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POPRODUCT OF LATTICES AND SORKIN'S THEOREM

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Balbes and Horn [1] introduced and studied in the class of distributive lattices a new operator, the so-called "order sum", which is a generalization of the free product and the ordinal sum of distributive lattices. This paper introduces such an operator for an arbitrary equational class $\mathscr{K}$ of lattices, this operator will be called a $\mathscr{K}$ — poproduct. In part two the word problem for the poproduct in the class of all lattices is solved. The main result, the generalization of well known Sorkin's theorem, can be found in part three of the paper. Several ideas and methods have been borrowed from paper [4].

1. Introduction. Let P be a poset and let L_p, $p \in P$ be pairwise disjoint lattices. The lattice operations in each L_p will be denoted by $\cup$, $\cap$. Let $Q = \bigcup_{p \in P} L_p$ be partially ordered in the following way:

for $a, b \in Q$ we put $a \leqq b$ if and only if one of the conditions

(1) and (2) holds:

(1) there is a $p \in P$ such that $a, b \in L_p$ and the relation $a \leqq b$ in L_p holds;

(2) there are $p, r \in P$ such that $a \in L_p$, $b \in L_r$ and the relation $p < r$ in the poset P holds.

If f is a mapping from Q into M, then f_p denotes its restriction on L_p.

Definition 1. *Let $\mathscr{K}$ be an equational class of lattices. Let $L, L_p \in \mathscr{K}$ for $p \in P$ and let P be a poset. The lattice L is said to be the $\mathscr{K}$ — poproduct of the lattices L_p if:*

1. *there is an isotone injection $i \colon Q \to L$ such that for each $p \in P$ i_p is a lattice homomorphism;*
2. *if $M \in \mathscr{K}$, then for every isotone mapping $f \colon Q \to M$ such that for each $p \in P$ f_p is a lattice homomorphism, there exists uniquely a lattice homomorphism Ψ: $L \to M$ such that $\Psi \circ i = f$.*

It is obvious that the definitions of the order sum from [1] and of the poproduct in the class $\mathscr{D}$ of distributive lattices coindice.

The $\mathscr{L}$ — poproduct, i.e. the poproduct in the class of all lattices, will be briefly called poproduct (of lattices).

The existence of the poproduct follows from [3]: R. A. Dean has constructed the free lattice $FL(P, \mathfrak{A}, \mathfrak{B})$ generated by the poset P and preserving the ordering of P, those lub's of a family $\mathfrak{A}$ of finite subsets of P which possess lub's in P and those glb's of a family $\mathfrak{B}$ of finite subsets of P which possess glb's in P.

It is easy to see that a $\mathscr{K}$ — poproduct L of lattices is uniquely determined up to isomorphism by the poset P and the lattices L_p, $p \in P$.

The following theorem will explain the relation between the $\mathscr{K}$ — poproduct, the $\mathscr{K}$ — free product and the ordinal sum, respectively. For the class $\mathscr{D}$ of distributive lattices, see [1].

Theorem 1. *Let $\mathscr{K}$ be a nontrivial equational class of lattices. Let L be the $\mathscr{K}$ — poproduct of the lattices $L_p \in \mathscr{K}$, $p \in P$ and let P be a poset. Then*

(1) *L forms the $\mathscr{K}$ — free product of the lattices $L_p (p \in P)$ if and only if P is an anti-chain.*

(2) *Lforms the ordinal sum of lattices $L_p (p \in P)$ if and only if P is a chain.*

Proof. (1) If P is an anti-chain, then according to the definition of the poproduct

1. there exists a family of lattice homomorphisms $i_p : L_p \to L$ for each $p \in P$;
2. if $M \in \mathscr{K}$, then for every family of lattice homomorphisms $f_p : L_p \to M$ there exists uniquely a lattice homomorphism $\Psi : L \to M$ such that $\Psi \circ i_p = f_p$ for each $p \in P$

That means that L is a $\mathscr{K}$ — free product of $L_p (p \in P)$.

Conversely, let L be a $\mathscr{K}$ — free product of $L_p (p \in P)$. Suppose that P is not an anti-chain, hence there exist q, $r \in P$ such that q $< r$. Let $M = \{0, 1\}$ be a two-element chain and let $f_p (p \in P)$ be a family of lattice homomorphism $f_p : L_p \to M$. Let $f_q(L_q) = 1$, $f_r(L_r) = 0$. Then there exists a homomorphism $\Psi : L \to M$ such that $\Psi \circ i_q = f_q$, $\Psi \circ i_r = f_r$. Since i is isotone and Ψ is a homomorphism, $\Psi \circ i_q(L_q) \leqq \Psi \circ i_r(L_r)$, contradicting the fact that $f_q(L_q) = 1 > 0 = f_r(L_r)$.

(2) If P is a chain, then Q is a lattice which obeys the conditions 1, 2 of definition 1, hence $L = Q$. Q is clearly an ordinal sum of L_p, $p \in P$.

Conversely, let L be an ordinal sum of L_p, $p \in P$. Then $L = Q$ and therefore P is a chain.

2· The word problem. Throughout the paper Q will denote the partially ordered set mentioned in the introduction. Let us denote by $\mathscr{P}(Q)$ the set of lattice polynomials (terms) over Q. These polynomials are formed from symbols denoting elements of Q and from the symbols $\cup$, $\cap$. For each $A \in \mathscr{P}(Q)$ we define a natural number $l(A)$ — the length of A — as follows:

if $A \in Q$, then $l(A) = 1$;

if $A_0, A_1 \in \mathscr{P}(Q)$, then $l(A_0 \cup A_1) = l(A_0 \cap A_1) = l(A_0) + l(A_1)$.

For each $A \in \mathscr{P}(Q)$ and each $\lambda \in P$, the existence and the value of the upper λ — cover, $A^{(\lambda)}$, $A^{(\lambda)} \in L_\lambda$, and the lower λ — cover, $A_{(\lambda)}$, $A_{(\lambda)} \in L_\lambda$, are defined as follows:

Definition 2.

(i) *if* $A \in L_\lambda$, *then* $A_{(\lambda)}$, $A^{(\lambda)}$ *exist and* $A_{(\lambda)} = A^{(\lambda)} = A$.

$A_{(\mu)}$, $A^{(\mu)}$ *do not exist for* $\mu \neq \lambda$.

(ii) *if* $A = B \sqcap C$, *then* $A^{(\lambda)}$ *exists if and only if* $B^{(\lambda)}$ *and* $C^{(\lambda)}$ *both exist and in this event* $A^{(\lambda)} = B^{(\lambda)} \cup C^{(\lambda)}$.

$A_{(\lambda)}$ *exists if and only if at least one of* $B_{(\lambda)}$, $C_{(\lambda)}$ *exists;* $A_{(\lambda)} = B_{(\lambda)}$ *(respectively* $C_{(\lambda)}$*) if only* $B_{(\lambda)}$ *(respectively* $C_{(\lambda)}$*) exists, and* $A_{(\lambda)} = B_{(\lambda)} \cup C_{(\lambda)}$ *if both* $B_{(\lambda)}$, $C_{(\lambda)}$ *exist.*

(iii) *if* $A = B \sqcap C$, *then* $A_{(\lambda)}$ *exists if and only if* $B_{(\lambda)}$ *and* $C_{(\lambda)}$ *both exist and in this event* $A_{(\lambda)} = B_{(\lambda)} \cap C_{(\lambda)}$.

$A^{(\lambda)}$ *exists if and only if at least one of* $B^{(\lambda)}$, $C^{(\lambda)}$ *exists;* $A^{(\lambda)} = B^{(\lambda)}$ *(respectively* $C^{(\lambda)}$*) if only* $B^{(\lambda)}$ *(respectively* $C^{(\lambda)}$*) exists, and* $A^{(\lambda)} = B^{(\lambda)} \cap C^{(\lambda)}$ *if both* $B^{(\lambda)}$, $C^{(\lambda)}$ *exist.*

Definition 3. *For any* A, $B \in \mathscr{P}(Q)$ *we define by induction on* $l(A) + l(B)$ *the relation* $A \subseteqq B$ *to hold if and only if at least one of the conditions* (1) *to* (6) *below holds*:

(1) $A = B$;

(2) *there are* λ, $\mu \in P$ *such that* $A^{(\lambda)}$, $B_{(\mu)}$ *exist and* $A^{(\lambda)} \leqq B_{(\mu)}$ *(in the ordering of* Q*)*;

(3) $A = A_0 \sqcup A_1$, *where* $A_0 \subseteqq B$ *and* $A_1 \subseteqq B$;

(4) $A = A_0 \sqcap A_1$, *where* $A_0 \subseteqq B$ *or* $A_1 \subseteqq B$;

(5) $B = B_0 \sqcup B_1$, *where* $A \subseteqq B_0$ *or* $A \subseteqq B_1$;

(6) $B = B_0 \sqcap B_1$, *where* $A \subseteqq B_0$ *and* $A \subseteqq B_1$;

Set $A \cong B$ *if* $A \subseteqq B$ *and* $B \subseteqq A$.

The relation $\cong$ is reflexive and symmetric, we shall show $\cong$ to be transitive.

Lemma 1. *If* $A_{(\lambda)}$, $A^{(\mu)}$ *both exist, then* $\lambda = \mu$.

Proof. Cf. [4], Lemma 1.

Lemma 2. *If* $A^{(\lambda)}$ *exists, then* $A \subseteqq A^{(\lambda)}$. *If* $A_{(\lambda)}$ *exists, then* $A_{(\lambda)} \subseteqq A$.

Proof. It follows from (2): $A^{(\lambda)} = [A^{(\lambda)}]_{(\lambda)}$, $[A_{(\lambda)}]^{(\lambda)} = A_{(\lambda)}$.

Theorem 2. $A \subseteqq B$ *and* $B \subseteqq C$ *implies* $A \subseteqq C$.

Proof. Let $A \subseteqq B$ and $B \subseteqq C$. If both of these relations are due to (2), we apply Lemma 1 and so $A \subseteqq C$ follows by (2). Otherwise we proceed by induction on $l(A) + l(B) + l(C)$.

Thus $\cong$ is an equivalence relation. Given $A \in \mathscr{P}(Q)$, denote $\langle A \rangle$ the equivalence class of A under $\cong$ and let $L = \mathscr{P}(Q)/\cong$. Define the binary relation $\leqq$ on L by $\langle A \rangle \leqq \langle B \rangle$ if and only if $A \subseteq B$. Then $\leqq$ is a partial order on L with respect to which L is a lattice. Moreover, by (3) and (5), $\langle A \rangle \cup \langle B \rangle = \langle A \sqcup B \rangle$ and dually $\langle A \rangle \cap \langle B \rangle = \langle A \sqcap B \rangle$.

Theorem 3. *L is the poproduct of L_λ, $\lambda \in \mathrm{A}$.*

Proof. $Q \subset L$, thus $i\colon Q \to L$ is an isotone injection. Let M be a lattice, let $f\colon Q \to M$ be an isotone mapping such that f_λ is a homomorphism. Let $g\colon L \to M$ be defined inductively as follows:

$$\text{if } A \in Q,\ g(A) = f(A);$$
$$g(A \cup B) = g(A) \cup g(B);$$
$$g(A \cap B) = g(A) \cap g(B).$$

Then the following statements hold:

If $A^{(\lambda)}$ exists, then $g(A) \leqq g(A^{(\lambda)})$. If $A_{(\lambda)}$ exists, then $g(A_{(\lambda)}) \leqq g(A)$. The proof is omitted, because, similarly as in [4], the formal computations are a special case of computations in the proof of Theorem 4. Now we have to show that g factors through $\cong$. It is enough to prove that $A \subseteqq B$ implies $g(A) \leqq g(B)$.

Let $A \subseteqq B$.

If $A \subseteqq B$ is due to (2), $A^{(\lambda)} \leqq B_{(\mu)}$, then since f is isotone on Q, there holds $g(A) \leqq g(A^{(\lambda)}) = f(A^{(\lambda)}) = f(B_{(\mu)}) = g(B_{(\mu)}) \leqq g(B)$.

If $A \subseteqq B$ is due to (3)—(6), the proof is by induction on $l(A) + l(B)$. Thus g is well — defined. g is isotone on Q, because f and g are equal on Q. g is an isomorphism by definition. Therefore $g \circ i = f$. Finally, g is unique, because it is a homomorphism and L is generated by Q.

Theorem 3 is proved.

3. Sorkin's theorem.

Theorem 4. *Let L be the poproduct of L_λ, $\lambda \in P$, let K be a lattice and let $i\colon Q \to L$ be an isotone injection such that for each $p \in P$, i_p is a lattice homomorphism. Let $f\colon Q \to K$ be an isotone mapping. Then there is an isotone mapping $g\colon L \to K$ such that $g \circ i = f$.*

Proof. We define $h\colon \mathscr{P}(Q) \to K$ by induction on the length of the polynomials in $\mathscr{P}(Q)$:

(i) if $A \in Q$, then $h(A) = f(A)$;

(ii) if $A = B \cup C$ and $A_{(\lambda)}$ is defined for no $\lambda \in P$, then $h(A) = h(B) \cup h(C)$, otherwise $h(A) = h(B) \cup h(C) \cup \bigcup\, (f(A_{(\lambda)})/\lambda \in P$ and $A_{(\lambda)}$ exists);

(iii) if $A = B \cup C$ and $A^{(\lambda)}$ is defined for no $\lambda \in P$, then $h(A) = h(B) \cap h(C)$, otherwise $h(A) = h(B) \cap h(C) \cap \bigcap\, (f(A^{(\lambda)})/\lambda \in P$ and $A^{(\lambda)}$ exists).

By definition 2, $A^{(\lambda)}$ or $A_{(\lambda)}$ exists for only finitely many $\lambda \in P$ and thus the definition makes sense. We define $g\colon L \to K$ by requiring that $g(\langle A \rangle) = h(A)$. Now the following statement holds:

If $A^{(\lambda)}$ is defined, then $h(A) \leqq f(A^{(\lambda)})$ (and dually).

The proof is by induction on $l(A)$, cf. [4]. Similarly as in [4], it can be shown by induction on $l(A) + l(B)$ that if $A, B \in \mathscr{P}(Q)$, $A \subseteqq B$, then $h(A) \leqq h(B)$. Therefore g is well defined and isotone. From the definitions of h and g it follows that $g \circ i = f$.

Remark. If f is a $\cup$-morphism, then g is $\cup$-morphism (and dually).

Proof. Cf. [4].

From Theorems 1 and 4 there immediately follows the

Corollary (Sorkin's theorem, [4]). *Let L be the free product of L_λ, $\lambda \in \Lambda$, let K be a lattice and let for each $\lambda \in \Lambda$, $f_\lambda : L_\lambda \to K$ be an isotone mapping. Then there is an isotone mapping $g \colon L \to K$ extending all the f_λ.*

4. Sublattices and irreducible elements.

Theorem 5. *For each $p \in P$ let L_p^* be a sublattice of L_p and let L^* be the sublattice of the poproduct of the L_p generated by L_p^*. Then L^* is the poproduct of the L_p^*.*

Proof. Cf. [4], Theorem 2.

Similar arguments as in the proof of Theorem 3 of [4] prove.

Theorem 6. *Let P consist of more than one element and let $P' \subseteq P$. Let L be the poproduct of L_p, $p \in P$. Then $L - \bigcup_{p \in P'} L_p$ is a sublattice of L if and only if for each $p \in P'$ there holds:*

L_p is a chain or p is comparable with each $q \in P$.

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