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ON A REPRESENTATION OF LATTICES BY CONGRUENCE RELATIONS[1])

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Introduction

In this paper we mean by a lattice $\mathscr{L}$ always a lattice with the least element 0 and the greatest element 1. Given a set A, $\mathscr{E}(A)$ denotes the lattice of all equivalence relations on A and Δ its least element.

Definition 1 [6]. *A lattice $\mathscr{L}$ $(0,1 \in \mathscr{L})$ is said to be strongly representable as a congruence lattice if whenever $\mathscr{L}$ is isomorphic to a sublattice $\mathscr{L}'$ of $\mathscr{E}(A)$ for some A, where $\Delta, A \times A \in \mathscr{L}'$, then there is an algebra based on A whose congruence lattice is $\mathscr{L}$.'*

In [6] it is shown that every finite distributive lattice is strongly representable. The above notion of strong representability seems to be designed for finite lattices. The class of infinite strongly representable lattices is relatively small. E. g. the infinite chain $a_0 < a_1 < a_2 < \ldots < u$ is not strongly representable because it suffices to find a chain of equivalence relations $\Delta < \alpha_1 < < \alpha_2 < \ldots < A \times A$ on a set A, such that $\bigvee_{i=1}^{\infty} \alpha_i \neq A \times A$. There is no algebra based on A having this chain as a congruence lattice. Moreover the stronger assertion holds, see Theorem 1 below. The following definition seems to be useful.

Definition 2. *A complete algebraic lattice* [2] *$\mathscr{L}$ is said to be quasi strongly representable as a congruence lattice if whenever $\mathscr{L}$ is isomorphic to a closed sublattice* [2] *$\mathscr{L}'$ of $\mathscr{E}(A)$ for some A, where Δ and $A \times A$ belong to $\mathscr{L}'$, then there is an algebra based on A whose congruence lattice is $\mathscr{L}'$.*

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[1]) A part of the results of this paper has been presented in the Summer Session On the Theory of Ordered Sets and General Algebra held at Horní Lipová 1972 and will appear without proofs in a special number of Acta Fac. rerum natur. Univ. Comenianae Math. 1973.

If $\mathscr{L}$ is a closed sublattice of $\mathscr{E}(A)$ containing Δ and if $a, b \in A$, $a \neq b$, we denote $\alpha(a, b)$ the equivalence-theoretic join of all $\gamma \in \mathscr{L}$ such that $(a, b) \notin \gamma$. We shall use the following obvious corollary of Armbrust's theorem [1, Th.3]:

Theorem A. *Let $\mathscr{L}$ be a closed sublattice of $\mathscr{E}(A)$ containing Δ. There is an algebra $\mathfrak{A}$ on A with unary and two-valued operations* [1] *whose congruence lattice is $\mathscr{L}$ if and only if*
(1) *for every equivalence relation $\beta \in \mathscr{E}(A)$ if $\beta \leqq \alpha(a, b)$ for all $a, b \in A$ such that $(a, b) \notin \beta$, then $\beta \in \mathscr{L}$.*

Definition 3 (see e. g. [2, p. 128]). *A complete lattice $\mathscr{L}$ is called Brouwerian if the identity $a \wedge \bigvee \{b_i : i \in I\} = \bigvee \{(a \wedge b_i) : i \in I\}$ holds (for an arbitrary set I) in $\mathscr{L}$. For the definition of a general Brouwerian lattice see* [2, p. 45].

Definition 4 (see e. g. [2, p. 119]). *A complete lattice $\mathscr{L}$ is called completely distributive if the identity*[2]
$$\bigwedge \{ \bigvee \{a_{ij} : j \in I_i\} : i \in K\} = \bigvee \{ \bigwedge \{a_{i\varphi(i)} : i \in K\} : \varphi \in \prod_{i \in K} I_i\}$$
(or the dual one) holds (for arbitrary sets I_i, K) in $\mathscr{L}$.

Definition 5. *Let $\mathscr{L}$ be a complete lattice. An element a is called completely join irreducible*[3]) *if $a = \bigvee \{x_i : i \in I\}$ implies $x_i = a$ for some $i \in I$ (I is an arbitrary set).*

Definition 6 [5, Chap. 2, Problem 4(*f*)]. *A lattice $\mathscr{L}$ is called weakly atomic if it has the property: if $a < b$, $a, b \in \mathscr{L}$, then c and d exist in $\mathscr{L}$ such that $a \leqq c < d \leqq b$ and $\{x \in \mathscr{L} : c < x < d\}$ is the empty set.*

Results

Theorem 1. *No infinite distributive lattice $\mathscr{L}$ is strongly representable.*

Lemma 1. *Every lattice $\mathscr{L}$ is isomorphic to a sublattice $\mathscr{L}'$ of $\mathscr{E}(A)$ for some A, such that $\Delta, A \times A \in \mathscr{L}'$.*

Corollary 1. *Every strongly representable lattice is quasi strongly representable.*

Remark 1. From Theorem 1, Corollary 1 and Corollary 2 below it follows

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2) In a complete lattice this identity is equivalent to its dual (G. N. Raney, Completely distributive complete lattices, Proc. Amer. Math. Soc., *3*, 1952, 677—680; see also [2, p. 120]).

3) In [2], or in [5], the notion "strictly join irreducible", or "join irreducible", is used.

that the class of quasi strongly representable lattices is larger than that of strongly representable lattices.

Theorem 2. *Let $\mathscr{L}$ be a completely distributive closed sublattice of $\mathscr{E}(A)$ containing Δ and $A \times A$. Then there is an algebra $\mathfrak{A}$ based on A whose congruence lattice $\mathscr{C}(\mathfrak{A}) = \mathscr{L}$. The algebra $\mathfrak{A}$ can be chosen in such a way that all its operations are unary and two-valued* [1].

Corollary 2. *Every complete algebraic and completely distributive lattice is quasi strongly representable.*

Remark 2. If $\mathscr{L}$ is a closed sublattice of $\mathscr{E}(A)$ containing Δ and $A \times A$ which is distributive but not completely distributive then it is possible that there is no algebra $\mathfrak{A}$ based on A having only unary and two-valued operations such that the congruence lattice $\mathscr{C}(\mathfrak{A}) = \mathscr{L}$, as the following example shows.

Example 1. Let $\mathfrak{A} = \langle A; +, .\rangle$ be the ring of all integers. It is known that the congruence lattice $\mathscr{C}(\mathfrak{A})$ of $\mathfrak{A}$ is Brouwerian but not dually Brouwerian, hence not completely distributive. We will show that the condition (1) of Theorem A is not fulfilled. Consider an arbitrary equivalence relation β on A not belonging to $\mathscr{C}(\mathfrak{A})$. For every $(a, b) \notin \beta$ there are only finite numbers of congruence relations $\gamma \in \mathscr{C}(\mathfrak{A})$ such that $(a, b) \in \gamma$. Hence $\alpha(a, b) = A \times A$. Thus $\beta \leqq \alpha(a, b)$ for all $a, b \in A$ such that $(a, b) \notin \beta$ but $\beta \notin \mathscr{C}(\mathfrak{A})$.

Remark 3. Note that the lattice of all congruence relations of an algebra with only unary two-valued operations need not be distributive as the following example shows: Let A be a set, $a, b \in A$, $a \neq b$. Denote by $\mathscr{L}$ the lattice consisting of Δ and of all equivalence relations δ on A such that $(a, b) \in \delta$. Let F be the set of all unary two-valued operations such that if $f \in F$ and $x \in A$, then $f(x) = a$ or $f(x) = b$. Then $\mathscr{L}$ is the congruence lattice of the algebra $\mathfrak{A} = \langle A; F\rangle$ and it is not distributive if card $A > 3$.

Theorem 3. *Let $\mathscr{L}$ be a complete Brouwerian lattice in which every element is a join of completely join irreducible elements. Then $\mathscr{L}$ is completely distributive.*

Corollary 3. *Let $\mathscr{L}$ be a complete Brouwerian atomic lattice. Then $\mathscr{L}$ is completely distributive.*

Corollary 4. *Let $\mathscr{L}$ be a Brouwerian lattice satisfying the descending chain condition (DCC). Then $\mathscr{L}$ is complete and completely distributive. In particular, every algebraic* [2] *distributive lattice $\mathscr{L}$ satisfying DCC is completely distributive.*

Theorem 4. *Let $\mathscr{L}$ be a complete dually Brouwerian lattice which is weakly atomic. Then every element of $\mathscr{L}$ is a join of completely join irreducible elements*

Corollary 5. *Let $\mathscr{L}$ be a complete Brouwerian, dually Brouwerian and weakly atomic lattice. Then $\mathscr{L}$ is completely distributive.*

Remark 4. The converse assertion does not hold. For example, the interval $[0, 1]$ of real numbers with the usual ordering is a completely distributive lattice but not weakly atomic.

Theorem 5. *Let $\mathscr{L}$ be a complete Brouwerian and weakly atomic lattice. Then the following conditions are equivalent:*
(2) *$\mathscr{L}$ is dually Brouwerian.*
(3) *$\mathscr{L}$ is completely distributive.*
(4) *Every element of $\mathscr{L}$ is a join of completely join irreducible elements.*

Theorem 6. *Let $\mathscr{L}$ be a complete algebraic lattice. Then the conditions* (2), (3), (5), *are equivalent, where*
(5) *$\mathscr{L}$ is distributive and every element of $\mathscr{L}$ is a join of completely join irreducible elements.*

Corollary 6. *Let $\mathscr{L}$ be a complete algebraic lattice satisfying one of the conditions* (2), (5) *of Theorem* 6. *Then $\mathscr{L}$ is quasi strongly representable. The algebra can be chosen in the same way as in Theorem 2.*

Corollary 7. *Let $\mathscr{L}$ be an algebraic distributive lattice satisfying DCC. Then $\mathscr{L}$ is quasi strongly representable.*

Proofs

Proof of Theorem 1. Let $\mathscr{L}$ be an infinite distributive lattice. $\mathscr{L}$ is isomorphic to a lattice $\langle \mathscr{S}, \cap, \cup \rangle$ of sets, where $\cap$ and $\cup$ denote the set--theoretic intersection and union (see e. g. [2]). We can suppose that the least element of $\mathscr{S}$ is the empty set $\emptyset$ (if this element is $U \neq \emptyset$, it suffices to replace every element $A \in \mathscr{S}$ by $A - U$). Let M be the greatest element of $\mathscr{S}$ and $u \notin M$. We associate with each $A \in \mathscr{S}$ the equivalence relation $\bar{A}$ on $M \cup \{u\}$ all blocks of which are one-element blocks except the block $A \cup \{u\}$. It can be easily seen that the equivalence relations $\bar{A}$ $(A \in \mathscr{S})$ form a lattice $\langle \mathscr{S}_1, \ , \vee \rangle$ isomorphic to $\mathscr{S}$ ($\wedge$ and $\vee$ are equivalence-theoretic meet and join). Because $\mathscr{S}_1$ is infinite and distributive there is an infinite chain $\{\bar{A}_n : n \in N\}$ (N is the set of all natural numbers) in $\mathscr{S}_1$ such that a) $\bar{A}_n < \bar{A}_{n+1}$ for each n or b) $\bar{A}_n > \bar{A}_{n+1}$ for each n. If one of the elements $\bigvee \{\bar{A}_n : n \in N\} = \bar{B}$, $\bigwedge \{\bar{A}_n : n \in N\} = \bar{C}$ (equivalence-theoretic join and meet) does not belong to $\mathscr{S}_1$, then $\mathscr{S}_1$ cannot be a congruence lattice of an algebra based on $M \cup \{u\}$. Suppose $\bar{B} \in \mathscr{S}_1$ (or $\bar{C} \in \mathscr{S}_1$). In case a) there is a dual prime ideal D in $\mathscr{S}_1$ which contains $\bar{B}$ and does not meet the set $\{\bar{A}_n : n \in N\}$ (by M. H.

Stone's theorem, see e. g. [3]). The set $J = \mathscr{S}_1 - D$ forms a prime ideal. In case b) there is a prime ideal J containg $\bar{C}$ which does not meet $\{\bar{A}_n : n \in N\}$. Its complement $D = \mathscr{S}_1 - J$ is a dual prime ideal. We associate with each equivalence relation $\bar{A} \in \mathscr{S}_1$ the equivalence relation A^* on $M \cup \{u, v\} = M_1$ ($v \neq u$, $v \notin M$) all blocks of which are one-element blocks except the block $A \cup \{u, v\}$ if $\bar{A} \in D$ and except the block $A \cup \{u\}$ if $\bar{A} \in J$. One can easily verify that the correspondence $\bar{A} \mapsto A^*$ is a lattice isomorphism of the lattice $\mathscr{S}_1$ with the sublattice $\mathscr{S}_2$ of $\mathscr{E}(M_1)$ consisting of all A^*, $\bar{A}$ in $\mathscr{S}_1$. Moreover the equivalence-theoretic join $\bigvee \{A_n^* : n \in N\}$ in case a), or the equivalence-theoretic meet $\bigwedge \{A_n^* : n \in N\}$ in case b) does not belong to $\mathscr{S}_2$. Hence there is no algebra based on M_1 whose congruence lattice is $\mathscr{S}_2$. This proves the Theorem.

Proof of Lemma 1. Without the conditions $0,1 \in \mathscr{L}$ and Δ, $A \times A \in \mathscr{L}'$ this Lemma is proved in [7] and [4]. The proof in [4] is based on the following assertions:

(i) To every lattice $\mathscr{L}$ a set A and a mapping $F : \mathscr{L} \to \mathscr{E}(A)$ exist such that $F(x \wedge y) = F(x) \wedge F(y)$ for all x, $y \in \mathscr{L}$. Such a representation is called a weak representation.

(ii) Let $F : \mathscr{L} \to \mathscr{E}(A)$ be a weak representation of a lattice $\mathscr{L}$. Then there is a set T and a mapping $G : \mathscr{L} \to \mathscr{E}(T)$ such that G is a lattice isomorphism, $A \subset T$ and $G(x) \cap (A \times A) = F(x)$ for each $x \in \mathscr{L}$.

By a detailed inspection of the construction of the proof of (ii) in [4] one can easily state that if in (ii) $F(0) = \Delta$, $F(1) = A \times A$, then $G(0) = \Delta$ and $G(1) = T \times T$ too. Hence it suffices to show that a weak representation F of $\mathscr{L}$ ($0,1 \in \mathscr{L}$) exists such that $F(0) = \Delta$, $F(1) = A \times A$. This can be easily done in a simple way: For each $x \in \mathscr{L}$ let $F(x)$ be the equivalence relation in $\mathscr{L}$ all blocks of which are one-element blocks except the block $\{y : 0 \leqq y \leqq x\}$.

Proof of Corollary 1. From the assumption and Lemma 1 we get that $\mathscr{L}$ is isomorphic to a congruence lattice of an algebra. Hence $\mathscr{L}$ is complete and algebraic. The rest is obvious.

Proof of Theorem 2. It suffices to show that $\mathscr{L}$ satisfies condition (1) of Theorem A. Let $\beta \in \mathscr{E}(A)$ and $\beta \leqq \alpha(a, b)$ for all $a, b \in A$ such that $(a, b) \notin \beta$. Denote $M = \{(a, b) \in A \times A : (a, b) \notin \beta\}$. If $(a, b) = m$, we shall write $\alpha(m)$ instead of $\alpha(a, b)$. Then $\beta \leqq \bigwedge \{\alpha(m) : m \in M\}$. For each $m \in M$, $\alpha(m) = \bigvee \{\alpha_m^i : i \in I_m\}$, where $\alpha_m^i \in \mathscr{L}$ and $m \notin \alpha_m^i$ for each $i \in I_m$. Using the completely distributive law we get:

$$\beta \leqq \bigwedge \{ \bigvee \{\alpha_m^i : i \in I_m\} : m \in M\} =$$
$$= \bigvee \{ \bigwedge \{\alpha_m^{\varphi(m)} : m \in M\} : \varphi \in \prod_{m \in M} I_m\}.$$

Obviously if $(a, b) \notin \beta$ (i. e. $(a, b) = k \in M$), then for every $\varphi \in \prod_{m \in M} I_m$, $(a, b) \notin$ $\notin \bigwedge \{\alpha_m^{\varphi(m)} : m \in M\}$. It follows $\bigwedge \{\alpha_m^{\varphi(m)} : m \in M\} \leqq \beta$ for every $\varphi \in \prod_{m \in M} I_m$. Hence $\bigvee \{ \bigwedge \{\alpha_m^{\varphi(m)} : m \in M\} : \varphi \in \prod_{m \in M} I_m\} \leqq \beta$, thus $\beta \in \mathscr{L}$.

Proof of Corollary 2. It follows from Theorem 2.

Proof of Theorem 3. It suffices to prove that

$$\text{(a)} \quad b = \bigwedge \{ \bigvee \{a_{ij} : j \in I_i\} : i \in K\} \leqq$$
$$\leqq \bigvee \{ \bigwedge \{a_{i\varphi(i)} : i \in K\} : \varphi \in \prod_{i \in K} I_i\} = c,$$

because the converse inequality holds in any complete lattice. To prove (a) we shall show that every completely join irreducible element which is below b is below c too. Let d be completely join irreducible and $d \leqq b$. Then for each $i \in K$, $d \leqq \bigvee \{a_{ij} : j \in I_i\}$. Since $\mathscr{L}$ is Brouwerian $d = d \wedge \bigvee \{a_{ij} : j \in I_i\}$ $= \bigvee \{(d \wedge a_{ij}) : j \in I_i\}$ for each $i \in K$. Using completely join irreducibility of d we get that $d \leqq a_{i\Psi(i)}$ for each $i \in K$ and some $\Psi \in \prod_{i \in K} I_i$. Hence $d \leqq c$ and (a) holds.

Proof of Corollary 4. In any lattice satisfying DCC, every element can be expressed as a join of a finite number of join irreducibles (see e. g. [2, Chap. VIII, § 1]) and every complete algebraic distributive lattice is Brouwerian (see e. g. [2. Chap. VIII., § 5, Ex. 9]), hence in both assertions Theorem 3 can be used. Note that the completeness of $\mathscr{L}$ ($1 \in \mathscr{L}$) follows from the assumptions of the Corollary.

Proof of Theorem 4. Let $a \in \mathscr{L}$ and let b the join of all completely join irreducible elements z such that $z \leqq a$. To prove $a = b$ suppose $b < a$. Weak atomicity implies that c and d exist in $\mathscr{L}$ such that $b \leqq c < d \leqq a$ and the set $\{y : c < y < d\}$ is empty. Then d is not completely join irreducible, hence $d = \bigvee \{x_i : i \in I\}$, $x_i < d$ for each $i \in I$. There is some $j \in I$ with $x_j \nleqq c$, hence $c \vee x_j = d$ since d covers c. Denote by Q the set of all such elements x_j. There is the least element $x = \bigwedge \{x_j : x_j \in Q\}$ in Q since $c \vee x = \bigwedge \{(c \vee x_j) : x_j \in Q\} = d$. This element x is completely join irreducible. For if not, then $x = \bigvee \{y_p : p \in P\}$ and $y_p < x$ for each $p \in P$. Let $p \in P$. Obviously $c \leqq c \vee y_p \leqq d$. But $c \vee y_p = d$ contradicts the choice of x. Hence $c \vee y_p = c$ for each $p \in P$, which implies $c \vee x = c$. This contradiction proves x to be completely join irreducible. Since $c \vee x = d$, $x \leqq a$ and $x \nleqq b$, which is impossible.

Proof of Corollary 5. It suffices to use Theorem 4 and Theorem 3.

Proof of Theorem 5. By Theorem 4, (2) implies (4). Using Theorem 3, (4) implies (3). The implication (3) $\Rightarrow$ (2) is trivial.

Proof of Theorem 6. Every complete algebraic lattice is weakly atomic (see e. g. [5, Chap. 2, Problem 4(f)]), hence (2) implies (5) by Theorem 4. Every complete algebraic distributive lattice is Brouwerian (see e. g. [2, Chap. VIII., § 5, Ex. 9]) hence (5) implies (3) by Theorem 3. The last implication (3) $\Rightarrow$ (2) is obvious.

Proof of Corollary 6. By Corollary 2 and Theorem 6.

Proof of Corollary 7. $\mathscr{L}$ is complete because $1 \in \mathscr{L}$. By [2, Chap. VIII., § 1] the condition (5) of Theorem 6 is fulfilled, hence using Corollary 6 our assertion follows.

Added in proof. This paper was accepted for publication before the author knew (written communication of A. Day) that S. Burris, H. Crapo, A. Day, D. Higgs and W. Nichols had proved in another way (unpublished result) that every Brouwerian and dually Brouwerian closed sublattice of $\mathscr{E}(A)$ containing Δ and $A \times A$ is a congruence lattice for some algebra based on A. Moreover, the author was informed that a part of the results of Theorems 3—6 can be deduced from the papers: G. Bruns, Verbandstheoretische Kennzeichung vollständiger Mangeringe, Archiv d. Math. *10*, 1959, 109—112; J. R. Buchi, Representation of complete lattices by sets, Portugaliae Math. *11*, 1952, 151—167. Proofs in these papers are based on a representation of lattices by complete rings of sets.

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