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ON THE EXISTENCE OF SOLUTION OF A SINGULAR BOUNDARY VALUE PROBLEM

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In the present paper the nonlinear, singular boundary value problem

$$L(u) = xu'' + u' = f(x, u, u'), \qquad x \in (0, 1) \tag{1}$$

$$\lim_{x \to 0+} u(x) < \infty, \qquad u(1) = 0 \tag{2}$$

is studied and some sufficient conditions for the existence of its solution are established. With the help of the Green function of the operator $L(u)$ we transform the problem (1), (2) into an integro-differential equation of the Hammerstein type. That enables us to investigate the existence by the following Tychonoff fixed point theorem:

Theorem 1. (F. Hartman [1].) *Let X be a locally convex, linear, complete topological Hausdorff space and M be a bounded, closed and convex subset of X. Let T be a continuous mapping defined on M into itself such that the closure of TM is compact. Then the equation $Tu = u$ has at least one solution in M.*

(In [1], p. 476 the assumption of continuity of T is omitted.)

1. To solve the problem (1), (2), first of all we have to find a solution of the linear equation

$$L(u) = f(x) \tag{3}$$

on (0, 1) satisfying conditions (2).

The Green function of this singular problem

$$G(x, \xi) = \begin{cases} \ln \xi & \text{if} \quad 0 \leq x < \xi \\ \ln x & \text{if} \quad \xi \leq x \leq 1 \end{cases}$$

is constructed in [2], p. 279 as an exercise of the classic theory. We see easily that this function may be obtained as a limit of the Green function

$$G_y(x, \xi) = \begin{cases} -\dfrac{\ln \xi}{\ln y} \ln x + \ln \xi & \text{if} \quad y \leq x < \xi \\ -\dfrac{\ln \xi}{\ln y} \ln x + \ln x & \text{if} \quad \xi \leq x \leq 1 \end{cases}$$

of the regular problem

$$xu'' + u' = 0, \qquad x \in (y, 1), \ y > 0 \tag{4}$$

$$u(y) = u(1) = 0 \tag{5}$$

for y tending to zero from the right. Consequently, if $f(x)$ is a continuous and bounded function on $(0, 1\rangle$, then the unique solution of the nonhomogeneous problem (3), (5) is given by the formula

$$u_y(x) = \int_y^1 G_y(x, \xi) f(\xi)\, \mathrm{d}\xi \qquad x \in \langle y, 1\rangle . \tag{6}$$

Hence we have at every point $x \in (0, 1\rangle$

$$u(x) = \lim_{y \to 0+} u_y(x) = \int_0^1 G(x, \xi) f(\xi)\, \mathrm{d}\xi < \infty . \tag{7}$$

Furthermore, there exists a uniform limit of $u_y(x)$ for $y \to 0+$ on each interval $\langle b, 1\rangle$, $b > 0$.

Since the function $u(x)$ of (7) has finite derivatives

$$u'(x) = \frac{1}{x} \int_0^x f(\xi)\, \mathrm{d}\xi$$

and

$$u''(x) = -\frac{1}{x^2} \int_0^x f(\xi)\, \mathrm{d}\xi + \frac{1}{x} f(x)$$

for any $x \in (0, 1\rangle$, the previous considerations enable to formulate the following

Lemma. *Let $f(x)$ be a continuous and bounded function on the half-closed interval* $(0, 1\rangle$. *Then there is one and only one solution $u(x)$ of the problem* (3), (2), *which is bounded together with its first derivative $u'(x)$ on* $(0, 1\rangle$. *This solution is given by formula* (7) *and on each interval* $\langle b, 1\rangle$, $b > 0$ *the function $u_y(x)$ given in* (6) *uniformly converges to this solution as y tends to zero from the right.*

2. In this section we shall prove the existence theorem for the nonlinear problem (1), (2).

Theorem 2. *Let $f(x, u, v)$ be a continuous and bounded function on* $E = (0, 1\rangle \times (-\infty, \infty) \times (-\infty, \infty)$. *Then there exists at least one solution of* *the problem* (1), (2). *This solution and its first derivative are bounded on the interval* $(0, 1\rangle$.

Proof. From the previous Lemma it follows that the problem (1), (2) and integro-differential equation:

$$u(x) = \int_0^1 G(x, \xi) f[\xi, u(\xi), u'(\xi)] \, d\xi \tag{8}$$

are mutually equivalent for $x \in (0, 1\rangle$. Thus the solution of (1), (2) is bounded and its first derivative on $(0, 1\rangle$ is bounded, too. The existence of the solution of (8) will be proved by applying Theorem 1.

Consider the linear space X of all real-valued functions defined on $(0, 1\rangle$, which have continuous first derivatives and put $I_n = \langle 1/(n + 1). 1$. Then the sequence of the functionals

$$p_n(u) = \max_{x \in I_n} [|u(x)| + |u'(x)|], \qquad n = 1, 2, \ldots$$

constitutes a countable, monotone family of semi-norms on X satisfying the axiom of separation, that is, for any $u_0 \in X$, $u_0 \neq 0$ there is $p_{n_0}(u)$ in the family such that $p_{n_0}(u_0) \neq 0$. The linear space X, topologized by the family of semi-norms $\{p_n(u)\}_{n=1}^{\infty}$ in such a way that an arbitrary neighbourhood of the element 0 of X is determined by the set

$$U(0, n, \varepsilon) = \{u \in X : p_n(u) < \varepsilon\}, \quad n = 1, 2, \ldots, \quad \varepsilon > 0$$

is a locally convex, linear topological Hausdorff space. This space will be denoted as (X, τ), where τ is the topology on X.

The space (X, τ) is complete. Indeed, let $\{u_k\}_{k=1}^{\infty}$ be a fundamental sequence of X, then for any neighbourhood $U(0, n, \varepsilon)$ there is an index $k_0(n, \varepsilon)$ such that for each $k > k_0$ and $l > k_0$ we have $u_k - u_l \in U(0, n, \varepsilon)$. Consequently for any $k > k_0$, $l > k_0$ and $\varepsilon > 0$, $n = 1, 2, \ldots$ the inequalities

$$|u_k(x) - u_l(x)| < \varepsilon/2, \ |u_k'(x) - u_l'(x)| < \varepsilon/2$$

hold on the interval I_n. These inequalities guarantee the existence of an element $u \in X$ with $p_n(u_k - u) < \varepsilon$ for $k > k_0$. Hence we obtain that $\lim_{k \to \infty} u_k = u$ in (X, τ).

If we put $K = \sup_E |f(x, u, v)| < \infty$, then the set $M = \{u(x) \in X : |u(x)| \leq K, \ |u'(x)| \leq K, \ x \in (0, 1\rangle\}$ is bounded, closed and convex in (X, τ).

In view of (8) it is suitable to choose the operator T on M as follows:

$$Tu(x) = \int_0^1 G(x, \xi) f[\xi, u(\xi), u'(\xi)] \, d\xi . \tag{9}$$

For any $u(x) \in M$ the estimates

$$(10) \qquad |Tu(x)| = |\int_0^1 G(x, \xi) f[\xi, u(\xi), u'(\xi)] \, \mathrm{d}\xi| \le$$

$$\le K\{x|\ln x| + \int_x^1 |\ln \xi| \, \mathrm{d}\xi\} \le K,$$

$$(11) \qquad |[Tu(x)]'| \le \left| \frac{1}{x} \int_0^x f[\xi, u(\xi), u'(\xi)] \, \mathrm{d}\xi \right| \le K$$

are fulfilled on $(0, 1\rangle$ and so the operator T maps the set M into itself, $TM \subset \subset M$.

Further we prove the continuity of the operator T on M.

From the assumption of continuity of $f(x, u, v)$ it follows that this function is uniformly continuous on the compact set $I_n \times \langle -K, K\rangle \times \langle -K, K\rangle$ for any positive integer n. Then for every fixed element $u_0(x)$ from M and for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for each $u \in M$ with $u - u_0 \in U(0, n_0, \delta)$, where $n_0 = (n + 1)^2 - 1$, the inequality

$$|f[\xi, u(\xi), u'(\xi)] - f[\xi, u_0(\xi), u_0'(\xi)]| < \varepsilon/5$$

is satisfied on the whole interval I_{n_0}. Then for $n > (10K/\varepsilon) - 1$ and $x \in I_n$ using the estimate $|\ln x| \le \ln (n + 1) \le n + 1$ we obtain

$$(12) \quad |Tu(x) - Tu_0(x)| \le \int_0^{1/(n+1)^2} |G(x, \xi)| \, |f[\xi, u(\xi), u'(\xi)] - f[\xi, u_0(\xi), u_0'(\xi)] \, \mathrm{d}\xi +$$

$$+ \int_{1/(n+1)^2}^1 |G(x, \xi)| \, |f[\xi, u(\xi), u'(\xi)] - f[\xi, u_0(\xi), u_0'(\xi)]| \, \mathrm{d}\xi \le$$

$$\le \frac{2K}{(n+1)^2} |\ln x| + \frac{\varepsilon}{5} \left[x - \frac{1}{(n+1)^2} \right] |\ln x| + \frac{\varepsilon}{5} \int_x^1 |\ln \xi| \, \mathrm{d}\xi \le$$

$$\le 2K/(n + 1) + \varepsilon/5(n + 1) + \varepsilon/5 < 3\varepsilon/5.$$

By the same condition $n > (10K/\varepsilon) - 1$ we have

$$(13) \quad |[Tu(x)]' - [Tu_0(x)]'| \le \frac{1}{x} \int_0^x |f[\xi, u(\xi), u'(\xi)] - f[\xi, u_0(\xi), u_0'(\xi)]| \, \mathrm{d}\xi \le$$

$$\le \frac{1}{x} \int_0^{1/(n+1)^2} |f[\xi, u(\xi), u'(\xi)] - f[\xi, u_0(\xi), u_0'(\xi)]| \, \mathrm{d}\xi +$$

$$+\frac{1}{x}\int\limits_{1/(n+1)^2}^{x} |f[\xi, u(\xi), u'(\xi)] - f[\xi, u_0(\xi), u_0'(\xi)]|\, d\xi \leq$$

$$\leq 2K/(n+1) + (\varepsilon/5x)\{x - [1/(n+1)^2]\} < 2\varepsilon/5.$$

By means of (12) and (13) we may conclude that if $u - u_0 \in U(0, n_0, \delta)$, then $Tu - Tu_0 \in U(0, n, \varepsilon)$ for any $\varepsilon > 0$ and every positive integer n. This completes the proof of the continuity of Tu on M.

To prove the relative compactness of the image set TM in (X, τ) we use the Ascoli-Arzela theorem.

From (10) and (11) we see that the set TM and the set $(TM)' = \{[Tu(x)]' : : u(x) \in M\}$ are uniformly bounded on each interval I_n. Let n be a positive integer and $\varepsilon > 0$. Take $x_1, x_2 \in I_n$ such that $x_1 < x_2$ and $|x_1 - x_2| < \varepsilon/4K(n + + 1)$. The equicontinuity of the system TM on I_n follows by the relation:

$$|Tu(x_1) - Tu(x_2)| = |\int\limits_0^{x_1} [G(x_1, \xi) - G(x_2, \xi)] f[\xi, u(\xi), u'(\xi)]\, d\xi +$$

$$+ \int\limits_{x_1}^{x_2} [G(x_1, \xi) - G(x_2, \xi)] f[\xi, u(\xi), u'(\xi)]\, d\xi +$$

$$+ \int\limits_{x_2}^{1} [G(x_1, \xi) - G(x_2, \xi)] f[\xi, u(\xi), u'(\xi)]\, d\xi| \leq Kx_1(\ln x_2 - \ln x_1) +$$

$$+ K(x_2 - x_1)\ln x_2 - K\int\limits_{x_1}^{x_2} \ln \xi\, d\xi = K|x_1 - x_2| < \varepsilon/2.$$

In the same way the inequality

$$|[Tu(x_1)]' - [Tu(x_2)]' \leq \left|\frac{1}{x_1} - \frac{1}{x_2}\right| \left|\int\limits_0^{x_1} f[\xi, u(\xi), u'(\xi)]\, d\xi\right| +$$

$$+ \left|\frac{1}{x_2}\int\limits_{x_1}^{x_2} f[\xi, u(\xi), u'(\xi)]\, d\xi\right| \leq \frac{2K}{x_2}|x_1 - x_2| \leq$$

$$\leq 2K(n+1)\,|x_1 - x_2| < \varepsilon/2$$

proves the equicontinuity of the set $(TM)'$ on I_n.

Hence to each sequence $\{v_k\}_{k=1}^{\infty} \subset TM$ there exists a subsequence $\{v_{k_i}\}_{i=1}^{\infty}$ of $\{v_k\}_{k=1}^{\infty}$ and an element $v(x) \in X$ such that

$$|v_{k_i}(x) - v(x)| < \varepsilon/2, \qquad |v_{k_i}'(x) - v'(x)| < \varepsilon/2$$

for $x \in I_n$, $l > l_0(n, \varepsilon)$, where l_0 is a sufficiently large positive integer. Thus for any n, any $l > l_0$ and $\varepsilon > 0$ we have $p_n(v_{k_l} - v) < \varepsilon$, which implies the convergence of $\{v_{k_l}(x)\}_{l=1}^{\infty}$ to $v(x)$ as $l \to \infty$ in the space (X, τ). The compactness is proved.

All the assumptions of the Tychonoff fixed point theorem are fulfilled and so the equation $Tu = u$ has at least one solution $u(x)$ in M. Since $u(x)$ satisfies (1) and (2) Theorem 2 holds true.

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