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Matematicko-fyzikálny časopis, Vol. 16 (1966), No. 4, 349--352

Persistent URL: `http://dml.cz/dmlcz/126977`

MATEMATICKO-FYZIKÁLNY ČASOPIS SAV, 16, 4, 1966

ON CYCLIC DECOMPOSITIONS OF THE COMPLETE GRAPH INTO $(4m + 2)$-GONS

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The construction of a cyclic decomposition of the complete graph into p-gons, where $p \equiv 0 \pmod 4$, was given in paper [1]; the case $p \equiv 1 \pmod 2$ was investigated in [2]. This article gives the solution of the problem of a cyclic decomposition of the complete graph in the remaining case $p \equiv 2 \pmod 4$.

Let k be natural, and let p of the form $n = 4m + 2$ be given, where m is natural. Denote $n = 2kp + 1$. In agreement to [2] the $(k \times p)$-matrix $\mathbf{A} = \|a_{ij}\|$ will be called a matrix of type (1), if $\{a_{11}, \ldots, a_{kp}\} = \{1, 2, \ldots, kp\}$ holds.

Theorem 1. *For arbitrary k and p of the form $p = 4m + 2$ there exists a $(k \times p)$-matrix $\mathbf{A} = \|a_{ij}\|$ of the type (1) and constants $\varepsilon_{ij} = 1$ or -1 such that*

$$\sum_{j=1}^{p} a_{ij}\varepsilon_{ij} \equiv 0 \pmod n$$

holds for all $i = 1, \ldots, k$.

Proof. The matrix $\mathbf{A} = \|a_{ij}\|$ and the constants ε_{ij} satisfying the conditions of the theorem can be determined as follows:

$$a_{ij} = \begin{cases} (i-1)p + j & 1 \leqq j \leqq p - 2 \\ (k - i + 1)p - 1 & j = p - 1 \\ (k - i + 1)p & j = p, \end{cases}$$

where $\varepsilon_{i,1}$ equals -1 and all remaining ε_{ij} equal $+1$ if $m = 1$, $\varepsilon_{i,4}$; $\varepsilon_{i,6}$, $\varepsilon_{i,7}$; $\varepsilon_{i,10}$, $\varepsilon_{i,11}$, $\ldots$, $\varepsilon_{i,p-4}$, $\varepsilon_{i,p-3}$ equal -1 and all remaining ε_{ij} equal $+1$ if $m \geq 2$.

One can see easily that the conditions of the theorem are satisfied. Obviously each of the numbers $1, 2, \ldots, kp$ appears in the matrix $\mathbf{A}$ exactly once. The i-th row of the matrix $\mathbf{A}$ is of the form:

$(i-1)p + 1$, $(i-1)p + 2$, $\ldots$, $ip - 4$, $ip - 3$, $ip - 2$, $(k - i + 1)p - 1$, $(k - i + 1)p$. We obtain

$$\sum_{j=1}^{p} a_{ij}\,\varepsilon_{ij} = [(i-1)p+1] + [(i-1)p+2] + [(i-1)p+3] -$$

$$- [(i-1)p+4] + \{(i-1)p+5] - [(i-1)p+6] -$$

$$- [(i-1)p+7] + [(i-1)p+8]\} + \{(i-1)p+9) -$$

$$- [(i-1)p+10] - [(i-1)p+11] + [(i-1)p+12]\} + \ldots$$

$$\ldots + \{(ip-5) - (ip-4) - (ip-3) + (ip-2)\} +$$

$$+ [(k-i+1)p-1] + (k-i+1)p = 2(i-1)p + 2 +$$

$$+ (k-i+1)p - 1 + (k-i+1)p = 2kp + 1.$$

Let there be given a complete graph $\langle n \rangle$ with n vertices $v_1, \ldots, v_n$, where n is of the form $n = 2kp + 1$, p is of the form $p = 4m + 2$, k is natural.

The length of an edge v_iv_j in the graph $\langle n \rangle$ is defined as a minimum of the numbers $|i - j|$, $n - |i - j|$. By the turning of an edge v_iv_j in the graph $\langle n \rangle$ we mean the adding of a 1 to the indices, whereby we get the edge $v_{i+1}v_{j+1}$ from the edge v_iv_j (the indices are taken modulo n). By the turning of a polygon in the graph $\langle n \rangle$ we mean a simultaneous turning of all edges of the polygon.

A decomposition $\mathscr{R} = \{K_1, \ldots, K_r\}$ of the complete graph into r polygons $K_1, \ldots, K_r$ is called cyclic if the following holds: If $\mathscr{R}$ contains a polygon K, then $\mathscr{R}$ contains also the polygon K' obtained from K by turning.

Theorem 2. *For an arbitrary natural k and for an arbitrary p of the form $p = 4m + 2$, where m is natural, there exists a cyclic decomposition of the graph $\langle 2kp + 1 \rangle$ into p-gons.*

Proof. Let in the graph $\langle 2kp + 1 \rangle$ be given k polygons, with p edges each:

$$K_j = \{v_{i_{j1}}v_{i_{j2}}, v_{i_{j2}}v_{i_{j3}}, \ldots, v_{i_{jp}}v_{i_{j1}}\};\ \{i_{j1}, \ldots, i_{jp}\} \subset \{1, 2, \ldots, 2kp + 1\}, j = 1, 2, \ldots, k.$$

If each of the possible lengths $1, 2, \ldots, kp$ in the graph $\langle 2kp + 1 \rangle$ is the length of exactly one of kp edges of the p-gons $K_1, \ldots, K_k$, then call the system of p-gons $\mathscr{K} = \{K_1, \ldots, K_k\}$ a basic system of p-gons in the graph $\langle 2kp + 1 \rangle$. We obtain a cyclic decomposition of the graph $\langle 2kp + 1 \rangle$ into p-gons if any of the p-gons of the basic system is turned successivelly $2kp$ times.

The basic system of p-gons in the graph $\langle 2kp + 1 \rangle$ can be determined with the help of the matrix of the type (1) satisfying the condition of Theorem 1. Let $\mathbf{A} = \|a_{ij}\|$ be such a matrix and let $\mathbf{E} = \|\varepsilon_{ij}\|$ be the corresponding matrix of constants constructed to prove Theorem 1. Denote by $\mathbf{A}'$ and $\mathbf{E}'$ the matrix which arises from the matrix $\mathbf{A}$ and $\mathbf{E}$ if the elements of each row of the matrix $\mathbf{A}$ and $\mathbf{E}$ respectively are permuted:

a) for $m = 1$ under the identic permutation

b) for $m \geq 2$ under the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots & 2m+2 & 2m+3 & 2m+4 \\ 1 & 2 & 3 & 4 & 5 & 4m+1 & 4m-1 & 4m-3 & \dots & 9 & 7 & 4m \end{pmatrix}$$

$$\begin{matrix} 2m+5 & \dots & 4m & 4m+1 & 4m+2 \\ 4m-2 & \dots & 8 & 6 & 4m+2 \end{matrix}\Bigg).$$

The matrix $\boldsymbol{A'}$ clearly also satisfies the conditions of Theorem 1 with the constants ε'_{ij}.

Choose an arbitrary vertex v_x ($x \in \{1, 2, \dots, 2kp + 1\}$). The p-gon K_i will be determined as follows:

$$K_i = \{v_x v_{x+c_{i1}}, v_{x+c_{i1}} v_{x+c_{i2}}, \dots, v_{x+c_{i,p-1}}, v_x\},$$

where

$$c_{ij} = \sum_{\nu=1}^{j} a'_{i\nu} \varepsilon'_{i\nu}, \; i = 1, \dots, k; \; j = 1, \dots, p.$$

It is easy to verify that no vertex appears in the sequence of the edges of K_i more than two times. Namely, it is easy to verify an equivalent statement that no pair of numbers a, b, for which $a \equiv b \pmod{2kp + 1}$, appears in the following sequence of p numbers:

$(i-1)p + 1$, $2(i-1)p + 3$, $3(i-1)p + 6$, $2(i-1)p + 2$, $3(i-1)p + 7$, $p(2i + k - 2) + 6$, $p(i + k - 2) + 9$, $p(2i + k - 2) + 4$, $p(i + k - 2) + 11, \dots, p(i + k - 2) + 2m + 3, p(2i + k - 3) + 2m + 12, p(i + k - 2) + 2m + 5$, $p(2i + k - 2) + 2m + 3$, $p(i + k - 2) + 2m + 7$, $p(2i + k - 2) + 2m + 1$, $p(i + k - 2) + 2m + 9$, $\dots$, $p(i + k - 1) - 1$, $p(2i + k - 2) + 7$, $p(i + k - 1) + 1$, $2kp + 1$.

This completes the proof of Theorem 2.

Example 1. The cyclic decomposition of the complete graph $\langle 61 \rangle$ into 10-gons will be obtained if each of the 10-gons K_1, K_2, K_3 is turned successivelly 60 times (the vertices are denoted by v_i, $i = 1, \dots, 61$):

$$K_1 = \{v_1v_2, v_2v_4, v_4v_7, v_7v_3, v_3v_8, v_8v_{37}, v_{37}v_{30}, v_{30}v_{38}, v_{38}v_{32}, v_{32}v_1\}$$
$$K_2 = \{v_1v_{12}, v_{12}v_{24}, v_{24}v_{37}, v_{37}v_{23}, v_{23}v_{38}, v_{38}v_{57}, v_{57}v_{40}, v_{40}v_{58}, v_{58}v_{42}, v_{42}v_1\}$$
$$K_3 = \{v_1v_{22}, v_{22}v_{44}, v_{44}v_6, v_6v_{43}, v_{43}v_7, v_7v_{16}, v_{16}v_{50}, v_{50}v_{17}, v_{17}v_{52}, v_{52}v_1\}.$$

By Theorem 2 with $p \equiv 2 \pmod 4$ there exists for an arbitrary $n \equiv 1 \pmod{2p}$ a cyclic decomposition of the graph $\langle n \rangle$ into p-gons. Obviously if $p \equiv 2 \pmod 4$ there exists no x, $x \not\equiv 1 \pmod{2p}$ so that for an arbitrary $n \equiv x \pmod{2p}$ there exists a cyclic decomposition of the graph $\langle n \rangle$ into p-gons. However, it is easy to verify that for some p, $p \equiv 2 \pmod 4$ there exist n and x, $x \neq 1$ so that $n \equiv x$ (mod 2p) and there exists a cyclic decomposition of the graph $\langle n \rangle$ into p-gons. This fact is shown by the following example.

Table 1

K_1		K_2		K_3		K_4		K_5		K_6	
[illegible]	6	[illegible]	8	[illegible]	9	[illegible]	10	[illegible]	11	[illegible]	12
6	15	8	[illegible]	9	7	10	[illegible]	11	7	12	
15	23	[illegible]	[illegible]	7	16	7	[illegible]	7	18	[illegible]	19
26	40	[illegible]	11	16	14	1	1	18	11	13	[illegible]
40	7	11	[illegible]	11	23	14	[illegible]	11	[illegible]	11	[illegible]
7	22	[illegible]	21	23	[illegible]	22	[illegible]	23	21	[illegible]	21
[illegible]	39	31	[illegible]	21	29	21	[illegible]	21	32	[illegible]	[illegible]
39	8	29	[illegible]	[illegible]	28	31	[illegible]	32	28	[illegible]	[illegible]
8	27	28	36	28	[illegible]	28	38	28	39	[illegible]	[illegible]
[illegible]	1	[illegible]	35	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1	21	[illegible]	42	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	16	[illegible]	[illegible]	[illegible]	[illegible]	47	[illegible]	[illegible]	42	[illegible]	[illegible]
16	25	42	1	42	[illegible]	[illegible]	[illegible]	42	1	[illegible]	[illegible]
25		[illegible]		2		3		[illegible]		[illegible]	

Example 2. A cyclic decomposition of the graph $\langle 49 \rangle$ into 14-gons (in the case $49 = 21$ (mod 28)). The decomposition is given in Table 1. In this table the vertices v_i are denoted briefly as i (v_0 denotes an arbitrary vertex v_i of the graph $\langle 49 \rangle$ (all numbers in Table 1 are taken modulo 49). The cyclic decomposition of the graph $\langle 49 \rangle$ into 14-gons will be obtained if the 14-gon K_1 is turned successively 48 times, and each of the 14-gons K_2, K_3, K_4, K_5, K_6 successively 6 times, which makes together 84 of 14-gons.

Now Theorem 2 can be combined with Theorem 1 of [1]:

Theorem 3. *For an arbitrary natural k and for an arbitrary even $p \geq 2$ there exists a cyclic decomposition of the graph $\langle 2kp + 1 \rangle$ into p-gons.*

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Received September 9, 1965.

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