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On Vector Measures in Cartesian Products

Matematický časopis, Vol. 21 (1971), No. 4, 241,242--247

Persistent URL: http://dml.cz/dmlcz/127069

[6, Theorem 33E] that $\mathscr{B}_0(S) \otimes \mathscr{B}_0(T)$, respectively $\mathscr{B}(S) \otimes \mathscr{B}(T)$ consists of the sets of the form

$$G = \bigcup_{i=1}^{k} E_i \times F_i, \tag{1}$$

where $\{E_i \times F_i\}_{i=1}^{k}$ is a finite system of mutually disjoint sets in $\mathscr{B}_0(S) \times \mathscr{B}_0(T)$, respectively in $\mathscr{B}(S) \times \mathscr{B}(T)$. The symbol $\mathscr{B}_0(S) \otimes_\sigma \mathscr{B}(T)$, respectively $\mathscr{B}(S) \otimes_\sigma \mathscr{B}(T)$ stands for the sigma ring generated by $\mathscr{B}_0(S) \otimes \mathscr{B}_0(T)$, respectively $\mathscr{B}(S) \otimes \mathscr{B}(T)$.

Let Z denote a locally convex topological vector space, its topology being determined by a system $\{|\cdot|_p\}_{p\in P}$ of seminorms. Denote by $\check{Z}$ a completion of Z and by Z' the dual space of Z.

3. Results. If l is an additive function on $\mathscr{B}_0(S) \times \mathscr{B}_0(T)$, respectively on $\mathscr{B}(S) \times \mathscr{B}(T)$ with values in Z, then on $\mathscr{B}_0(S) \otimes \mathscr{B}_0(T)$, respectively on $\mathscr{B}(S) \otimes \mathscr{B}(T)$ there exists one and only one additive set function n with values in Z such that n coincides with l on $\mathscr{B}_0(S) \times \mathscr{B}_0(T)$, respectively $\mathscr{B}(S) \times \mathscr{B}(T)$. The function n is defined by the equality

$$n(G) = \bigcup_{i=1}^{k} l(E_i \times F_i), \tag{2}$$

for every set G in $\mathscr{B}_0(S) \otimes \mathscr{B}_0(T)$, respectively $\mathscr{B}(S) \otimes \mathscr{B}(T)$ of the form (1). The proof is similar to the proof in case l is a real valued set function [6, Exercise 8.5].

We now proceed axiomatically, assuming the following axiom:

Axiom A. *There exists, for each $z' \in Z'$, a positive number $K(z') < \infty$ such that*

$$|z' \circ n(G)| \leqq K(z'),$$

for an arbitrary set G in $\mathscr{B}_0(S) \otimes \mathscr{B}_0(T)$ or in $\mathscr{B}(S) \otimes \mathscr{B}(T)$.

We emphasize that it is possible to exhibit a counter-example in which H is a separable real Hilbert space, $S = T = [0, 1]$, $m : \mathscr{B}[0, 1] \to H$ is regular Borel vector-valued measure, hence bounded on $\mathscr{B}[0, 1]$ [5, III. 4. 5], however the function n, corresponding to the function l, defined by means of a scalar product $(.,.)$ on H, i. e. $l(E \times F) = (m(E), m(F))$, is not bounded over $\mathscr{B}([0, 1]) \otimes \mathscr{B}([0, 1])$. It follows that, in generally, Axiom A is not satisfied. The corresponding measure m can be constructed using a result by D. A. Edwards, who proved the following [14, Theorem 3].

There exists a vector-valued function z defined on $[0, 1]$ and such that

(i) *the range of z spans a separable real Hilbert space H,*

(ii) *z is strongly continuous on $[0, 1]$,*

(iii) *z is of Dunford bounded variation on* [0, 1],

(iv) *the function $u : R^2 \to R^1$ defined by*

$$u(t, s) = (z(t), z(s))$$

is of Fréchet bounded variation but is not of Vitali bounded variation on $[0, 1] \times [0, 1]$.

If we take arbitrary real numbers a and b such that we have $0 \leqslant a \leqslant b \leqslant 1$ and put

$$m((a, b]) = z(b) - z(a),$$

then we can obtain a unique extension of m to a vector-valued measure $\overline{m} : \mathscr{B}([0, 1]) \to H$ [9, Theorem 5.1]. Since the function u is not of Vitali bounded variation on $[0, 1] \times [0, 1]$, the set function n obtained from the function l, defined by

$$l(E \times F) = (\overline{m}(E), \overline{m}(F)), \ E, F \in \mathscr{B}([0, 1]),$$

using the equality (2), can not be bounded over $\mathscr{B}([0, 1]) \otimes \mathscr{B}([0, 1])$.

A Borel vector-valued measure $m : \mathscr{B}(S) \to Z$ is said to be regular if, for every $z' \in Z'$, the complex measure $E \to \langle m(E), z' \rangle$, $E \in \mathscr{B}(S)$, is regular, i. e. its variation is regular in the sense of [6]. It is known that every Baire vector-valued measure $m_0 : \mathscr{B}_0(S) \to Z$ is regular and can be extended uniquely to a regular Borel vector-valued measure $m : \mathscr{B}(S) \to \check{Z}$ and every additive regular vector-valued set function on the ring $\mathscr{R}(S)$ with values in Z is sigma additive [3, (R_1), Theorem 3], [11].

Theorem 1. *Let l be a set function defined on $\mathscr{B}(S) \times \mathscr{B}(T)$ with the following properties:*

(i) *the values of l are in Z,*

(ii) *for every $E \in \mathscr{B}(S)$ the function ${}_E l$ is additive and regular, hence sigma additive on $\mathscr{B}(T)$,*

(iii) *for every $F \in \mathscr{B}(T)$ the function l_F is additive and regular, hence sigma additive on $\mathscr{B}(S)$.*

Then if Axiom A holds, the function n is regular and hence weakly sigma additive on $\mathscr{B}(S) \otimes \mathscr{B}(T)$. Moreover, there exists a unique extension of n to a measure $\overline{n}$ defined on $\mathscr{B}(S) \otimes_\sigma \mathscr{B}(T)$ with values in Z'' and sigma additive for the topology $\sigma(Z'', Z')$.

Proof. We shall show that the function n is regular on $\mathscr{B}(S) \otimes \mathscr{B}(T)$. Take any $z' \in Z'$. Then the function $z' \circ n$ satisfies the assumptions of the Corollary 3 proved by Kluvánek [8] and hence $z' \circ n$ is regular and sigma additive on $\mathscr{B}(S) \otimes \mathscr{B}(T)$ for every z'. Since $n: \mathscr{B}(S) \otimes \mathscr{B}(T) \to Z$ is weakly sigma additive and weakly bounded (Axiom A) thus by the result of Métivier

[12], [13] there exists a unique extension of n to a measure $\bar{n}$ defined on $\mathscr{B}(S) \otimes_\sigma \mathscr{B}(T)$ with values in Z'' and sigma additive for the topology $\sigma(Z'', Z')$.

Theorem 2. *Let l be a function defined on $\mathscr{B}_0(S) \times \mathscr{B}_0(T)$ with the following properties*:

(i) *the values of l are in Z,*

(ii) *for every $E \in \mathscr{B}_0(S)$ the function ${}_El$ is a Baire vector-valued measure on $\mathscr{B}_0(T)$.*

(iii) *for every $F \in \mathscr{B}_0(T)$ the function l_F is a Baire vector-valued measure on $\mathscr{B}_0(S)$.*

Then if Axiom A holds, n is regular and hence weakly sigma additive on $\mathscr{B}_0(S) \otimes \mathscr{B}_0(T)$. Moreover, there exists a unique extension of n to a measure defined on $\mathscr{B}_0(S \times T)$ with values in Z'' and sigma additive for the topology $\sigma(Z'', Z')$.

The proof follows from the proof of Theorem 1 and from the fact that every vector-valued Baire measure is regular [3], [11].

Theorem 3. *Let the assumptions* (i), (ii), (iii) *of Theorem* 1 *be satisfied. Let the function n satisfy Axiom A. Let the space Z be weakly sequentially complete.*

Then on $\mathscr{B}(S \times T)$ there exists one and only one regular Borel measure $\bar{n}$ with values in Z such that $\bar{n}(E \times F) = n(E \times F) = l(E \times F)$ for $E \times F \in \mathscr{B}(S) \times \times \mathscr{B}(T)$. In particular, l is sigma additive on $\mathscr{B}(S) \times \mathscr{B}(T)$.

Proof. a) Since n is bounded on $\mathscr{B}(S) \otimes \mathscr{B}(T)$ and Z is weakly sequentially complete there exists one and only one measure again denoted by n defined on the sigma ring $\mathscr{B}(S) \otimes_\sigma \mathscr{B}(T)$ generated by $\mathscr{B}(S) \otimes \mathscr{B}(T)$ with values in Z [9, Theorem 5.1].

b) Now as for each compact set C_1 in S, the correspondence

$$F \to n(C_1 \times F), \quad F \in \mathscr{B}(T),$$

is the regular vector-valued Borel measure on $\mathscr{B}(T)$ with values in Z, and for each compact set C_2 in T, the correspondence

$$E \to n(E \times C_2), \quad E \in \mathscr{B}(S),$$

is the regular Borel vector-valued measure on $\mathscr{B}(S)$ with values in Z, thus n can be extended to the one and only regular Borel measure $\bar{n}$ on $\mathscr{B}(S \times T)$ with values in Z [4, Theorem 3], (cf. also [1]).

Remark. Note that if one of the spaces S or T is such that every bounded subspace is second countable (equivalently, in view of the Urysohn theorem, each bounded subspace of S, resp. T is metrisable), especially if one of the spaces S or T is metrisable, then

$$\mathscr{B}(S \times T) = \mathscr{B}(S) \otimes_\sigma \mathscr{B}(T), \qquad \text{[7, Theorem 8.1 and Corollary]},$$

i. e. the Borel sets in S and T „multiply". In this case it suffices to give only the part a) of the proof of Theorem 3.

Let $\mathscr{R}$ be a ring of sets and $m : \mathscr{R} \to Z$ a vector-valued measure. The p-variation of m over E is the set function $|m|_p$ defined for every seminorm $|\cdot|_p$ by the relation

$$|m|_p(E) = \sup \sum_{i=1}^{k} |m(E_i)|_p, \quad E \in \mathscr{R},\ p \in P,$$

where the supremum is taken for all finite disjoint families $\{E_i\} \subset \mathscr{R}$ such that $\bigcup_{i=1}^{k} E_i = E$.

Theorem 4. *Let the assumptions* (i), (ii), (iii) *of Theorem* 1 *be satisfied. Let the function n have a bounded p-variation for every $p \in P$ on $\mathscr{B}(S) \otimes \mathscr{B}(T)$. Let the space Z be sequentially complete.*

Then on $\mathscr{B}(S \times T)$ there exists one and only one regular Borel measure $\bar{n}$ with values in Z such that $\bar{n}(E \times F) = l(E \times F)$ for $E \times F \in \mathscr{B}(S) \times \mathscr{B}(T)$ and $\bar{n}$ has a bounded p-variation for every $p \in P$.

Proof. It is easy to see that $|n|_p$ is a bounded nonnegative measure on $\mathscr{B}(S) \otimes \mathscr{B}(T)$ with the property: $|n(G)|_p \leqslant |n|_p(G)$ for every $G \in \mathscr{B}(S) \otimes \mathscr{B}(T)$. Hence by [9, Theorem 4.2] there exists one and only one measure $\bar{n}$ on $\mathscr{B}(S) \otimes_\sigma \mathscr{B}(T)$ extending n and we have $|n(H)|_p \leqslant |n|_p(H)$ for every $H \in \in \mathscr{B}(S) \otimes_\sigma \mathscr{B}(T)$. Now we may proceed as in the part b) of the proof of Theorem 3.

Using [9, Theorem 4.2] and [4] we may prove the following.

Theorem 5. *Let the assumptions* (i), (ii), (iii) *of Theorem* 1 *be satisfied and let Z be sequentially complete. Let there exist for every $p \in P$ a bounded nonnegative measure ν_p on $\mathscr{B}(S) \otimes \mathscr{B}(T)$ such that $\nu_p(G) \to 0$, $G \in \mathscr{B}(S) \otimes \mathscr{B}(T)$ implies $|n(G)|_p \to 0$.*

Then on $\mathscr{B}(S \times T)$ there exists one and only one regular Borel measure $\bar{n}$ with values in Z extending n.

4. The case of Banach spaces. For the rest of the paper suppose that Z is a Banach space. In this case we have the following.

Theorem 6. *Let the assumptions* (i), (ii), (iii) *from Theorem* 1 *be satisfied. Let the set $M = \{n(G) : G \in \mathscr{B}(S) \otimes \mathscr{B}(T)\}$ be conditionally weakly compact in Z.*

Then n can be extended to a regular Borel measure on $\mathscr{B}(S \times T)$ with values in Z, in particular, l is sigma additive on $\mathscr{B}(S) \otimes \mathscr{B}(T)$.

Proof. Since M is conditionally weakly compact, M is bounded and therefore Axiom A holds. Moreover, by the result of Kluvánek [9] the set function n can be extended to a measure on the sigma ring $\mathscr{B}(S) \otimes_\sigma \mathscr{B}(T)$ with values in Z and this measure can be extended uniquely to a regular Borel vector-valued measure on $\mathscr{B}(S \times T)$ with values in Z [4].

The following theorem is a generalization of Theorem 3.

Theorem 7. *Let Z be a Banach space containing no subspace isomorphic to* c_0 (e. g. *a weakly complete space*). *Let the assumptions* (i), (ii), (iii) *of Theorem* 1 *be satisfied.*

Then if Axiom A holds there exists a unique vector-valued regular Borel measure on $\mathscr{B}(S \times T)$ extending l. In particular, l is sigma additive on $\mathscr{B}(S) \times \times \mathscr{B}(T)$.

Proof. The set function $z' \circ n$ is sigma additive and of bounded variation on $\mathscr{B}(S) \otimes \mathscr{B}(T)$ for every $z' \in Z'$. Take any sequence (E_i) of mutually disjoint sets in $\mathscr{B}(S) \otimes \mathscr{B}(T)$. Since $z' \circ n$ can be extended to a measure $\overline{z' \circ n}$ on $\mathscr{B}(S) \otimes_\sigma \mathscr{B}(T)$, the series $\sum_{i=1}^{\infty} z'n(E_i)$ is weakly unconditionally convergent and hence by the result of Bessaga and Pelczyński [2, Theorem 5] the series $\sum_{i=1}^{\infty} n(E_i)$ is unconditionally convergent, and thus by the result of Kluvánek [10, Theorem] n can be extended to a measure $\bar{n}$ on $\mathscr{B}(S) \otimes_\sigma \mathscr{B}(T)$ with values in Z. Now we may proceed similarly as in proving the preceding theorem.

Added in proof.

The topic of this paper is related to that of the paper of R. M. Dudley and Lewis Pakula, *A counter-example on the inner product of measures* (preprint). They consider the measures m and n with values in a real Hilbert space H, and their inner product (m, n). M. Dudley and L. Pakula say that P. Masani asked in September 1970 whether (m, n) necessarily has a countably additive extension to the sigma ring $\mathscr{S} \otimes_\sigma \mathscr{T}$. The counter-example given in our paper shows that the answer is in the negative. In the paper of Dudley and Pakula another counter-example is given. Their counter-example shows that there exist countable additive measures m and n with values in a separable real Hilbert space H such that both m and n are purely atomic with countably many atoms and orthogonal values on disjoint sets. The inner product (m, n) is unbounded both above and below, is not countably additive on $\mathscr{S} \otimes \mathscr{T}$ and hence has no countably additive extension to the sigma ring $\mathscr{S} \otimes_\sigma \mathscr{T}$. The measures m and n are so called "orthogonally scattered" measures, cf. P. Masani, *Orthogonally scattered measures*, Advances in Math. *2* (1967), 61—117. Orthogonally scattered Hilbert-values measures seem to be among the best behaved Banach-valued measures with infinite total variation. Thus our example and that of Dudley and Pakula suggest that it may be difficult to find any reasonable broad conditions under which (m, n) would be countably additive while $|m|(S) = |n|(T) = +\infty$.

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Received January 2, 1969

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ROČNÍK 21 1971 ČÍSLO 4

ON VECTOR MEASURES IN CARTESIAN PRODUCTS

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1. Introduction. Let $\mathscr{S}$ and $\mathscr{T}$ be sigma rings of subsets of the sets S, T respectively. Let l be a finite nonnegative set function defined on the system $\mathscr{S} \times \mathscr{T}$ of all sets of the form $E \times F$, $E \in \mathscr{S}$, $F \in \mathscr{T}$. Assume that $l(. \times F)$, for every fixed $F \in \mathscr{T}$, is sigma additive on $\mathscr{S}$ as a function of E and, for every fixed $E \in \mathscr{S}$, $l(E \times .)$ is sigma additive on $\mathscr{T}$ as a function of F. If S and T are locally compact (Hausdorff) spaces, $\mathscr{S}$ and $\mathscr{T}$ sigma rings of Borel sets in S, T respectively, and $l(. \times F)$ for every fixed $F \in \mathscr{T}$, resp. $l(E \times .)$ for every fixed $E \in \mathscr{S}$, is regular [6], then

(1) l is sigma additive on $\mathscr{S} \times \mathscr{T}$ as a function of $E \times F$, and

(2) l can be extended to a sigma ring of Borel sets in $S \times T$, this extension being a regular Borel measure on $S \times T$ [8].

In this article we investigate several cases in which analogues of (1) and (2) can be obtained for the set function on $\mathscr{S} \times \mathscr{T}$ with values in a locally convex topological vector space Z. This is the case, for example, if Axiom A is satisfied and Z is weakly sequentially complete.

2. Notations and definitions. In the following S and T stand for locally compact (Hausdorff) topological spaces. Further $\mathscr{B}_0(S)$, respectively $\mathscr{B}(S)$ denotes a sigma ring of all Baire, resp. Borel sets in S. Similarly $\mathscr{B}_0(T)$ and $\mathscr{B}(T)$ in T, and $\mathscr{B}_0(S \times T)$ and $\mathscr{B}(S \times T)$ in $S \times T$.

Denote

$$\mathscr{B}_0(S) \times \mathscr{B}_0(T) = \{E \times F: E \in \mathscr{B}_0(S),\ F \in \mathscr{B}_0(T)\},$$

$$\mathscr{B}(S) \times \mathscr{B}(T) = \{E \times F: E \in \mathscr{B}(S),\ F \in \mathscr{B}(T)\}.$$

If l is a set function on $\mathscr{B}_0(S) \times \mathscr{B}_0(T)$, then for any $E \in \mathscr{B}_0(S)$ the symbol ${}_El$ denotes the function on $\mathscr{B}_0(T)$ defined by the equality ${}_El(F) = l(E \times F)$. Similarly for $F \in \mathscr{B}_0(T)$ we denote l_F the function on $\mathscr{B}_0(S)$ for which $l_F(E) = l(E \times F)$. We use similar notations for a function l on $\mathscr{B}(S) \times \mathscr{B}(T)$ [cf. 8].

Let us denote by $\mathscr{B}_0(S) \otimes \mathscr{B}_0(T)$, respectively $\mathscr{B}(S) \otimes \mathscr{B}(T)$ the smallest ring containing $\mathscr{B}_0(S) \times \mathscr{B}_0(T)$, respectively $\mathscr{B}(S) \times \mathscr{B}(T)$. It is well known