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ATOMS AND GENERATORS IN BOOLEAN $\mathfrak{m}$-ALGEBRAS

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The present note should be considered as a by-product of the research done by the authors. However, the main results are not, at least not explicitly, stated in any publication (known to the authors), moreover, some of them are improvements of the results proved in [4] (e. g. p. 134).

Let $\mathfrak{n}$, $\mathfrak{m}$ be cardinal numbers, $\mathfrak{m}$ being infinite. A Boolean $\mathfrak{m}$-algebra is a Boolean algebra which is $\mathfrak{m}$-complete (every subset containing at most $\mathfrak{m}$ elements has the join). $\mathfrak{A}_{\mathfrak{m}\mathfrak{n}}$ will denote the free Boolean $\mathfrak{m}$-algebra with $\mathfrak{n}$ free $\mathfrak{m}$-generators (the notions and the denotation used here are those of Sikorski [4]).

In this note a relation between the number of atoms and that of generators in a Boolean algebra is shown. That allows us to determine the minimal power of a set of $\mathfrak{m}$-generators in $\mathrm{P}(\mathfrak{m})$, on the other hand we are able to give the exact number of atoms in $\mathfrak{A}_{\mathfrak{m}\mathfrak{n}}$. At the end the cardinality of $\mathfrak{A}_{\mathfrak{m}\mathfrak{n}}$ is shown to be $\mathfrak{n}^{\mathfrak{m}}$ for $\mathfrak{n}$ infinite.

Let us denote $1.b = b$, $0.b = -b$ for any element b cf a Boolean algebra.

Lemma 1. *Let $\mathfrak{A}$ be a Boolean $\mathfrak{m}$-algebra (algebra) with a set S of $\mathfrak{m}$-generators (generators). Then*

(a) *for any $\varepsilon \in {}^{S}2$ the meet*

$$s_\varepsilon = \bigwedge_{s \in S} \varepsilon(s) \, . \, s \tag{1}$$

exists and is an atom or zero,

(b) *every atom $a \in A$ is of the form* (1).

Proof. For $\alpha \in \mathfrak{m}^+$ let us define by induction:

$$A_0 = \bigcup_{s \in S} \{s, -s\}, \quad A'_\alpha = \bigcup_{\beta \in \alpha} A_\beta, \quad A_\alpha = \bigcup_{\varphi \in {}^{\mathfrak{m}}A'_\alpha} \{\bigwedge_{\xi \in \mathfrak{m}} \varphi(\xi), \; - \bigwedge_{\xi \in \mathfrak{m}} \varphi(\xi)\}. \tag{2}$$

By assuption we have $\mathfrak{A} = \bigcup_{\alpha \in \mathfrak{m}^+} A_\alpha$ and by induction through $\alpha \in \mathfrak{m}^+$ we immediately see that, if $x \in \mathfrak{A}$ fulfils $x \leqslant s$ or $x \leqslant -s$ for all $s \in S$, then also

$x \leqslant b$ or $x \leqslant -b$ for all $b \in \mathfrak{A}$. Therefore every lower bound of $\{\varepsilon(s) . s;\ s \in S\}$ is an atom or zero and while the join of two lower bounds is again a lower bound but the join of two different atoms not, (a) is proved. To prove (b) it suffices to define $\varepsilon(s) = 1$ if $a \leqslant s$, $\varepsilon(s) = 0$ if $a \leqslant -s$.

If S only generates $\mathfrak{A}$, the proof is quite analogous (we take $\aleph_0$ instead of $\mathfrak{m}^+$).

Note 1. A special case of Lemma 1 (a) (for card $S \leqslant \mathfrak{m}$) is formulated in the proof of Theorem 24.5 [4].

The following four statements are immediate consequences of the previous lemma.

Corollary 1. *If $\mathfrak{n} \leqslant \mathfrak{m}$, then in a Boolean $\mathfrak{m}$-algebra with $\mathfrak{n}$ $\mathfrak{m}$-generators every $\mathfrak{m}$-ultrafilter is principal.*

Corollary 2. *If a Boolean algebra $\mathfrak{A}$ with $\mathfrak{n}$ generators* ($\mathfrak{m}$-*generators, complete generators*) *is $\mathfrak{n}$ distributive, then* card $\mathfrak{A} \subseteq 2^{-\mathfrak{n}}$.

Proof. The $\mathfrak{n}$-distributivity of $\mathfrak{A}$ gives us $1 = \bigvee \{s_\varepsilon;\ \varepsilon \in {}^S2\}$, therefore by Lemma 1, $\mathfrak{A}$ is atomic and the number of atoms is $\leqslant 2^n$. As every element in $\mathfrak{A}$ is a join of atoms, the statement is proved.

Corollary 3. *If a Boolean 2^n-algebra $\mathfrak{A}$ with $\mathfrak{n}$ generators* ($\mathfrak{m}$-*generators, complete generators*) *is $\mathfrak{n}$-distributive, then $\mathfrak{A}$ is of the form* $\mathrm{P}(X)$.

Proof. By the same reason as in Corollary 2, $\mathfrak{A}$ is atomic and for X we take the set of all atoms in $\mathfrak{A}$.

Theorem 1. *A Boolean algebra with $\mathfrak{n}$ generators* ($\mathfrak{m}$-*generators, complete generators*) *has at most $2^{\mathfrak{n}}$ atoms.*

Note 2. In Corollary 2, the $\mathfrak{n}$-distributivity of $\mathfrak{A}$ is essential. For $\mathfrak{n}$ regular and for any cardinal $\mathfrak{k}$ there exists a Boolean algebra $\mathfrak{A}_\mathfrak{k}$ with $\mathfrak{n}$ complete generators, which is $\mathfrak{m}$-distributive for all $\mathfrak{m} < \mathfrak{n}$ and card $\mathfrak{A}_\mathfrak{k} \geqslant \mathfrak{k}$ (see Vopěnka [7], Solovay [5]).

Let us denote $\log \mathfrak{m} = \min \{\omega_\alpha;\ 2^{\omega_\alpha} \geqslant \mathfrak{m}\}$, let $\mathrm{P}^\mathfrak{m}(X)$ denotes the Boolean algebra of all the subsets of X of the power $\leqslant \mathfrak{m}$ and their complements (with set-theoretical operations).

Theorem 2. *The least possible number of $\mathfrak{m}$-generators of the Boolean algebra* $\mathrm{P}(\mathfrak{m})$ *is* $\log \mathfrak{m}$.

Proof. Let us denote $\log \mathfrak{m} = \mathfrak{k}$. Then $\mathrm{P}^\mathfrak{m}(\mathrm{P}(\mathfrak{k}))$ has $\mathfrak{k}$ $\mathfrak{m}$-generators $s_\xi = \{p;\ \xi \in p\ \&\ p \subseteq \mathfrak{k}\}$ for $\xi \in \mathfrak{k}$. Therefore $P(\mathfrak{m})$ has also at most $\mathfrak{k}$ generators. By Theorem 1, $\mathrm{P}(\mathfrak{m})$ cannot have less than $\mathfrak{k}$ generators.

Theorem 3. *If $\mathfrak{n} \leqslant \mathfrak{m}$, $\mathfrak{A}_{\mathfrak{m}\mathfrak{n}}$ has exactly $2^\mathfrak{n}$ atoms; if $\mathfrak{m} < \mathfrak{n}$, $\mathfrak{A}_{\mathfrak{m}\mathfrak{n}}$ is atomless.*

Proof. Let be $\varepsilon \in {}^S2$ (using the denotation from Lemma 1). If $\mathfrak{n} \leqslant \mathfrak{m}$,

ε can be extended to a $\mathfrak{m}$-homomorphism φ of $\mathfrak{A}_{\mathfrak{mn}}$ onto the two-element Boolean algebra 2, where clearly $\varphi(s_\varepsilon) = 1$. Therefore, by Lemma 1 (a), s_ε is an atom. Evidently, for $\varepsilon \neq \varepsilon'$ $s_\varepsilon \neq s_{\varepsilon'}$ holds.

Now, let $\mathfrak{m} < \mathfrak{n}$ and let $\mathfrak{A}_{\mathfrak{mn}}$ have an atom a. Using the denotation (2), $a \in A_\alpha$ holds for some α. Therefore there exists $S' \subseteq S$, card $S' \leqslant \mathfrak{m}$ such that $a \in \mathfrak{A}'$, where $\mathfrak{A}'$ is the $\mathfrak{m}$-subalgebra of $\mathfrak{A}_{\mathfrak{mn}}$ generated by S'. By Lemma 1 (b) we have

$$a = s_\varepsilon = \bigwedge_{s \in S} \varepsilon(s) \,.\, s \tag{3}$$

and

$$a = s_{\varepsilon'} = \bigwedge_{s \in S'} \varepsilon'(s) \,.\, s \tag{4}$$

for proper $\varepsilon \in {}^{S}2$, $\varepsilon' \in {}^{S'}2$.

If we define $\varphi(s) = \varepsilon'(s)$ for $s \in S'$,

$$\varphi(s) = -\varepsilon(s) \text{ for } s \in S - S',$$

then the mapping $\check{n}$ can be extended to a $\mathfrak{m}$-homomorphism f of $\mathfrak{A}_{\mathfrak{mn}}$ onto 2. As $S - S' \neq 0$, by (3) we have $f(a) = 0$. On the other hand from (4) we get $f(a) = 1$. Thus $\mathfrak{A}_{\mathfrak{mn}}$ is atomless.

If $\mathfrak{A}$ is a Boolean algebra, let us define (see also Vopěnka [6]):

$$\mu(\mathfrak{A}) = \min \{\omega_\alpha;\ (\forall b \subseteq \mathfrak{A})\,[\text{card}\, b \geqslant \omega_\alpha \rightarrow (\exists x, y \in b)\,[x \neq y\ \&\ \&\ x \wedge y \neq 0]]\}. \tag{5}$$

Lemma 2. *If* $\mathfrak{n} \geqslant \aleph_0$, *then* $\mu(\mathfrak{A}_{\mathfrak{mn}}) > \mathfrak{m}^+$.

Proof. Let us suppose that every system of disjoint elements of $\mathfrak{A}_{\mathfrak{mn}}$ is of power $\leqslant \mathfrak{m}$ (i. e. $\mu(\mathfrak{A}_{\mathfrak{mn}}) \leqslant \mathfrak{m}^+$). Then $\mathfrak{A}_{\mathfrak{mn}}$, being $\mathfrak{m}$-complete, is also complete and moreover it is the free complete Boolean algebra with $\mathfrak{n}$ free complete generators. But from the results of Gaifman [1] and Hales [2] follows that for $\mathfrak{n}$ infinite such an algebra does not exist (see also Sikorski [4]).

Theorem 4. *If* $\mathfrak{n} \geqslant \aleph_0$, *then* card $\mathfrak{A}_{\mathfrak{mn}} = \mathfrak{n}^{\mathfrak{m}}$.

Proof. Different sets of generators of the cardinality $\mathfrak{m}$ have different joins, therefore, if $\mathfrak{n} \geqslant \mathfrak{m}$, card $\mathfrak{A}_{\mathfrak{mn}} \geqslant \mathfrak{n}^{\mathfrak{m}}$. Using Lemma 2 we have a set b of disjoint elements of $\mathfrak{A}_{\mathfrak{mn}}$ with card $b = \mathfrak{m}^+$. For different subsets b', b'' of b, the joins $\bigvee b'$, $\bigvee b''$ are different, too. Therefore, for $\mathfrak{m}^+ \geqslant \mathfrak{n}$ we have card $\mathfrak{A}_{\mathfrak{mn}} \geqslant (\mathfrak{m}^+)^{\mathfrak{m}} \geqslant \mathfrak{n}^{\mathfrak{m}}$. The statement card $\mathfrak{A}_{\mathfrak{mn}} \leqslant \mathfrak{n}^{\mathfrak{m}}$ is trivial.

Theorem 5. *If* $\mathfrak{n} \geqslant \aleph_0$, $\mathfrak{m} \geqslant \aleph_1$, *then* $\mathfrak{A}_{\mathfrak{mn}}$ *is not* $\aleph_1$*-representable.*

Proof. The complete Boolean algebra $\mathfrak{A}$ of all open regular subsets of the topological space which is the Cartesian product of $\aleph_0$ topological spaces of the cardinality $\aleph_1$ with the discrete topology each has, by Solovay [5], $\aleph_0$ complete generators. They are also $\aleph_1$-generators, because, by Vopěnka [7], $\mu(\mathfrak{A}) = \omega_2$. By the method of Karp [3] we easily show that $\mathfrak{A}$ is not $\aleph_1$-representable. But $\mathfrak{A}$ is a homomorph image of $\mathfrak{A}_{\mathfrak{mn}}$ for $\mathfrak{n} \geqslant \aleph_0$, $\mathfrak{m} \geqslant \aleph_1$ and therefore $\mathfrak{A}_{\mathfrak{mn}}$ is not $\aleph_1$-representable, either.

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