in \tilde{N}(J(y))$ and $y \in \tilde{N}(J(x))$. Hence $x^m = ayb$ for any positive integer m and any $a, b \in S$ and $y^n = cxd$ for any positive integer n and any $c, d \in S$.

b) Let $z \in \tilde{N}(J(x))$. We shall prove that $z \in \tilde{N}(J(y))$. Clearly $z^r = exf$ for some positive integers r. Let k be a positive integer such that $m \leqslant 3^k$. According to Lemma 3 $z^{r.3^k} = (exf)^{3^k} = \alpha x^{3^k}\beta = \alpha' x^m \beta' = (\alpha' a)y(b\beta')$. Hence $z^{r.3^k} = (\alpha' a)y(b\beta')$ i. e. $z^{r.3^k} \in J(y)$. This implies that $z \in \tilde{N}(J(y))$. In a simiar way it can be proved that $z \in \tilde{N}(J(y))$ implies $z \in \tilde{N}(J(x))$. This gives $\tilde{N}(J(x))$ 1= $= \tilde{N}(J(y))$.

From Theorem 5 we have:

Corollary ([1], [3], [11]). *Let S be a commutative semigroup. Then the elements x and y of S are contained in the same class of the MSL decomposition of S if and only if there exist positive integers m, n and elements $a, b \in S$ such that $x^m = ay$ and $y^n = bx$.*

7. THE SCHWARZ RADICAL OF K-SEMIGROUPS

J. E. Kuczkowski [2] showed that if S is a K-semigroup and J is an ideal of S, then $R(J) \subseteq M(J) = L(J) = R^*(J) = \tilde{N}(J) = C(J)$. The following example shows the existence of a K-semigroup S such that $R(J) \neq \tilde{N}(J)$.

Example. Let S be the semigroup generated by $X = \{O, a, b_1, b_2, b_3, \ldots\}$ subject to the generating relations:

$O . x = x . O = O$, for all $x \in S$

$x^2 = O$, for all $x \in S$

$xyzyx = yxzxy$, for all $x, y, z \in S$.

Clearly $\tilde{N}(\{O\}) = S$. We shall show that the principal ideal $J(a)$ generated by a is not $\{O\}$-potent, i. e. $a \notin R(\{O\})$. Hence $R(\{O\}) \neq \tilde{N}(\{O\})$.

Since $ab_n \in J(a)$ for $n = 1, 2, 3, \ldots$ we have
$(ab_1)(ab_2) \ldots (ab_n) \in (J(a))^n$. However, $(ab_1)(ab_2) \ldots (ab_n) \neq O$. This implies that $J(a)$ is not $\{O\}$-potent.

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