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PERMUTABILITY, DISTRIBUTIVITY OF EQUIVALENCE RELATIONS AND DIRECT PRODUCTS (1)

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Notations. In the whole paper the symbol $\Pi(M)$ will denote the lattice of all equivalence relations on the set $M \neq \emptyset$. We shall use the symbols $\wedge$, $\vee$, $\bigwedge$, $\bigvee$ for lattice-theoretical operations. We shall define some equivalence relations by quoting their blocks, e.g. C: $\{1, 2\}$, $\{3\}$ will denote the equivalence relation whose blocks are $\{1, 2\}$, $\{3\}$. The greatest and the least equivalence relation on a set M will be denoted by I and O, respectively. In the whole paper the symbol AB or $A \,.\, B$ for given equivalence relations A, B will denote their product in its usual meaning (cf. [1]). The symbol $L(\mathscr{S})$ will denote the sublattice of the lattice $\Pi(M)$ generated by the set $\mathscr{S}$ of equivalence relations ona set M. The symbol $C(\mathfrak{A})$ will denote the lattice of all congruence relations on an algebra $\mathfrak{A}$. The descending chain condition will be abbreviated by DCC.

Introduction. There exists a one-one correspondence between direct decompositions of an algebra into two factors and couples (A, B) of permutable congruence relations such that A is a complement of B. It is not sufficient toassume the pairwise permutability of corresponding congruence relations toextend this result for an arbitrary number of factors. In papers [6], [8] the concept of complete permutability and absolute permutability of a system of congruence relations is introduced to this purpose. In [11] the absolute permutability is used to an external characterization of a special kind of subdirect representations of algebras. In the present paper another kind of permutability ("weak" and "strong") is introduced. Both those concepts of permutability are also sufficient to the characterization of direct products of algebras (see § 5) and to external characterizations of certain subdirect representations of algebras (see [3]).

In the present paper interrelations among the mentioned concepts of permutability are investigated. It is further shown that some of these per-

(1) Most results (without proofs) of the present paper were published in Acta Fac. rer. natur. Univ. Comenianae Math., special number, 1971, 3—11.

mutabilities imply certain distributive identities for the equivalence relations which form the given permutable set $\mathscr{S}$ or the distributivity of the lattice $L(\mathscr{S})$ generated by $\mathscr{S}$. An absolutely permutable set of equivalence relations generates a distributive lattice whose elements are pairwise permutable and a Boolean algebra 2^n if this set is finite. In this direction also some G. H. Wenzel's results [11] are corrected and improved. In the case of weakly or strongly permutable set $\mathscr{S}$ of equivalence relations the situation is more complicated. If card $\mathscr{S} = 3$ then $L(\mathscr{S})$ is distributive but this is not the case when card $\mathscr{S} > 3$. Sufficient conditions for a strongly permutable set of equivalence relations to generate a distributive lattice are given.

The concept of "finite strong permutability" of equivalence relations is introduced analogously as the Chinese remainder theorem in algebras (see e.g. [4]). Some known theorems connected with the Chinese remainder theorem ([4], [13]) are generalized and completed (see Remark 3.1, Theorem 3.3, Theorem 3.6, Corollary 3.8, Corollary 3.9, Remark 3.2). It is possible to enlarge the Chinese remainder theorem in algebras to the case of an arbitrary (possibly infinite) number of congruence relations. The corresponding notion of this "unrestricted Chinese remainder theorem" for equivalence relations is that of "strong permutability". The class of algebras in which any set $\mathscr{S}$ of congruence relations is strongly permutable (i.e. the unrestricted Chinese remainder theorem is satisfied) is described (Corollary 3.8).

The results of [11] concerning a special kind of subdirect products of algebras are corrected. Some results of the present paper are useful to characterize some types of subdirect representations of algebras (more general than that in [11]; see [3]) using the weak permutability.

1. Various Notions of Permutability of Equivalence Relations

Two equivalence relations A_1, A_2 on a set M are said to be permutable if $A_1A_2 = A_2A_1$.

Definition 1.1. *A set $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ of equivalence relations on a set M is called weakly permutable if for any family $(x^\gamma : \gamma \in \Gamma)$ of elements of M such that $x^\alpha(A_\alpha \vee A_\beta)x^\beta$ for any $\alpha, \beta \in \Gamma$ there exists $x \in M$ such that $xA_\gamma x^\gamma$ for any $\gamma \in \Gamma$.*

Definition 1.2. *A set $\mathscr{S}$ of equivalence relations on M is called finitely strongly permutable*(2) *if any finite subset of $\mathscr{S}$ is weakly permutable.*

(2) The assertion that, for an algebra $\mathfrak{A}$, the set $C(\mathfrak{A})$ is finitely strongly permutable means in terminology of [4] that $\mathfrak{A}$ satisfies the Chinese remainder theorem.

Definition 1.3. *A set $\mathscr{S}$ of equivalence relations on M is called strongly permutable if any subset of $\mathscr{S}$ is weakly permutable.*

Definition 1.4. [6] (cf. [4]). *A set $\mathscr{S}$ of equivalence relations on M is called completely permutable if whenever we are given $\{A_\lambda : \lambda \in \Lambda\}$, $A_\lambda \in \mathscr{S}$, and set $C_\lambda = \bigwedge\{A_\nu : \nu \neq \lambda, \nu \in \Lambda\}$ and we are given $(x^\lambda : \lambda \in \Lambda)$, $x^\lambda \in M$ with $x^\lambda(C_\lambda \vee \vee C_\nu)x^\nu$ for all λ, $\gamma \in \Lambda$, then we get that there exists an $x \in M$ such that $xA_\lambda x^\lambda$ for all $\lambda \in \Lambda$.*

Definition 1.5 [8], [11]. *A set $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ of equivalence relations on M is called absolutely permutable*(3) *if for any family $(x^\gamma : \gamma \in \Gamma)$ of elements of M such that $x^\alpha(\bigvee\{A_\gamma : \gamma \in \Gamma\})x^\beta$ for any α, $\beta \in \Gamma$ there exists $x \in M$ such that $xA_\gamma x^\gamma$ for any $\gamma \in \Gamma$.*

The following Lemma can be easily proved.

Lemma 1.1. *Given a set $\mathscr{S}$ of equivalence relations on M, each of the following properties implies the next one:* absolutely permutable, strongly permutable, completely permutable. *Strong permutability implies weak permutability. None of the converse implications holds.*

Remark 1.1. The following examples show that the converse implications do not hold: a) If the set $\mathscr{S}$ is weakly permutable it need not be pairwise permutable and not even strongly permutable as the next example shows. Let $B_1, B_2, B_1 \wedge B_2$ be equivalence relations on a set M such that B_1, B_2 are not permutable. b) A completely permutable set of equivalence relations need not be weakly (nor strongly) permutable as the example in Remark 2.3 shows. c) Let $M = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A_1 : \{1, 7\}$, $\{2, 4\}$, $\{3, 8\}$, $\{5, 6\}$; $A_2 : \{1, 4\}$, $\{2, 7\}$, $\{3, 6\}$, $\{5, 8\}$; $A_3 : \{1, 3\}$, $\{2, 5\}$, $\{4, 6\}$, $\{7, 8\}$. The set $\mathscr{S} = \{A_1, A_2, A_3\}$ is strongly permutable (this is proved in Example 3.1) but it is not absolutely permutable, for if we take $x^1 = 1$, $x^2 = 3$, $x^3 = 5$ then $x^i(A_1 \vee A_2 \vee A_3)x^j$ holds for any $i, j \in \{1, 2, 3\}$ but there does not exist $x \in M$ such that x^iA_ix for each $i \in \{1, 2, 3\}$.

Remark 1.2. For any natural number n $(n > 2)$ there exists a set $\mathscr{S}$ of n equivalence relations which is not weakly permutable but any its proper subset is weakly (even absolutely) permutable. Example: Let M be a set of all sequences $a = (a_1, \ldots, a_n)$, where $a_i \in Z$ (the set of all integers), and $a_1 + a_2 + \ldots + a_n$ is even. We set aA_ib if $a_i = b_i$. Then the set $\{A_1, \ldots, A_n\}$ is not weakly permutable but any its proper subset is weakly permutable.

Let us give some examples of strongly permutable sets.

Example 1.1. The set of all congruence relations on an algebra $\mathfrak{A}$ having

(3) In [8] the term „assoziiert" is used.

the lattice $C(\mathfrak{A})$ distributive and any two congruence relations permutable is finitely strongly permutable (see Remark 3.1). In particular it is true for relative complemented lattices, l-groups, Brouwerian algebras [7], the ring Z of integers. On the other hand an infinite set of congruence relations on Z need not be weakly permutable as the following example shows: Let Θ_p be a congruence relation on Z modulo p, where p is a prime. The set $\{\Theta_p : p \in \in P\}$ (P is the set of all primes) is not weakly permutable because for the elements $x^p = 0$ for $p > 2$ and $x^2 = 1$ there does not exist an element x such that $x^p \, \Theta_p \, x$ for any $p \in P$. Corollary 3.8 describes a class of algebras in which any set of congruence relations is strongly permutable. Further examples of strongly permutable sets can be easily constructed using the results of § 5.

2. Distributivity and Weak Permutability

The next Lemma is obvious.

Lemma 2.1. *Two equivalence relations are weakly (absolutely, completely) permutable if and only if they are permutable.*

Lemma 2.2. *Let $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ be a weakly permutable set of equivalence relations on M. Then for any $\iota \in \Gamma$:*

(2.1) $A_\iota \vee (\bigwedge\{A_\gamma : \gamma \neq \iota,\ \gamma \in \Gamma\}) = \bigwedge\{(A_\iota \vee A_\gamma) : \gamma \neq \iota, \gamma \in \Gamma\}$.

(2.2) A_ι *is permutable with* $\bigwedge\{A_\gamma : \gamma \neq \iota,\ \gamma \in \Gamma\}$.

Proof. Let $x[\bigwedge\{(A_\iota \vee A_\gamma) : \gamma \neq \iota, \gamma \in \Gamma\}]y$, then $x(A_\iota \vee A_\gamma)y$ for any $\gamma \neq \iota$. Let us take $x^\iota = x$, $x^\gamma = y$ for any $\gamma \neq \iota$, $\gamma \in \Gamma$. Obviously $x^\alpha(A_\alpha \vee \vee A_\beta)x^\beta$ for any α, $\beta \in \Gamma$. From the weak permutability of $\mathscr{S}$ it follows that there exists $t \in M$ such that $tA_\iota x$, $tA_\gamma y$ for any $\gamma \neq \iota$. Thus $xA_\iota t$ and $t(\bigwedge\{A_\gamma : :\gamma \neq \iota,\ \gamma \in \Gamma\})y$. Consequently, $x[A_\iota(\bigwedge\{A_\gamma\colon\ \gamma \neq \iota,\ \gamma \in \Gamma\})]y$. Hence $\bigwedge\{(A_\iota \cdot A_\gamma)\colon\ \gamma \neq \iota,\ \gamma \in \Gamma\} \leqq A_\iota(\bigwedge\{A_\gamma\colon\ \gamma \neq \iota,\ \gamma \in \Gamma\}) \leqq A_\iota \vee (\bigwedge\{A_\gamma\colon\ \gamma \neq \iota,\ \gamma \in \in \Gamma\})$. From this (2.1) and the identity $A_\iota \cdot (\bigwedge\{A_\gamma\colon\ \gamma \neq \iota,\ \gamma \in \Gamma\}) = A_\iota \vee \vee (\bigwedge\{A_\gamma\colon\ \gamma \neq \iota,\ \gamma \in \Gamma\})$ hold. By [4, Chap. 0, Ex. 15], (2.2) holds.

Lemma 2.3. *Let $\{A_1, A_2, A_3\}$ be a set of equivalence relations on M satisfying the next conditions:*

(2.1a) $A_1 \vee (A_2 \wedge A_3) = (A_1 \vee A_2) \wedge (A_1 \vee A_3)$.

(2.2a) A_1 *is permutable with* $A_2 \wedge A_3$.

(2.3a) $A_1 \wedge (A_2 \vee A_3) = (A_1 \wedge A_2) \vee (A_1 \wedge A_3)$.

Then the median identity holds:

(2.4) $(A_1 \vee A_2) \wedge (A_1 \vee A_3) \wedge (A_2 \vee A_3) = (A_1 \wedge A_2) \vee (A_1 \wedge A_3) \vee (A_2 \wedge A_3).$

Proof. By successive using of the assumptions (2.1a), (2.2a), the modular identity for permutable equivalence relations [1, Chap. IV., Th. 13, p. 95] and (2.3a) we get:

$$(A_1 \vee A_2) \wedge (A_1 \vee A_3) \wedge (A_2 \vee A_3) = [A_1 \vee (A_2 \wedge A_3)] \wedge (A_2 \vee A_3) =$$

$$[A_1 \wedge (A_2 \vee A_3)] \vee (A_2 \wedge A_3) = (A_1 \wedge A_2) \vee (A_1 \wedge A_3 \vee (A_2 \wedge A_3).$$

Lemma 2.4. *Let* $\mathscr{S} = \{A_1, A_2, A_3\}$ *be a weakly permutable set of equivalence relations on* M. *Then (2.4) and the next conditions hold*:

(2.3) $A_i \wedge (A_j \vee A_k) = (A_i \wedge A_j) \vee (A_i \wedge A_k)$ *for* $i, j, k \in \{1, 2, 3\}$.

(2.5) $A_i \wedge A_j$ *is permutable with* $A_i \wedge A_k$.

Hence $L(\mathscr{S})$ *is distributive.*

Proof. In view of the symmetry of Definition 1.1 it suffices to prove the assertion (2.3) and (2.5) only for one triple $i = 1$, $j = 2$, $k = 3$. (If some of indices i, j, k are equal then the assertion is trivial.) Let $x[A_1 \wedge (A_2 \vee A_3)]y$, then xA_1y and $x(A_2 \vee A_3)y$. Let us take $x^1 = x = x^2$, $x^3 = y$. It is evident that $x^i(A_i \vee A_j)x^j$ for $i, j \in \{1, 2, 3\}$. From the weak permutability it follows that there exists $t \in M$ such that xA_1t, xA_2t, yA_3t. It follows yA_1t. Thus $x(A_1 \wedge A_2)t$, $t(A_1 \wedge A_3)y$, and consequently $x(A_1 \wedge A_2)(A_1 \wedge A_3)y$. It follows $A_1 \wedge (A_2 \vee A_3) \leqq (A_1 \wedge A_2)(A_1 \wedge A_3) \leqq (A_1 \wedge A_2) \vee (A_1 \wedge A_3)$. Hence (2.3) and (2.5) hold. By Lemma 2.2 and 2.3, the median identity (2.4) holds. By [9] it follows that $L(\mathscr{S})$ is distributive.

Lemma 2.5. *Let* $\mathscr{S} = \{A_1, A_2, A_3\}$ *be a set of equivalence relations on* M *satisfying conditions (2.1), (2.3) and (2.2a). Then* $L(\mathscr{S})$ *is distributive.*

Proof. The median identity (2.4) holds by Lemma 2.3, hence the assertion of the Lemma follows by [9].

Remark 2.1. If $\mathscr{S} = \{A_1, A_2, A_3, A_4\}$ is a weakly permutable set of equivalence relations on M then $L(\mathscr{S})$ need not be distributive as the Example 3.1 shows.

Remark 2.2. If $L(\mathscr{S})$, generated by the set $\mathscr{S} = \{A_1, A_2, A_3\}$ of equivalence relations, is distributive, then $\mathscr{S}$ need not be weakly permutable, not even if the elements of the set $\mathscr{S}$ are pairwise permutable and $A_i \vee A_j = I$ for $i \neq j$, $i, j \in \{1, 2, 3\}$ as the Example 4.1 shows. (In the Example 4.1, $L(\mathscr{S})$ is even a Boolean lattice.)

Remark 2.3. The assertion of Lemma 2.2 or Lemma 2.4 does not hold if we replace "weakly permutable" by "completely permutable" as the next example shows: $M = \{1, 2, 3, 4\}$; A_1: $\{1, 2\}$, $\{3, 4\}$; A_2: $\{1, 3\}$, $\{2, 4\}$; A_3: :$\{1, 4\}$, $\{2, 3\}$. The following identities hold: $A_i \cdot A_j = A_i \vee A_j = I$ for $i \neq j$, $i, j \in \{1, 2, 3\}$, $C_i = A_j \wedge A_k = 0$ for $i \neq j \neq k \neq i$. If $x^i(C_i \vee C_j)x^j$ then $x^i = x^j$ (for any i, j) thus the set $\{A_1, A_2, A_3\}$ is completely permutable but none of the distributive laws (2.1) and (2.3) holds: $A_i = A_i \vee (A_j \wedge A_k) \neq \neq (A_i \vee A_j) \wedge (A_i \vee A_k) = I$, $A_i = A_i \wedge (A_j \vee A_k) \neq (A_i \wedge A_j) \vee (A_i \wedge A_k) = = 0$. It follows that $\{A_1, A_2, A_3\}$ is not weakly permutable.

Lemma 2.6. *Let $\mathscr{S} = \{A_1, A_2, A_3\}$ be a set of equivalence relations on M satisfying the conditions (2.1a), (2.2a) and*
(2.6) A_2 is permutable with A_3.
Then $\mathscr{S}$ is weakly permutable.

Proof. Let x^1, x^2, x^3 be arbitrary elements of M satisfying $x^i(A_i \vee A_j)x^j$, $i, j \in \{1, 2, 3\}$. Then by (2.6) there exists $t \in M$ such that tA_2x^2, tA_3x^3. It follows $x^1(A_1 \vee A_2)t$, $x^1(A_1 \vee A_3)t$ and thus $x^1[(A_1 \vee A_2) \wedge (A_1 \vee A_3)]t$. Using (2.1a) we get $x^1[A_1 \vee (A_2 \wedge A_3)]t$. From (2.2a) we get $x^1[A_1(A_2 \wedge A_3)]t$ hence there exists $z \in M$ such that x^1A_1z, $z(A_2 \wedge A_3)t$. It follows x^iA_iz for any $i \in \in \{1, 2, 3\}$.

Remark 2.4. The condition (2.6) of Lemma 2.6 is not necessary to the weak permutability (not even if $L(\mathscr{S})$ is a Boolean lattice) as the next example shows: $M = \{1, 2, 3, 4\}$; A_1: $\{1, 2\}$, $\{3\}$, $\{4\}$; A_2: $\{1\}$, $\{2, 3\}$. $\{4\}$; A_3: $\{1\}$, $\{2\}$, $\{3, 4\}$. Conditions (2.1a), (2.2a) are necessary.

Remark 2.5. Conditions (2.1a), (2.2a), (2.3a) do not imply the weak permutability of $\{A_1, A_2, A_3\}$ as the following example shows: $A \vee B$, A, B where $AB \neq BA$.

3. Systems Generated by Permutable Sets of Equivalence Relations

Definition 3.1. *By a complete lattice of equivalence relations on a set M w mean an a closed* [1] *sublattice of the lattice* $\Pi(M)$.

Theorem 3.1. *Any complete lattice L of equivalence relations on M is algebraic* [1]. *The set $\mathscr{K}$ of all compact elements* [1] *in L is a $\vee$-subsemilattice of L and L is isomorphic to the lattice of all ideals in $\mathscr{K}$. Any element of L is a join of elements of $\mathscr{K}$.*

Proof. The Theorem can be proved analogously as in paper [6, Lemma 2.3 and Theorem 2.1] where a complete lattice of congruence relations is considered. The second part of Theorem see also in [1, Chap. VIII., Th. 8].

Lemma 3.1. *Let $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ be a set of equivalence relations on M in which for any $A_i \in \mathscr{S}$, $i = 1, \ldots, n$,* (3.1) $A_1 \wedge (\bigvee \{A_i : i = 2, \ldots, n\}) = \bigvee \{(A_1 \wedge A_i) : i = 2, \ldots, n\}$.

Then

(3.2) $A_\iota \wedge (\bigvee \{A_\gamma : \gamma \in \Lambda\}) = \bigvee \{(A_\iota \wedge A_\gamma) : \gamma \in \Lambda\}$ *for any* $\Lambda \subset \Gamma$ *and any* $\iota \in \Gamma$.

Proof. Let $x[A_\iota \wedge (\bigvee \{A_\gamma : \gamma \in \Lambda\})]y$, then $xA_\iota y$ and $x(\bigvee \{A_\gamma : \gamma \in \Lambda\})y$. Then there exist $\gamma(1), \ldots, \gamma(m) \in \Lambda$ such that $x(\bigvee \{A_{\gamma(i)} : i = 1, \ldots, m\})y$ and by (3.1) we get $x[\bigvee \{(A_\iota \wedge A_{\gamma(i)}) : i = 1, \ldots, m\}]y$ and thus $x[\bigvee \{(A_\iota \wedge A_\gamma) : \gamma \in \Lambda\}]y$. The converse inequality holds in any complete lattice, thus (3.2) holds.

Corollary 3.1. [1, Chap. VIII, §5, Ex. 9] *A complete distributive lattice of equivalence relations is Brouwerian.*

Lemma 3.2. *Let* $\mathcal{S} = \{A_\gamma : \gamma \in \Gamma\}$ *be a strongly permutable set of equivalence relations on* M. *Let* $\mathcal{S}_1 = \{B_\iota : \iota \in \Lambda\}$ *be a set of equivalence relations on* M *such that any element* $B_\iota \in \mathcal{S}_1$ *is a meet of elements of the set* $\mathcal{S}$. *Then* $\mathcal{S}_1$ *is strongly permutable.*

Proof. Let $\mathcal{S}_2 \subset \mathcal{S}_1$. The elements of the set $\mathcal{S}_2$ can be ordered to a transfinite sequence $B_0, B_1, \ldots, B_k, \ldots, k < \alpha$, where α is an ordinal number. Let $N = (x^k : k < \alpha)$ be a family of elements of M such that $x^j(B_j \vee B_k)x^k$ for any $j, k < \alpha$. Let $\mathcal{S}_0 \subset \mathcal{S}$ be the system consisting of all those A_γ which occur in expressions of elements B_i. With each $A_\gamma \in \mathcal{S}_0$ we associate the least index $k(\gamma)$ such that the given representation of element $B_{k(\gamma)}$ as the meet of elements of $\mathcal{S}_0$ includes A_γ. Then $x^{k(\gamma)}(A_\gamma \vee A_\delta)x^{k(\delta)}$ holds for any A_γ, $A_\delta \in \mathcal{S}_0$, because $x^{k(\gamma)}(B_{k(\gamma)} \vee B_{k(\delta)})x^{k(\delta)}$ and $B_{k(\gamma)} \leqq A_\gamma$, $B_{k(\delta)} \leqq A_\delta$. Since the set $\mathcal{S}$ is strongly permutable there exists an element $x \in M$ such that $x^{k(\gamma)}A_\gamma x$ for any $A_\gamma \in \mathcal{S}_0$. We shall show that $x^h B_h x$ for any $h < \alpha$. Let A_γ be an arbitrary element of S_0 such that the given representation of B_h as the meet of elements of $\mathcal{S}_0$ includes A_γ. Since $A_\gamma \geqq B_{k(\gamma)}$, $A_\gamma \geqq B_h$, $x^{k(\gamma)}(B_{k(\gamma)} \vee B_h)x^h$, thus $x^{k(\gamma)}A_\gamma x^h$. Since $x^{k(\gamma)}A_\gamma x$, $x^h A_\gamma x$ holds. This holds for any A_γ occurring in the representation of the element B_h thus $x^h B_h x$. Hence $\mathcal{S}_1$ is strongly permutable.

Corollary 3.2. *Let* $\mathcal{S} = \{A_\gamma : \gamma \in \Gamma\}$ *be a strongly permutable set of equivalence relations on* M. *Let* $\mathcal{S}_1 = \{B_\iota : \iota \in \Gamma_1\}$ *be the same as in Lemma 3.2. Then the following conditions are fulfilled.*

(3.3) $B_\iota \vee \bigwedge \{B_\lambda : \lambda \in \Lambda\}) = \bigwedge \{(B_\iota \vee B_\lambda) : \lambda \in \Lambda\}$ *for any* $\iota \in \Gamma_1$ *and any* $\Lambda \subset \Gamma_1$.

(3.4) *The elements of* $\mathcal{S}_1$ *are pairwise permutable.*

(3.5) *Any two elements* $\bigvee\{A_\iota : \iota \in \Lambda_1 \subset \Gamma\}$, $\bigvee\{A_\lambda : \lambda \in \Lambda_2 \subset \Gamma\}$ *are permutable.*

Proof. (3.3) and (3.4) follow from Lemma 3.2 and Lemma 2.2. (3.5) follows from [2, Th. 2.2].

Theorem 3.2. *Let* $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ *be a set of equivalence relations on* M *and* $\mathscr{S}_1 = \{B_\iota : \iota \in \Gamma_1\}$ *the set of all finite meets of elements of* $\mathscr{S}$. *Then* $\mathscr{S}$ *is finitely strongly permutable if and only if the following conditions are fulfilled:*

(3.6) *Any element* $A_\iota \in \mathscr{S}$ *is permutable with any* $B_\lambda \in \mathscr{S}_1$.

(3.7) $A_1 \vee (\bigwedge\{A_i : i = 2, \ldots, m\}) = \bigwedge\{(A_1 \vee A_i) : i = 2, \ldots, m\}$ *holds for any finite number of elements* $A_i \in \mathscr{S}$, $i = 1, \ldots, m$.

Proof. If $\mathscr{S}$ is finitely strongly permutable then (3.6) and (3.7) hold by Corollary 3.2. Conversely, let (3.6) and (3.7) hold. We shall show by induction that $\{A_1, \ldots, A_n\} \subset \mathscr{S}$ is strongly permutable. For $n = 2$ it holds by (3.6) and Lemma 2.1. Let $(x^i : i = 1, \ldots, n)$ be a family of elements of M such that $x^i(A_i \vee A_j)x^j$ for any $i, j \in \{1, \ldots, n\}$. By the induction assumption $\{A_1, \ldots, A_{n-1}\}$ is strongly permutable and thus there exists an element $z \in M$ such that $x^k A_k z$ for any $k \in \{1, \ldots, n-1\}$. Obviously also $z(A_k \vee A_n)x^n$ for $k = 1, \ldots, n-1$. Then $z[\bigwedge\{(A_k \vee A_n) : k = 1, \ldots, n-1\}]x^n$. By (3.7), $x^n[A_n \vee (\bigwedge\{A_k : k = 1, \ldots, n-1\})]z$. By (3.6), there exists an element $t \in M$ such that $x^n A_n t$ and $t(\bigwedge\{A_k : k = 1, \ldots, n-1\})z$. Then $tA_k z$ for any $k = 1, \ldots, n-1$. It follows $tA_k x^k$ for $k \in \{1, \ldots, n\}$. Thus $\{A_1, \ldots, A_n\}$ is strongly permutable and $\mathscr{S}$ is finitely strongly permutable.

Corollary 3.3. *Let* L *be a lattice of equivalence relations on* M. *The elements of the lattice* L *form a finitely strongly permutable set if and only if* L *is distributive and the elements of* L *are pairwise permutable.*

Remark 3.1. In particular the assertion of Corollary 3.3 holds if L is the lattice of all congruence relations of an algebra. As a Corollary we get the assertion [4, Chap. V., Ex. 68] (see the footnote[2])).

Theorem 3.3. *Let* $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ *be a set of equivalence relations on* M *and let the meet - subsemilattice* $\mathscr{S}_1$ *generated by* $\mathscr{S}$ *satisfy DCC. Then* $\mathscr{S}$ *is strongly permutable if and only if* $\mathscr{S}$ *is finitely strongly permutable, i. e. if the conditions (3.6) and (3.7) are fulfilled.*

Proof. Let $\mathscr{S}$ be finitely strongly permutable. Let $\mathscr{S}_0 = \{A_\gamma : \gamma \in \Gamma_1\}$, $\Gamma_1 \subset \Gamma$ and $(x^\gamma : \gamma \in \Gamma_1)$ be such a family of elements of M that $x^\gamma(A_\gamma \vee A_\delta)x^\delta$ for any $\gamma, \delta \in \Gamma_1$. DCC implies that $\bigwedge\{A_\gamma : \gamma \in \Gamma_1\} = A_{\alpha(1)} \wedge A_{\alpha(2)} \wedge \ldots \wedge A_{\alpha(n)}$ for some $\alpha(i) \in \Gamma_1$. By the assumptions, there exists $x \in M$ such that $xA_{\alpha(i)}x^{\alpha(i)}$ for any $i = 1, \ldots, n$. Now let $A_\beta \neq A_{\alpha(i)}$ for $i = 1, \ldots, n$, $A_\beta \in \mathscr{S}_0$. Since the set $\{A_\beta, A_{\alpha(1)}, \ldots, A_{\alpha(n)}\}$ is weakly permutable, there exists an element $t \in M$ such that $x^\beta A_\beta t$ and $x^{\alpha(i)} A_{\alpha(i)} t$, $i = 1, \ldots, n$. This implies $x A_{\alpha(i)} t$ for each $i = 1, \ldots, n$, hence $x(\bigwedge \{A_{\alpha(i)} : i = 1, \ldots, n\}) t$ and $x A_\beta t$ because $A_\beta \geqq A_{\alpha(1)} \wedge \ldots \wedge A_{\alpha(n)}$. This implies $x^\beta A_\beta x$, hence $\mathscr{S}_0$ is weakly permutable. Thus $\mathscr{S}$ is strongly permutable. The converse assertion is obvious.

Question 1. Find necessary and sufficient conditions for a set of equivalence relations to be strongly permutable.

Theorem 3.4. *Let $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ be a finitely strongly permutable set of equivalence relations on M. Let $\mathscr{F}$ be the set of all finite joins of elements of $\mathscr{S}$ and $\mathscr{F}_1$ the set of all finite meets of elements of $\mathscr{F}$. Let the following conditions hold:*

(3.8) $A \vee (\bigwedge \{B_i : i = 1, \ldots, n\}) = \bigwedge \{(A \vee B_i) : i = 1, \ldots, n\}$ *for any* A, $B_i \in \mathscr{F}$.

(3.9) *Any $A_\gamma \in \mathscr{S}$ is permutable with any element of $\mathscr{F}_1$.*

Then $L(\mathscr{S})$ is distributive, even the unrestricted([4]) *distributive identity (3.2) is valid in $L(\mathscr{S})$ and the elements of $L(\mathscr{S})$ form a finitely strongly permutable set.*

Proof. From the conditions (3.8) and (3.9) it follows (by Theorem 3.2) that the set $\mathscr{F}$ is finitely strongly permutable, because if we replace $\mathscr{S}$ of Theorem 3.2 by $\mathscr{F}$ then conditions (3.8) and (3.7) are the same. By [2, Theorem 2.2] if $B \cdot A_i = A_i \cdot B$ for $i = 1, \ldots, n$ then B is permutable with $\bigvee \{A_i : i = 1, \ldots, n\}$ hence condition (3.9) implies the validity of (3.6) for $\mathscr{F}$. We shall use Lemma 3.2 to prove that $\mathscr{F}_1$ is finitely strongly permutable i. e. each finite subset of $\mathscr{F}_1$ is strongly permutable: Let $\mathscr{U}$ be a finite subset of elements of $\mathscr{F}_1$. Denote $\mathscr{V}$ the set of elements of $\mathscr{F}$ which are in the expressions of the elements of $\mathscr{U}$ (finite meets of elements of $\mathscr{F}$). Obviously $\mathscr{V}$ is finite and for $\mathscr{F}$ is finitely strongly permutable (Definition 1.2, Definition 1.3) then $\mathscr{V}$ is strongly permutable and by Lemma 3.2, $\mathscr{U}$ is strongly permutable too. Hence $\mathscr{F}_1$ is finitely strongly permutable and thus (by Corollary 3.2) the distributive identity (3.8) holds for any finite number of elements of $\mathscr{F}_1$. We shall now show that the elements of the set $\mathscr{F}_1$ form a sublattice of the lattice $\Pi(M)$ i. e.

$$[\bigwedge \{\bigvee \{A_{ij} : j = 1, \ldots, s(i)\} : i = 1, \ldots, m\}] \vee$$
$$\vee [\bigwedge \{\bigvee \{B_{ek} : k = 1, \ldots, t(e)\} : e = 1, \ldots, n\}] \in \mathscr{F}_1$$

where $A_{ij}, B_{ek} \in \mathscr{S}$.

([4]) The elements $A \wedge (\bigvee \{B_\iota : \iota \in \Lambda \subset \Gamma\}$ and $\bigvee \{(A \wedge B_\iota) : \iota \in \Lambda \subset \Gamma\}$ need not belong to $L(\mathscr{S})$.

But $\bigwedge\{\bigvee\{A_{ij} : j = 1, \ldots, s(i)\} : i = 1, \ldots, m\}$, $\bigvee\{B_{ek} : k = 1, \ldots, t(e)\}$ and $\bigvee\{A_{ij} : j = 1, \ldots, s(i)\}$ are elements of $\mathscr{F}_1$, thus we can use the identity (3.8) twice and we get: $[\bigwedge\{\bigvee\{A_{ij} : j = 1, \ldots, s(i)\} : i = 1, \ldots, m\}]$ $[\bigwedge\{\bigvee\{B_{ek} : k=1, \ldots, t(e)\} : e = 1, \ldots, n\}] = \bigwedge\{([\bigwedge\{\bigvee\{A_{ij} : j = 1, \ldots, s(i)\} : i = 1, \ldots, m\}] \vee [\bigvee\{B_{ek} : k = 1, \ldots, t(e)\}]) : e = 1, \ldots, n\} = \bigwedge\{[\bigwedge\{\bigvee\{A_{ij} : j = 1, \ldots, s(i)\} \vee \bigvee\{B_{ek} : k = 1, \ldots, t(e)\}\} : i = 1, \ldots, m] : e = 1, \ldots, n\} \in \mathscr{F}_1$. Thus $\mathscr{F}_1$ is a sublattice of the lattice $\Pi(M)$, which follows $\mathscr{F}_1 = L(\mathscr{S})$. Since (3.8) holds in $\mathscr{F}_1$, the lattice $\mathscr{F}_1 = L(\mathscr{S})$ is distributive. By Lemma 3.1, the unrestricted distributive identity (3.2) holds.

Example 3.1. We shall show that if we take a strongly permutable set $\mathscr{S} = \{A_1, A_2, A_3, A_4\}$ of equivalence relations, the lattice $L(\mathscr{S})$ need not be distributive. We shall show that the identity (3.8) from Theorem 3.4 need not be fulfilled. Let $M = \{1, 2, 3, 4, 5, 6, 7, 8\}$; A_1: $\{1, 7\}$, $\{2, 4\}$, $\{3, 8\}$, $\{5, 6\}$; A_2: $\{1, 4\}$, $\{2, 7\}$, $\{3, 6\}$, $\{5, 8\}$; A_3: $\{1, 3\}$, $\{2, 5\}$, $\{4, 6\}$, $\{7, 8\}$; A_4: $\{1, 5\}$, $\{2, 3\}$, $\{4, 8\}$, $\{6, 7\}$. It is evident that $A_i \wedge A_j = 0$ for $i \neq j$, $i, j \in \{1, 2, 3, 4\}$ and hence $A_i \wedge A_j \wedge A_k = 0$ for $i \neq j$ (or $j \neq k$ or $i \neq k$). We shall verify the strong permutability of $\mathscr{S}$ by Theorem 3.2: Condition (3.6) is satisfied, because any two elements A_i, A_j are permutable, and the meets of at least two elements of $\mathscr{S}$ are 0. For condition (3.7) it suffices to verify the next two identities only in the case that i, j, k, e are pairwise different.

(a) $A_i \vee (A_j \wedge A_k) = (A_i \vee A_j) \wedge (A_i \vee A_k)$.

(b) $A_i \vee (A_j \wedge A_k \wedge A_e) = (A_i \vee A_j) \wedge (A_i \vee A_k) \wedge (A_i \vee A_e)$.

But $A_i \vee (A_j \wedge A_k) = A_i \vee 0 = (A_i \vee A_j) \wedge (A_i \vee A_k)$, $A_i \vee (A_j \wedge A_k \wedge A_e) = A_i$ and $(A_i \vee A_j) \wedge (A_i \vee A_k) \wedge (A_i \vee A_e) = A_i \wedge (A_i \vee A_e) = A_i$ hold if i, j, k, e are pairwise different. Hence $\mathscr{S}$ is strongly permutable, but $L(\mathscr{S})$ is not distributive because the identity (3.8) is not fulfilled: $A_1 \vee A_2 = A_1 \vee [(A_2 \vee A_3) \wedge (A_2 \vee A_4)] \neq (A_1 \vee A_2 \vee A_3) \wedge (A_1 \vee A_2 \vee A_4) = I$.

Question 2. Whether $L(\mathscr{S})$, generated by a strongly permutable set $\mathscr{S}$ ($\operatorname{card} \mathscr{S} > 3$), is modular? When the answer would be positive, then the same question for $\mathscr{S}$ completely permutable.

Corollary 3.4. *Let $\mathscr{S}$ be a finitely strongly permutable $\vee$-semilattice of equivalence relations on M. Then $L(\mathscr{S})$ is distributive, even the infinite distributive identity (3.2) holds in $L(\mathscr{S})$ and $L(\mathscr{S})$ is finitely strongly permutable.*

Proof. This follows from Theorem 3.2 and Theorem 3.4.

Theorem 3.5. *Let L be a complete lattice of equivalence relations on M and $\mathscr{S} \subset L$ a $\vee$-subsemilattice such that any element $X \in L$ is a join of elements of $\mathscr{S}$ and let $\mathscr{S}$ be finitely strongly permutable. Then L is also finitely strongly permutable and Brouwerian.*

Proof. Let $\{A_1, \ldots, A_n\} \subset L$ and let $x^1, \ldots, x^n \in M$ such that $x^i(A_i \vee A_j)x^j$ for any $i, j \in \{1, \ldots, n\}$. It can be easily seen that the elements of L are pairwise permutable [2, Th. 2.2]. Thus for any couple (i, j) there exists an element y^{ij} such that $x^i A_i y^{ij}$ and $y^{ij} A_j x^j$. By the assumption $A_i = \bigvee \{B_i^k : k \in K_1\}$ and $A_j = \bigvee \{B_j^k : k \in K_2\}$, where $B_i^k, \; B_j^k \in \mathcal{S}$. Then there exist finite subsets $F_1 \subset K_1, F_2 \subset K_2$ such that $x^i(\bigvee \{B_i^k : k \in F_1\}) y^{ij}$ and $y^{ij}(\bigvee \{B_j^k : k \in F_2\}) \, x^j$. Setting $C_i^j = \bigvee \{B_i^k : k \in F_1\}$, $C_j^i = \bigvee \{B_j^k : k \in F_2\}$ we get $x^i(C_i^j \vee C_j^i)x^j$. Let us denote $C_i = \bigvee \{C_i^j : j = 1, \ldots, n\}$ for $i = 1, \ldots, n$. Obviously $C_i \in \mathcal{S}$ and $x^i(C_i \vee C_j) \, x^j$ for any i, j. Since $\mathcal{S}$ is finitely strongly permutable there exists an element $x \in M$ such that $x^i C_i x$ for any $i \in \{1, \ldots, n\}$. Since $C_i \leqq A_i$ we get $x^i A_i x$ for any i, thus the lattice L is finitely strongly permutable. By Theorem 3.2, L is distributive and by Corollary 3.1, L is Brouwerian.

Corollary 3.5. *Let L be a complete lattice of equivalence relations on M and $\mathcal{K} \subset L$ the set of all compact elements of L* [1]. *If $\mathcal{K}$ is finitely strongly permutable then L also is finitely strongly permutable and Brouwerian.*

Proof. The assertion follows from Theorem 3.5 and Theorem 3.1.

Theorem 3.6. *Let $\mathcal{S}$ be a $\wedge$-semilattice of equivalence relations on M and let any three elements of $\mathcal{S}$ form a strongly permutable set. Then $\mathcal{S}$ is finitely strongly permutable. In particular this holds if $\mathcal{S}$ is a sublattice of the lattice $\Pi(M)$.*

Proof. By Lemma 2.2, $A_\alpha \vee (A_\beta \wedge A_\gamma) = (A_\alpha \vee A_\beta) \wedge (A_\alpha \vee A_\gamma)$ for any $A_\alpha, \; A_\beta, \; A_\gamma \in \mathcal{S}$. Let $\{A_1, \ldots, A_n\} \subset \mathcal{S}$ be an arbitrary finite subset of $\mathcal{S}$. We shall show that conditions (3.6) and (3.7) are fulfilled. The condition (3.7) will be proved by induction. It holds for $m = 3$. Let it hold for $m = n - 1$. Since $A_2 \wedge \ldots \wedge A_{n-1} \in \mathcal{S}$ then $A_1 \vee (A_2 \wedge \ldots \wedge A_{n-1} \wedge A_n) = [A_1 \vee (A_2 \wedge \ldots \wedge A_{n-1})] \wedge (A_1 \vee A_n) = (A_1 \vee A_2) \wedge (A_1 \vee A_3) \wedge \ldots \wedge (A_1 \vee A_{n-1}) \wedge (A_1 \vee A_n)$. From the strong permutability of any triple of $\mathcal{S}$ it follows that the elements of $\mathcal{S}$ are pairwise permutable and so the condition (3.6) is fulfilled. By Theorem 3.2, $\mathcal{S}$ is finitely strongly permutable.

Corollary 3.6. *Let any three congruence relations on an algebra $\mathfrak{A}$ be strongly permutable, then $C(\mathfrak{A})$ is finitely strongly permutable.*

Remark 3.2. This Corollary for the case of rings is given in [13, Chap. V, § 7]. As to the Chinese remainder theorem in equational classes of algebras see [12].

Using Corollary 3.6 and Theorem 3.3 we get:

Corollary 3.7. *Let $\mathfrak{A}$ be such an algebra that the lattice $C(\mathfrak{A})$ satisfies DCC. If any three congruence relations on $\mathfrak{A}$ are strongly permutable then $C(\mathfrak{A})$ is strongly permutable.*

Using Corollary 3.3 and Corollary 3.7 we get:

Corollary 3.8. *Let $\mathfrak{A}$ be such an algebra that the lattice $C(\mathfrak{A})$ satisfies DCC. Then the following four conditions are equivalent:*

(i) *$C(\mathfrak{A})$ form a strongly permutable set.*

(ii) *$C(\mathfrak{A})$ form a finitely strongly permutable set (see the footnote[2]).*

(iii) *Any three elements of $C(\mathfrak{A})$ form a strongly permutable set.*

(iv) *$C(\mathfrak{A})$ is distributive and the elements of $C(\mathfrak{A})$ are pairwise permutable.*

Remark 3.3. The condition (i) of Corollary 3.8 can be interpreted in the following way: Any set $\mathscr{S}$ of congruence relations on $\mathfrak{A}$ satisfies the Chinese remainder theorem (without restriction to the finiteness of $\mathscr{S}$).

Corollary 3.9. ([10], see also [12, Th. 6.9]). *The following four conditions are equivalent in a primitive class $\mathscr{A}$ of algebras:*

(3.10) *For any $\mathfrak{A} \in \mathscr{A}$, the lattice $C(\mathfrak{A})$ is distributive and the elements of $C(\mathfrak{A})$ are pairwise permutable.*

(3.11) *Any three congruence relations on any $\mathfrak{A} \in \mathscr{A}$ form a strongly permutable set.*

(3.12) *Congruence relations on any $\mathfrak{A} \in \mathscr{A}$ form a finitely strongly permutable set.*

(3.13) *There exist ternary polynomials p, q in $\mathscr{A}$ such that:*

$$p(x, x, y) = y = p(y, x, x)$$

$$q(x, x, y) = q(x, y, x) = q(y, x, x) = x.$$

Proof. The conditions (3.10) and (3.13) are equivalent by [10]. The conditions (3.11) and (3.12) are equivalent by Corollary 3.6. By Corollary 3.3, the condition (3.12) is equivalent to the condition (3.10).

4. Absolutely Permutable Equivalence Relations

Lemma 4.1 [11]. *Let $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ be an absolutely permutable set of equivalence relations on M. Then $A_\varkappa \vee A_\delta = \bigvee \{A_\gamma : \gamma \in \Gamma\}$ for any $A_\varkappa, A_\delta \in \mathscr{S}$, $A_\varkappa \neq A_\delta$.*

Proof. (5) Let $x(\bigvee \{A_\gamma : \gamma \in \Gamma\})y$ and $\varkappa \in \Gamma$. Let us take a family $(x^\gamma : \gamma \in \Gamma$, $x^\varkappa = x$ and $x^\gamma = y$ for $\gamma \neq \varkappa)$, of elements of M. Then there exists $t \in M$ such that $xA_\varkappa t$ and $tA_\delta y$ and thus $x(A_\varkappa \vee A_\delta)y$ for any $\delta \neq \varkappa$.

(5) We give a shorter proof than that in [11].

Theorem 4.1. *Let $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ be a set of equivalence relations on M. Then the following conditions are equivalent:*

(4.1) *$\mathscr{S}$ is absolutely permutable.*

(4.2) *$A_\iota \vee A_\delta = \bigvee\{A_\gamma : \gamma \in \Gamma\}$ holds for any $A_\iota, A_\delta \in \mathscr{S}$, $A_\iota \neq A_\delta$ and $\mathscr{S}$ is weakly permutable.*

Proof. The condition (4.1) implies (4.2) by Lemma 1.1 and Lemma 4.1. Conversely, let $(x^\gamma : \gamma \in \Gamma)$ be a family of elements of M such that $x^\iota(\bigvee\{A_\gamma : : \gamma \in \Gamma\})x^\varkappa$ for any $\iota, \varkappa \in \Gamma$. Then $x^\iota(A_\iota \vee A_\varkappa)x^\varkappa$ holds (by (4.2)) for any A_ι, $A_\varkappa \in \mathscr{S}$, $A_\iota \neq A_\varkappa$. From the weak permutability it follows that there exists $x \in M$ such that $xA_\gamma x^\gamma$ for any $\gamma \in \Gamma$. Thus $\mathscr{S}$ is absolutely permutable.

Lemma 4.2. *Let $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ be an absolutely permutable set of equivalence relations on M. Then $\mathscr{S}$ is completely permutable and*

(4.3) *$(\bigwedge\{A_\iota : \iota \neq \delta,\ \iota \in \Gamma\}) \vee A_\delta = \bigvee\{A_\gamma : \gamma \in \Gamma\}$ holds for any $\delta \in \Gamma$.*

Proof. By Lemma 1.1, $\mathscr{S}$ is strongly permutable and also completely permutable. By Lemma 2.2 and Lemma 4.1, $(\bigwedge\{A_\iota : \iota \neq \delta,\ \iota \in \Gamma\}) \vee A_\delta = \bigwedge\{(A_\iota \vee A_\delta) : \iota \neq \delta,\ \iota \in \Gamma\} = \bigvee\{A_\gamma : \gamma \in \Gamma\}$.

The next Lemma is obvious.

Lemma 4.3. *Let $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ be a completely permutable set of equivalence relations on M and let the following conditions hold:*

(4.4) *$(\bigwedge\{A_\iota : \iota \neq \gamma,\ \iota \in \Gamma\}) \vee (\bigwedge\{A_\varkappa : \varkappa \neq \omega,\ \varkappa \in \Gamma\}) = \bigvee\{A_\gamma : \gamma \in \Gamma\}$ for any $\gamma, \omega \in \Gamma$, $\gamma \neq \omega$.*

Then $\mathscr{S}$ is absolutely permutable.

Lemma 4.4. *Let $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ be an absolutely permutable set of equivalence relations on M. Then the set $\mathscr{S} \cup \{\bigvee\{A_\gamma : \gamma \in \Gamma\}\}$ is also absolutely permutable.*

Proof. Let $\{y\} \cup (x^\gamma : \gamma \in \Gamma)$ be a family of elements of M such that $x^\iota(\bigvee\{A_\gamma : \gamma \in \Gamma\})x^\delta$ for any $\iota, \delta \in \Gamma$ and $x^\iota(\bigvee\{A_\gamma : \gamma \in \Gamma\})y$ for any $\iota \in \Gamma$. By the assumption there exists $x \in M$ such that $xA_\gamma x^\gamma$ for any $\gamma \in \Gamma$. Obviously, $x(\bigvee\{A_\gamma : \gamma \in \Gamma\})y$ also holds, hence it follows the assertion of Lemma.

Remark 4.1. Let $\Theta_1, \ldots, \Theta_n$ be congruence relations of an algebra $\mathfrak{A}$. In the paper [11, § 1] there is the next assertion: „$CRT(\Theta_1, \ldots, \Theta_n)$ is satisfied[6] in an algebra $\mathfrak{A}$ if and only if $\{\Theta_1, \ldots, \Theta_n\}$ are pairwise permutable and $C([\Theta_1, \ldots, \Theta_n])$ is distributive, where $C([\Theta_1, \ldots, \Theta_n])$ denotes the sublat-

[6] $CRT(\Theta_1, \ldots, \Theta_n)$ is satisfied in an algebra $\mathfrak{A}$ if and only if for arbitrary elements $x^1, \ldots, x^n$ of the algebra $\mathfrak{A}$ satisfying $x^i\ (\Theta_i \vee \Theta_j)x^j$ for all $i, j \in \{1, \ldots, n\}$ there exists $z \in \mathfrak{A}$ such that $x^i\Theta_i z$ for all $i \in \{1, \ldots, n\}$ (i. e. the set $\{\Theta_1, \ldots, \Theta_n\}$ is weakly permutable).

tice of $C(\mathfrak{A})$ generated by $\Theta_1, \ldots, \Theta_n$." This assertion is false in both directions as it can be seen in Example 3.1 and Example 4.1. In Example 3.1 the set $\mathscr{S}$ is strongly permutable, hence $CRT(A_1, A_2, A_3, A_4)$ holds, but the lattice $L(\mathscr{S}) = C([A_1, A_2, A_3, A_4])$ is not distributive. In [11] this assertion was used in the proof of the following Lemma W [11, § 2] which is false, too, as Example 4.1 shows.

Lemma W. *Let $\Theta_1, \ldots, \Theta_n$ be different congruences on an algebra $\mathfrak{A}$. Then $\{\Theta_1, \ldots, \Theta_n\}$ is absolutely permutable if and only if the following three conditions hold:*

(i) *$\{\Theta_1, \ldots, \Theta_n\}$ is pairwise permutable.*

(ii) *$C([\Theta_1, \ldots, \Theta_n])$ is distributive.*

(iii) *$\Theta_{i_0} \vee \Theta_{j_0} = \bigvee \{\Theta_i : i = 1, \ldots, n\}$ for all pairs $1 \leqq i_0 \neq j_0 \leqq n$.*

Example 4.1. We shall show that conditions (i), (ii), (iii) of Lemma W are not sufficient for the absolute permutability. Let $M = \{1, 2, 3, 4, 5, 6\}$; $A_1 : \{1, 2, 5\}, \{3, 4, 6\}$; $A_2 : \{2, 3, 4\}, \{1, 5, 6\}$; $A_3 : \{1, 3, 6\}, \{2, 5, 4\}$; $A_1 \wedge A_2 : \{1, 5\}, \{2\}, \{3, 4\}, \{6\}$; $A_2 \wedge A_3 : \{2, 4\}, \{1, 6\}, \{3\}, \{5\}$; $A_1 \wedge A_3 : \{3, 6\}, \{2, 5\}, \{1\}, \{4\}$. The lattice $C([A_1, A_2, A_3])$ generated by A_1, A_2, A_3 is the eight - element Boolean lattice, A_1, A_2, A_3 are pairwise permutable and $A_1 \vee A_2 \vee A_3 = A_i \vee A_j = I$ for any $i \neq j$, $i, j \in \{1, 2, 3\}$, but if we take $x^1 = 1$, $x^2 = 2$, $x^3 = 3$ (evidently $x^i(A_1 \vee A_2 \vee A_3)x^j$ for any $i, j \in \{1, 2, 3\}$) there does not exist such an element x that $x^i A_i x$ for each $i \in \{1, 2, 3\}$, hence $\{A_1, A_2, A_3\}$ is not absolutely permutable, not even weakly permutable (because weak permutability with (iii) implies absolute permutability by Theorem 4.1). Hence $CRT(A_1, A_2, A_3)$ does not hold.

Howewer, the assertion that conditions (i), (ii), (iii) of Lemma W are necessary, is true. Even the stronger assertion is true:

Theorem 4.2. *Let $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ be an absolutely permutable set of equivalence relations on M. Then $L(\mathscr{S})$ is finitely strongly permutable, hence distributive and its elements are pairwise permutable, and the unrestricted distributive identity (3.2) holds for the elements of $L(\mathscr{S})$. If* $\operatorname{card} \Gamma = n$ *is finite and $\mathscr{S}$ does not contain the element $\{\bigvee \{A_\gamma : \gamma \in \Gamma\}\}$ then $L(\mathscr{S})$ is the Boolean lattice 2^n.*

Remark. As to the complete distributivity of the complete sublattice of $\Pi(M)$ generated by an absolutely permutable set see Example 5.1.

We shall use the next obvious Lemma to the proof of Theorem 4.2.

Lemma 4.5. *Let L be a distributive lattice with the least and the greatest element, generated by a set $\mathscr{S}$. If each element $a \in \mathscr{S}$ has a complement in L, then L is a Boolean lattice.*

Proof of Theorem 4.2. We use Theorem 3.4 for $\mathscr{F} = \mathscr{S} \cup \{\bigvee\{A_\gamma : : \gamma \in \Gamma\}\}$. By Lemma 4.4, $\mathscr{F}$ is absolutely permutable and also strongly permutable (Lemma 1.1). By Theorem 3.2, the assumptions of Theorem 3.4 are satisfied and hence $L(\mathscr{S})$ is distributive, the unrestricted distributive identity (3.2) holds for the elements of $L(\mathscr{S})$ and $L(\mathscr{S})$ is finitely strongly permutable. Let $\mathscr{S} = \{A_1, \ldots, A_n\}$ be finite. Then $L(\mathscr{S})$ has the least element $\bigwedge\{A_i : i = 1, \ldots, n\}$ and the greatest one $\bigvee\{A_i : i = 1, \ldots, n\}$. By Lemma 4.2, the elements A_i have complements $\bigwedge\{A_j : j \neq i, j = 1, \ldots, n\}$. By Lemma 4.5, $L(\mathscr{S})$ is a Boolean lattice. Any element of $L(\mathscr{S})$ can be represented as a finite meet of finite joins of the elements A_i. Obviously A_i are exactly antiatoms of $L(\mathscr{S})$ and $L(\mathscr{S}) \cong 2^n$.

Corollary 4.1. *Let $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ be a set of equivalence relations such that each finite subset of $\mathscr{S}$ is absolutely permutable. Then $L(\mathscr{S})$ is finitely strongly permutable, hence distributive and its elements are pairwise permutable.*

Proof. It suffices to observe that to each finite subset $\mathscr{A}$ of $L(\mathscr{S})$ there is a finite subset $\mathscr{S}_1$ of $\mathscr{S}$ such that $\mathscr{A} \subset L(\mathscr{S}_1)$.

Remark 4.2. On the other hand, if $\mathscr{S}$ is strongly permutable and fulfils the conditions of Theorem 3.4, then $L(\mathscr{S})$ need not be a Boolean lattice as the next example shows: $M = \{1, 2, 3\}$; $A_1 : \{1\}, \{2\}, \{3\}$; $A_2 : \{1, 2\}, \{3\}$; $A_3 : \{1, 2, 3\}$.

Theorem 4.3. *Let $\mathscr{S} = \{A_1, \ldots, A_n\}$ be a finite set of equivalence relations on M. Then the following conditions are equivalent:*

(4.4) $\mathscr{S} = \{A_1, \ldots, A_n\}$ *is absolutely permutable.*

(4.5) $A_i \,.\, \bigwedge\{A_j : j \neq i, j = 1, \ldots, n\} = \bigvee\{A_j : j = 1, \ldots, n\}$ *holds for any* $i - 1, \ldots, n$.

To the proof of this Theorem we shall use the next Lemma:

Lemma 4.6. *Let $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ be an absolutely permutable set of equivalence relations on M. Then*

(4.6) $A_\alpha \,.\, \bigwedge\{A_\gamma : \gamma \neq \alpha, \gamma \in \Lambda \subset \Gamma\} = \bigvee\{A_\gamma : \gamma \in \Gamma\}$ *holds for any* $\alpha \in \Gamma$.

Proof of Lemma 4.6. By Lemma 4.1, $A_\alpha \vee A_\beta = \bigvee\{A_\gamma : \gamma \in \Gamma\}$ for $\alpha \neq \beta$, $\alpha, \beta \in \Gamma$ and by Lemma 1.1, $\mathscr{S}$ is strongly permutable. By Definition 1.3 and Lemma 2.2, $A_\alpha \,.\, \bigwedge\{A_\gamma : \gamma \neq \alpha, \gamma \in \Lambda \subset \Gamma\} = A_\alpha \vee (\bigwedge\{A_\gamma : \gamma \neq \alpha, \gamma \in \in \Lambda \subset \Gamma\}) = \bigwedge\{(A_\alpha \vee A_\gamma) : \gamma \neq \alpha, \gamma \in \Lambda \subset \Gamma\}$. Hence (4.6) holds.

Proof of Theorem 4.3. By Lemma 4.6, (4.4) implies (4.5). Conversely, from (4.5) it follows $A_i \vee A_j = \bigvee\{A_j : j = 1, \ldots, n\}$ for $j \neq i$. We shall show that $\mathscr{S}$ is strongly permutable i. e. (3.6) and (3.7) from Theorem 3.2 hold. First we shall show (3.6). Let $A_i \in \mathscr{S}$ and $B = \bigwedge\{A_k : k \in K \subset \{1, \ldots,$

$n\}\}$. If $i \in K$ then $B \leqq A_i$ and $B \,.\, A_i = A_i \,.\, B$ holds. If $i \notin K$ then $A_i \,.\, B = \bigvee\{A_j : j = 1, \ldots, n\}$, hence by [4, Chap. 0, Ex. 15] $A_i \,.\, B = B \,.\, A_i$. Now we shall show (3.7), i. e. $A_i \vee \bigwedge\{A_k : k \in K\} = \bigwedge\{(A_i \vee A_k) : k \in K\}$ where $K \subset \{1, \ldots, n\}$. If $i \in K$, then $A_i \vee \bigwedge\{A_k : k \in K\} = A_i = \bigwedge\{(A_i \vee A_k) : k \in K\}$. If $i \notin K$, then because of $A_i \vee A_k = \bigvee\{A_j : j = 1, \ldots, n\}$, we get $\bigwedge\{(A_i \vee A_k) : k \in K\} = \bigvee\{A_j : j = 1, \ldots, n\} = A_i \,.\, \bigwedge\{A_k : k \in K\} = A_i \vee \bigwedge\{A_k : k \in K\}$. Thus (3.7) holds. Hence $\mathscr{S}$ is strongly permutable and by Theorem 4.1, $\mathscr{S}$ is absolutely permutable.

Remark 4.3. The converse implication of Lemma 4.6 does not hold (in the infinite case) as the following example shows: Let M be the set of all sequences $a = (a_i : i \in N)$ (N is the set of all natural numbers) where a_i is 0 or 1 and the set $\{i : a_i = 1\}$ is finite. Given $a, b \in M$ set aB_ib if and only if $a_i = b_i$. Evidently $B_i \,.\, \bigwedge\{B_j : j \neq i, j \in N\} = I$ but the set $\{B_i : i \in N\}$ is not absolutely permutable: If for any $i \in N$, x^i is the element of M such that $x^i_j = 0$ for $i \neq j$ and $x^i_i = 1$, then there exists no element $x \in M$ such that x^iB_ix for any $i \in N$.

The next Theorem shows that it is not possible to characterize an infinite absolutely permutable set of equivalence relations on M analogously as the strongly permutable one in Theorem 3.3.

Theorem 4.4. *Let $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ be an absolutely permutable set of equivalence relations on M and let the meet-subsemilattice $\mathscr{S}_1$ generated by $\mathscr{S}$ satisfy DCC. Then $\mathscr{S}$ must be finite.*

Proof. By the assumptions $\bigwedge\{A_\gamma : \gamma \in \Gamma\} = \bigwedge\{A_{\gamma(i)} : \gamma(i) \in \Gamma, i = 1, \ldots, n\}$. If there exists $A_{\gamma(n+1)} \in \mathscr{S}$, $A_{\gamma(n+1)} \neq A_{\gamma(i)}$ for any $i = 1, \ldots, n$, $A_{\gamma(n+1)} \neq \bigvee\{A_\gamma : \gamma \in \Gamma\}$, then we can use Lemma 4.6 (or Theorem 4.3) (because any subset of $\mathscr{S}$ is absolutely permutable too) and we get $A_{\gamma(n+1)} = A_{\gamma(n+1)} \,.\, \bigwedge\{A_{\gamma(i)} : \gamma(i) \in \Gamma, i = 1, \ldots, n\} = \bigvee\{A_{\gamma(i)} : \gamma(i) \in \Gamma, i = 1, \ldots, n, n+1\}$. This is a contradiction.

The following Theorem shows that Lemma 3.2 fails to be true if we replace the condition „strongly permutable" by „absolutely permutable".

Theorem 4.5. *Let $\mathscr{S} = \{A_\gamma : \gamma \in \Gamma\}$ be an absolutely permutable set of equivalence relations on M. Then a set $\mathscr{S}_1$ of meets of elements of $\mathscr{S}$ need not be absolutely permutable. The sublattice $L(\mathscr{S})$ of the lattice $\Pi(M)$ is absolutely permutable only in trivial cases*: card $\mathscr{S} = 1$ *or $\mathscr{S}$ is an two - element chain.*

Proof. Let $\mathscr{S}$ be absolutely permutable and $A, B \in \mathscr{S}$, $A \nleq B$ and $B \nleq A$. Assume that $\mathscr{S}_1$ is absolutely permutable. Then $A \wedge B \in \mathscr{S}_1 \subset L(\mathscr{S})$, $A \wedge B < A$ but $A = (A \wedge B) \vee A \neq A \vee B$ which contradicts the Lemma 4.1. The set $\mathscr{S}$ cannot include a three-element chain because it contradicts the Lemma 4.1.

In the paper [11] a special kind of subdirect products (W — constructable) is characterized by using the absolute permutability. In [11, Theorem 2] the following assertion is included:

Assertion A. *Let an algebra $\mathfrak{A}$ be an internal subdirect product of $\mathfrak{A}/\Theta_\gamma$, where Θ_γ, $\gamma \in \Gamma$, are congruence relations on $\mathfrak{A}$. Then $\mathfrak{A}$ is W — constructable if and only if the system $\{\Theta_\gamma : \gamma \in \Gamma\}$ is absolutely permutable.*

In [11] the next Theorem W is proved (using Assertion A and Lemma W which is incorrect):

Theorem W [11, Theorem 3]. *Let $\Theta_1, \ldots, \Theta_n$ be congruences on $\mathfrak{A}$ yielding an internal subdirect representation of $\mathfrak{A}$. Then $\mathfrak{A}$ is W-constructable if and only if*

(i) $\Theta_1, \ldots, \Theta_n$ *are pairwise permutable.*

(ii) $C([\Theta_1, \ldots, \Theta_n])$ *is distributive.*

(iii) $\Theta_{i_0} \vee \Theta_{j_0} = \bigvee\{\Theta_i : i = 1, \ldots, n\}$ *for all different pairs* Θ_{i_0}, Θ_{j_0} *in* $\{\Theta_1, \ldots, \Theta_n\}$.

Theorem W is incorrect because the absolute permutability of the set $\{\Theta_1, \ldots, \Theta_n\}$ is not equivalent to conditions (i), (ii), (iii) of Theorem W (see Example 4.1). Theorem W can be modified (using Assertion A and Theorem 4.3) in the following way:

Theorem 4.6. *Let $\{\Theta_1, \ldots, \Theta_n\}$ be a finite set of congruence relations on an algebra $\mathfrak{A}$ yielding an internal subdirect representation of $\mathfrak{A}$. The $\mathfrak{A}$ is W-constructable if and only if the next condition holds:*

$\Theta_i \cdot \bigwedge\{\Theta_j : j \neq i,\ j = 1, \ldots, n\} = \bigvee\{\Theta_j : j = 1, \ldots, n\}$ *for any* $i = 1, \ldots, n$.

Remark 4.4. In accordance with Theorem 4.6, the results of § 3 [11] are to be modified.

5. Direct Decompositions

In the papers [8], [11] the next Theorem is proved:

Theorem B. *Let $\mathfrak{A}$ be an algebra. There exists a one-one correspondence between the non-trivial direct decompositions of the algebra $\mathfrak{A}$ and the sets $\mathscr{S} = \{\Theta_\gamma : \gamma \in \Gamma\}$ of non-trivial congruence relations (i.e. different from 0 and I) on $\mathfrak{A}$ having the following properties:*

$$\bigwedge\{\Theta_\gamma :\ \gamma \in \Gamma\} = 0. \tag{1}$$

$$\bigvee\{\Theta_\gamma :\ \gamma \in \Gamma\} = I. \tag{2}$$

$$\mathscr{S} \text{ is absolutely permutable.} \tag{3}$$

Using Theorem B, Lemma 4.1, Theorem 4.1 and Lemma 1.1 we get:

Theorem 5.1. *Let $\mathfrak{A}$ be an algebra. There exists a one-one correspondence between the non-trivial direct decompositions of the algebra $\mathfrak{A}$ and the sets $\mathscr{S} = \{\Theta_\gamma\colon \gamma \in \Gamma\}$ of non-trivial congruence relations on $\mathfrak{A}$ having the following properties:*

$$(1) \qquad \bigwedge\{\Theta_\gamma\colon \gamma \in \Gamma\} = 0.$$

$$(2') \qquad \Theta_\iota \vee \Theta_\varkappa = I \text{ for any } \Theta_\iota,\ \Theta_\varkappa \in \mathscr{S},\ \Theta_\iota \neq \Theta_\varkappa.$$

(3′) $\qquad\mathscr{S}$ *is weakly or strongly or absolutely permutable.*

Using Theorem 5.1 and Theorem 4.3 we get:

Theorem 5.2. *Let $\mathfrak{A}$ be an algebra. There exists a one-one correspondence between the finite non-trivial direct decompositions of $\mathfrak{A}$ and the finite sets $\{\Theta_1, \ldots, \Theta_n\}$ of non-trivial congruence relations on $\mathfrak{A}$ having the following properties:*

$$(1) \qquad \bigwedge\{\Theta_i\colon i = 1, \ldots, n\} = 0.$$

$$(2'') \qquad \Theta_i \,.\, \bigwedge\{\Theta_j : j \neq i,\ j = 1, \ldots, n\} = I \text{ for any } i = 1. \ldots, n.$$

Using Theorems 4.3, 4.2 and 5.1 we get:

Corollary 5.1. *The sublattice $L(\mathscr{S})$ of the congruence lattice generated by the set $\mathscr{S}$ of Theorem 5.1 is distributive and any two congruence relations in this sublattice are permutable. In particular, if $\mathscr{S}$ is finite set of Theorem 5.2, then $L(\mathscr{S})$ is a Boolean lattice of 2^n elements* [4, Chap. 3., Ex. 4., p. 154].

Example 5.1. The following example shows that an absolutely permutable set $\mathscr{S}$ of equivalence relations need not satisfy the completely distributive law $\bigwedge\{\bigvee\{A_{ij} : j \in K_i\} : i \in J\} = \bigvee\{\bigwedge\{A_{if(i)} : i \in J\} : f \in \times\{K_i : i \in J\}\}$. It follows that the complete lattice generated by $\mathscr{S}$ need not be completely distributive. In our example $\mathscr{S}$ is even the system of equivalence relations corresponding to a direct decomposition: Let M be the set of all sequences $(a_i : i \in N)$, where a_i is 0 or 1 and let for $a, b \in M$, $a\,\Theta_n b$ if $a_n = b_n$. Let $\mathscr{P}$ be the set of all two - element subsets of the set N. We assert that

$$\bigwedge\{(\Theta_i \vee \Theta_j) : \{i, j\} \in \mathscr{P}\} \neq \bigvee\{\bigwedge\{\Theta_{f(K)} : K \in \mathscr{P}\} : f \in F\}$$

where F is the Cartesian product of all elements of $\mathscr{P}$. Indeed, let $x, y \in M$, where $x_n = 0$ and $y_n = 1$ for all $n \in N$. Obviously $x(\Theta_i \vee \Theta_j)y$ if $i \neq j$, hence $x(\bigwedge\{(\Theta_i \vee \Theta_j) : \{i, j\} \in \mathscr{P}\})y$. The set $N - \{f(K) : K \in \mathscr{P}\}$ contains at most one element. Indeed, let i be an element of this set. Then $f(\{i, j\}) = j$ for all $j \neq i$, hence $N - \{f(K) : K \in \mathscr{P}\} = \{i\}$. Suppose next that $x(\bigvee\{\bigwedge\{\Theta_{f(K)} : K \in \mathscr{P}\} : f \in F\})y$. Then there are elements $x^0, x^1, \ldots, x^n$; $f_1, f_2, \ldots, f_n$ $(x^i \in M,\ f_i \in F)$ such that $x^0 = x$, $x^n = y$, and $x^{i-1}(\bigwedge\{\Theta_{f_i(K)} : K \in \mathscr{P}\})x^i$,

$i = 1, \ldots, n$ ($n \in N$). The sequences x^{i-1}, x^i differ in at most one index. Hence the set $\{h \in N\colon x_h \neq y_h\}$ is finite (in fact it contains at most n elements), which is a contradiction.

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