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Jozef Dravecký

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SPACES WITH MEASURABLE DIAGONAL

JOZEF DRAVECKÝ

It is known that the graph of any measurable real-valued function $f : X \to R$ is a measurable set, i.e. $f^{-1}(\mathscr{B}) \subset \mathscr{S}$ implies $\{[x, y];\ x \in X, y = f(x)\} \in \mathscr{S} \times \mathscr{B}$, where $\mathscr{S}$ is a σ-algebra on X and $\mathscr{B}$ is the Borel σ-algebra on the real line R (cf. [3, page 142]). If f maps a separable metric space X into a separable metric space Y, then the above result holds for Borel measurable functions with $\mathscr{S}$ the Borel σ-algebra on X — see [4, Theorem 3.3, page 16], as well as for Lebesgue measurable functions with $\mathscr{S}$ the Lebesgue σ-algebra on X — see [1]. In the special case when the discrete σ-algebra C_Z on a set Z with cardinality $\aleph_1$ is considered, it follows from a theorem by B. V. Rao [5, Theorem 2] that the graph of every $f : Z \to Z$ is measurable (i.e. in $C_Z \times C_Z$).

In this paper we establish a series of necessary and sufficient conditions for a measurable space $(Y, \mathscr{T})$ in order that every measurable mapping into Y have a measurable graph, the most remarkable of them being that Y have a measurable diagonal. The last sufficient condition can be weakened if we consider a given measurable mapping.

To show the applications of the general results for mappings into linearly ordered spaces and topological spaces, in which measurable sets are defined in a natural way, we give necessary and sufficient conditions under which such spaces have measurable diagonals.

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1. Measurable spaces

We call $(X, \mathscr{S})$ a measurable space iff X is a nonempty set and $\mathscr{S}$ a σ-ring on X, that is, a family closed under countable unions and set-theoretic difference such that $\bigcup \mathscr{S} = X$. Clearly every σ-algebra is a σ-ring.

We write $\mathscr{S} = \sigma(\mathscr{E})$ and say that $\mathscr{E}$ generates $\mathscr{S}$ iff $\mathscr{S}$ is the least σ-ring including $\mathscr{E}$. A σ-ring $\mathscr{S}$ is called countably generated iff it has a countable generator.

Let $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ be measurable spaces. A set $M \subset X \times Y$ is called measurable iff it is in the σ-ring $\mathscr{S} \times \mathscr{T}$ generated by the family of all sets $A \times B$ with $A \in \mathscr{S}$, $\mathscr{B} \in \mathscr{T}$. A measurable space $(Y, \mathscr{T})$ will be said to have a measurable diagonal iff $D = \{[x, y] \in Y \times Y;\ x = y\}$ is in $\mathscr{T} \times \mathscr{T}$.

A mapping $f : X \to Y$ is measurable iff $f^{-1}(B) \in \mathscr{S}$ for each $B \in \mathscr{T}$. The graph of f is the set

$$\{[x, y] \in X \times Y;\ y = f(x)\}.$$

We recall now the definition of a $\mathscr{P}$-system in [2].

Definition 1. *Let $(Y, \mathscr{T})$ be a measurable space and let $\mathscr{P} = \{P_n^k;\ \emptyset \neq P_n^k \in \mathscr{T}$, $n \in N_k$, $k = 1, 2, \ldots\}$, where N_k is either the set of all positive integers or a set $\{1, 2, \ldots, i_k\}$. $\mathscr{P}$ is called a $\mathscr{P}$-system on Y iff $\bigcup\{P_n^k;\ n \in N_k\} = Y$ for each $k = 1, 2, \ldots$.*

We shall say that a $\mathscr{P}$-system $\mathscr{P} = \{P_n^k;\ n \in N_k,\ k = 1, 2, \ldots\}$ is disjoint iff for every k and $n \neq m$ we have $P_n^k \cap P_m^k = \emptyset$.

Let $\mathscr{E}$ be a family of subsets of Y. We say that $\mathscr{E}$ separates points of Y iff for any two distinct points u and v in Y there is $E \in \mathscr{E}$ with $u \in E \not\ni v$ or $u \notin E \ni v$. This condition is a generalization of the T_0 separating known in topological spaces. However, it can easily be verified that if $\mathscr{E}$ is a σ-ring or a disjoint $\mathscr{P}$-system on Y, then the separating conditions of the types T_0, T_1 and T_2 are all e ¡uivalent.

Theorem 1. *Let $(Y, \mathscr{T})$ be a measurable space. The following statements are equivalent.*

a *$(Y, \mathscr{T})$ has a measurable diagonal.*
b *There is a countably generated σ-ring $\mathscr{D} \subset \mathscr{T}$ which separates points of Y.*
c *There is a countable family $\mathscr{C} \subset \mathscr{T}$ that separates points of Y.*
d *There is a disjoint $\mathscr{P}$-system on Y that separates points of Y.*
e *For every measurable space $(X, \mathscr{S})$ and every measurable mapping $f : X \to Y$, the graph of f is a measurable set.*

Proof. **a** ⇒ **b**. Let $\mathscr{K}$ denote the family of all sets $K \subset Y \times Y$ having the property that there is a countable $\mathscr{C} \subset \mathscr{T}$ with $K \in \sigma(\mathscr{C}) \times \sigma(\mathscr{C})$. First we show that $\mathscr{K}$ is a σ-ring. Let therefore $K_n \in \mathscr{K}$, $n = 1, 2, \ldots$, hence there are countable families $\mathscr{C}_n \subset \mathscr{T}$ such that $K_n \in \sigma(\mathscr{C}_n) \times \sigma(\mathscr{C}_n)$, $n = 1, 2, \ldots$. It follows that each K_n is in $\sigma(\bigcup_n \mathscr{C}_n) \times \sigma(\bigcup_n \mathscr{C}_n)$ and hence $\bigcup_n K_n \in \sigma(\bigcup_n \mathscr{C}_n) \times \sigma(\bigcup_n \mathscr{C}_n)$, which proves $\bigcup_n K_n \in \mathscr{K}$. The difference $K_1 - K_2$ is in $\sigma(\mathscr{C}_1 \cup \mathscr{C}_2) \times \sigma(\mathscr{C}_1 \cup \mathscr{C}_2)$ and therefore in $\mathscr{K}$.

Now we show that every measurable rectangle $A \times B$, $A, B \in \mathscr{T}$, is in $\mathscr{K}$ Denoting by $\mathscr{A}$ and $\mathscr{B}$, respectively, countable subfamilies of $\mathscr{T}$ with $A \in \sigma(\mathscr{A})$ and $B \in \sigma(\mathscr{B})$ (there are such, cf. (3, page 24]) we infer that $A \times B \in \sigma(\mathscr{A} \cup \mathscr{B}) \times \sigma(\mathscr{A} \cup \mathscr{B})$. Therefore $A \times B \in \mathscr{K}$ and it follows that $\mathscr{T} \times \mathscr{T} \subset \mathscr{K}$, in particular the diagonal $D \in \mathscr{K}$.

Let then $\mathscr{D} \subset \mathscr{T}$ be a countably generated σ-ring such that $D \in \mathscr{D} \times \mathscr{D}$. We prove that $\mathscr{D}$ separates points of Y. If it did not, there would exist distinct points u, $v \in Y$ such that

$$(\forall E \in \mathscr{D}) u \in E \Leftrightarrow v \in E. \tag{1}$$

There is no difficulty in proving that the family $\mathscr{L} = \{L \subset Y \times Y;\ [u, u] \in \in L \Leftrightarrow [u, v] \in L\}$ is a σ-ring. By (1), $\mathscr{L}$ contains all the sets $A \times B$ with A, $B \in \mathscr{D}$ and therefore $\mathscr{L} \subset \mathscr{D} \times \mathscr{D}$. It follows that $D \in \mathscr{L}$, a contradiction.

b $\Rightarrow$ **c**. Suppose that $\mathscr{D}$ is a countably generated σ-ring that separates points of Y. We prove that any generator $\mathscr{C}$ of $\mathscr{D}$ does the same. In fact, if there were distinct points u, $v \in Y$ with $(\forall M \in \mathscr{C})\ u \in M \Leftrightarrow v \in M$, then the σ-ring $\mathscr{M} = \{M \subset Y;\ u \in M \Leftrightarrow v \in M\}$ would include $\mathscr{C}$ and hence also $\mathscr{D}$, which contradicts the assumption that $\mathscr{D}$ separates points of Y.

c $\Rightarrow$ **d**. Let $\mathscr{C} \subset \mathscr{T}$ be a countable family that separates points of Y. We construct a disjoint $\mathscr{P}$-system $\mathscr{P}$ by induction as follows.

Put $\mathscr{C} = \{C_1, C_2, \ldots\}$ and define $\{P^1_n;\ n \in N_1\}$ by enumerating all nonempty sets out of $C_i - \bigcup \{C_j;\ j < i\}$, $i = 1, 2, \ldots$. Evidently $P^1_n \in \mathscr{T}$ for $n \in N_1$, $P^1_n \cap P^1_m = \emptyset$ for $n \neq m$ and $\bigcup\{P^1_n; n \in N_1\} = \bigcup\{C_i; i = 1, 2, \ldots\} = Y$.

The family $\{P^k_n;\ n \in N_k\}$ already defined, form $P^k_n \cap C_{k+1}$ and $P^k_n - C_{k+1}$ for each $n \in N_k$. Enumerating all nonempty sets thus obtained we define $\{P^{k+1}_n;\ n \in N_{k+1}\}$. Again P^{k+1}_n, $n \in N_{k+1}$ are pairwise disjoint measurable sets and $\bigcup\{P^{k+1}_n;\ n \in N_{k+1}\} = \bigcup\{P^k_n;\ n \in N_k\} = Y$. Observe that every P^k_n is either included in C_k or disjoint with it. This implies that $\mathscr{P}$ separates points of Y as well as $\mathscr{C}$ did. In fact, given $u \neq v$ in Y, there is C_k with $u \in C_k \not\ni v$ or $u \notin C_k \ni v$ and there is some $P^k_n \ni u$. Then $u \in P^k_n \subset C_k \not\ni v$ or $u \in P^k_n \subset Y - C_k \not\ni v$.

d $\Rightarrow$ **e**. Let $\mathscr{P} = \{P^k_n;\ n \in N_k,\ k = 1, 2, \ldots\}$ be a disjoint $\mathscr{P}$-system on Y which separates points. Let $(X, \mathscr{S})$ be a measurable space and $f: X \to Y$ a measurable mapping. Put $F = \bigcap\{\bigcup\{f^{-1}(P^k_n) \times P^k_n;\ n \in N_k\};\ k = 1, 2, \ldots\}$. Since f is measurable and each P^k_n is in $\mathscr{T}$, we have $F \in \mathscr{S} \times \mathscr{T}$. It is sufficient to prove that $F = \{[x, y] \in X \times Y;\ y = f(x)\}$. Clearly, for each k and every $x \in X$ there is exactly one n with $f(x) \in P^k_n$. Hence $[x, f(x)] \in f^{-1}(P^k_n) \times P^k_n$ and it follows that $\{[x, y] \in X \times Y;\ y = f(x)\} \subset F$. Now suppose that $[x, y]$ is not in the graph of f, that is, $y \neq f(x)$. Since $\mathscr{P}$ separates (being a disjoint $\mathscr{P}$-system, also in the sense T_1) points of Y, there is some P^k_n such that $f(x) \in P^k_n$ and $y \notin P^k_n$. For that k the pair $[x, y]$ does not belong to any $f^{-1}(P^k_n) \times P^k_n$, since $x \in f^{-1}(P^k_n)$ implies $f(x) \in P^k_n$ and hence $y \notin P^k_n$. Thus we have shown that the graph of f coincides with the measurable set F.

e $\Rightarrow$ a. Since $D = \{[x, y]; y = x\}$ is the graph of the identity mapping on Y, it is $\mathscr{T} \times \mathscr{T}$-measurable.

The proof is complete.

Remark 1. A measurable space $(X, \mathscr{S})$ is sometimes defined to be separable iff $\mathscr{S}$ is countably generated and contains all singletons (cf. [5], [4]). Observe that the statement b in Theorem 1 is equivalent with the following.

b′ *There is a σ-ring $\mathscr{D} \subset \mathscr{T}$ such that $(Y, \mathscr{D})$ is separable.*

In fact, if $\mathscr{D}$ is generated by a countable set $\mathscr{C}$ and separates points, then each $\{y\} = \bigcap\{C \in \mathscr{C}; y \in C\}$ is in $\mathscr{D}$, and conversely, every σ-ring containing singletons separates points.

As a consequence of the last remark we infer that whenever Y is a countable set, $(Y, \mathscr{T})$ has a measurable diagonal if and only if $\mathscr{T}$ contains all subsets of Y.

It is also easy to see that if $(Y, \mathscr{T})$ is a space with a measurable diagonal, then card $Y \leqslant 2^{\aleph_0}$.

Remark 2. In [6], B. V. Rao stated that the diagonal of Y belongs to $\mathscr{T} \times \mathscr{T}$ if and only if there is a countably generated σ-algebra $\mathscr{D} \subset \mathscr{T}$ with singletons as atoms. This proposition (given without proof) is evidently equivalent with a $\Rightarrow$ b of Theorem 1 of the present paper.

Remark 3. A measurable space $(Y, \mathscr{T})$ satisfying the condition b of Theorem 1 can be represented by a separable metrizable space with the Borel σ-algebra in the following way. Enumerate the sets of the generator $\mathscr{C} = \{C_1, C_2, \ldots\}$ and assign to every $y \in Y$ a sequence $s = \{s_n\}_{n=1}$ defined by $s_n = 1$ iff $y \in C_n$ and $s_n = 0$ otherwise. Since $\mathscr{C}$ separates points, Y is mapped one-to-one onto a subset Z of the separable metric space M of all zero-one sequences, which implies that Z is metrizable and separable. The image of each C_k is $\{s \in Z; s_k = 1\}$ and therefore the σ-ring generated by $\{C_1, C_2, \ldots\}$, corresponds to the relativization to Z of the σ-ring in M generated by the cylinders, i.e. to the Borel σ-algebra in Z. This shows a connection between our Theorem 1, proved set-theoretically, and earlier results on measurable graphs, established in a topological setting.

Remark 4. Trivial examples may show that there are measurable mappings $f : X \to Y$ with measurable graphs even if $(Y, \mathscr{T})$ is not a space with a measurable diagonal. A sufficient condition for the graph measurability of a given measurable mapping is established in the following.

Proposition. *Let $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ be measurable spaces and let f be a measurable mapping from X into Y. If there is a set $Z \in \mathscr{T}$ such that $f(x) \in Z$ for each $x \in X$ and such that Z with the relative σ-ring $\mathscr{U} = \{Z \cap E; E \in \mathscr{T}\}$ is a space with a measurable diagonal, then the graph of f is a measurable set.*

Proof. We may view upon f as a mapping from X into Z. Evidently f is $\mathscr{U}$-measurable and hence by Theorem 1 its graph is in $\mathscr{S} \times \mathscr{U}$. Since $Z \in \mathscr{T}$, we have $\mathscr{U} \subset \mathscr{T}$ which implies $\mathscr{S} \times \mathscr{U} \subset \mathscr{S} \times \mathscr{T}$ and completes the proof.

2. Ordered spaces

Throughout this section, Y will stand for a nonempty set linearly ordered by $<$. We shall use the notation $a \leqslant b$ iff $a < b$ or $a = b$ and the symbol $[y < a]$ for $\{y \in Y;\ y < a\}$. The symbols $[y > a]$, $[y \leqslant a]$ and $[y \geqslant a]$ are defined analogously. A pair $[u, v] \in Y \times Y$ will be called a gap iff $u < v$ and for no $y \in Y$ we have $u < y < v$. It is natural to define that a mapping $f: X \to Y$, where $(X, \mathscr{S})$ is a measurable space, be measurable iff for each $y \in Y$ the inverse images of the sets $[y < a]$ and $[y > a]$ are in $\mathscr{S}$. This leads to $\mathscr{T}$ being the σ-ring generated by the family $\mathscr{H}$ of all the sets $[y \subset a]$, $[y > a]$ for $a \in Y$. It has been shown in [2] that this measurability is in general weaker than the Borel measurability derived from the order topology on Y.

Definition 2. *We shall say that an ordered space Y is separable iff there is a countable set $Q \subset Y$ such that for any two points u, $v \in Y$ with $u < v$ there is a point $q \in Q$ with $u \leqslant q \leqslant v$.*

Theorem 2. *Let Y denote an ordered space and $\mathscr{T}$ the σ-ring on Y defined above. Then $(Y, \mathscr{T})$ has a measurable diagonal if and only if Y is separable.*

Proof. To prove sufficiency, let Q be the countable set from Definition 2 and put $\mathscr{C} - \{[y < q];\ q \in Q\} \cup \{[y > q];\ q \in Q\}$. Clearly $\mathscr{C} \subset \mathscr{T}$. Let u, v be any distinct points in Y. We may and do assume that $u < v$. Since Y is separable, there is $q \in Q$ with $u \leqslant q \leqslant v$. However, $u < v$ implies that $u < q$ or $q < v$. In the former case we have $u \in [y < q] \not\ni v$ and in the latter $u \notin [y > q] \ni v$ which proves that $\mathscr{C}$ separates points of Y and it follows by Theorem 1 that $(Y, \mathscr{T})$ has a measurable diagonal.

Suppose now that $(Y, \mathscr{T})$ is a space with a measurable diagonal. By Theorem 1 there is a σ-ring $\mathscr{D} \subset \mathscr{T}$ generated by a countable family $\mathscr{C} = \{C_1, C_2, \ldots\}$ and such that $\mathscr{D}$ separates points of Y. Since $\mathscr{T}$ is generated by $\mathscr{H} - \{[y < a],\ [y > a];\ a \in Y\}$, to every $C_i \in \mathscr{C}$ there is a countable $\mathscr{H}_i \subset \mathscr{H}$ with $C_i \in \sigma(\mathscr{H}_i)$. Denote by $\mathscr{E}$ the countable family $\bigcup\{\mathscr{H}_i;\ i = 1, 2, \ldots\}$. Since every C_i is in $\sigma(\mathscr{E})$, we have $\mathscr{D} = \sigma(\mathscr{E})$, $\mathscr{E}$ being a countable subfamily of $\mathscr{H}$, i.e. $\mathscr{E} = \{[y < q_i];\ i = 1, 2, \ldots\} \cup \{[y > q_j];\ j = 1, 2, \ldots\}$ for some q_i, $q_j \in Y$. If we denote by Q the countable set of all those q_i, q_j that $[y < q_i] \in \mathscr{E}$, $[y > q_j] \in \mathscr{E}$, we see that any $E \in \mathscr{E}$ is either a set $[y < q]$ or $[y > q]$ with some $q \in Q$.

As we have shown in part b $\Rightarrow$ c of the proof of Theorem 1, the generator

$\mathscr{E}$ of $\mathscr{D}$ separates points of Y as well as $\mathscr{D}$ does. Therefore to any two points $u < v$ of Y there is $E \in \mathscr{E}$ with $u \in E \not\ni v$ or $u \notin E \ni v$. In the former case we have $u < q \leqslant v$, in the latter $u \leqslant q < v$ for some $q \in Q$, which proves the separability of Y.

For an example of an ordered set Y such that $(Y, \mathscr{T})$ is not a space with a measurable diagonal, put $Y = I \times \{0, 1\}$, where I is any interval of real numbers, with the lexicographical ordering.

Remark 5. If $\mathscr{U}$ is any σ-ring on a separable ordered space Y such that all $[y < a]$ and $[y > a]$, $a \in Y$ are in $\mathscr{U}$, in particular if $\mathscr{U}$ is the Borel σ-algebra generated by the order topology, then $(Y, \mathscr{U})$ has a measurable diagonal since $D \in \mathscr{T} \times \mathscr{T} \subset \mathscr{U} \times \mathscr{U}$.

3. Topological spaces

Let us consider now a topological space $(Y, \mathscr{G})$. We say that $(Y, \mathscr{G})$ is a second-countable topological space iff the topology $\mathscr{G}$ has a countable basis. $(Y, \mathscr{G})$ is called a T_0-space iff to any two distinct points $u, v \in Y$ there is an open set G such that $u \in G \not\ni v$ or $u \notin G \ni v$.

Theorem 3. *Let $(Y, \mathscr{G})$ be a topological space and $\mathscr{T}$ a σ-algebra generated by $\mathscr{G}$.* Then *$(Y, \mathscr{T})$ has a measurable diagonal if and only if there is a topology $\mathscr{H} \subset \mathscr{G}$ such that $(Y, \mathscr{H})$ is a second-countable T_0-space.*

Proof. Let $(Y, \mathscr{T})$ have a measurable diagonal. By Theorem 1, there is $\mathscr{C} = \{C_1, C_2, \dots\} \subset \mathscr{T}$ which separates points. Since $\mathscr{T}$ is generated by $\mathscr{G}$, to each $C_i \in \mathscr{C}$ there is a countable $\mathscr{G}_i \subset \mathscr{G}$ with $C_i \in \sigma(\mathscr{G}_i)$. Let $\mathscr{H}$ be the topology whose subbasis is $\bigcup\{\mathscr{G}_i;\ i = 1, 2, \dots\}$. Evidently $\mathscr{H}$ has a countable basis and $\mathscr{H} \subset \mathscr{G}$. We have still to prove that $(Y, \mathscr{H})$ is a T_0-space. If it were not, for some distinct $u, v \in Y$ we would have $u \in E \Leftrightarrow v \in E$ for every $E \in \mathscr{H}$. Since by our assumption $\mathscr{C}$ separates points, there is $C_i \in \mathscr{C}$ with $u \in C_i \not\ni v$ or $u \notin C_i \ni v$. The family $\mathscr{M} = \{M \subset Y;\ u \in M \Leftrightarrow v \in M\}$ is a σ-algebra and $\mathscr{G}_i \subset \mathscr{H} \subset \mathscr{M}$. It follows that $u \in C_i \Leftrightarrow v \in C_i$, a contradiction.

To prove the converse, let $(Y, \mathscr{H})$ be a second-countable T_0-space with $\mathscr{H} \subset \mathscr{T}$. It is sufficient to show that the countable basis $\mathscr{B}$ of $\mathscr{H}$ separates points of Y. If there were distinct points $u, v \in Y$ such that for each $E \in \mathscr{B}$ we had $u \in E \Leftrightarrow v \in E$, then the same would hold for every E in $\sigma(\mathscr{B})$. Since however $\mathscr{B}$ is a countable basis for $\mathscr{H}$, we have $\mathscr{H} \subset \sigma(\mathscr{B})$ and therefore $u \in E \Leftrightarrow v \in E$ for each $E \in \mathscr{H}$, which contradicts the assumption that $(Y, \mathscr{H})$ is a T_0-space.

An anti-discrete topological space with at least two points may serve as an example of a topological space not having a measurable diagonal.

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Katedra matematickej analýzy
Prírodovedeckej fakulty UK
Mlynská dolina
816 31 Bratislava