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CONCERNING CONNECTIONS ON ASSOCIATED BUNDLES

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A connection in a principal fibre bundle induces a vertical projection in each of the tangent spaces to each associated fibre bundle, this projection being de facto the covariant differential of sections in this associated bundle. It is even this covariant differentiation which is important in practical computations with connections, e. g. on vector bundles. Suppose there are two fibre bundles associated to the same (or locally the same, cf. below) principal fibre bundle in which a connection is given. Then the two covariant differentiations induced in both associated bundles are in some way mutually related. If there is namely a bundle morphism of these bundles satisfying a general enough condition, then it preserves the covariant differentiations. This follows from the statements of propositions 1 and 1a which allow us to obtain a number of results concerning the behaviour of the covariant differential in concrete cases, usually proved independently (cf. the examples below).

We consider the case of C^∞ differentiability and finite dimension throughout the paper. Suppose $H(B, G)$ to be a principal bundle over B with structure group G and projection $b : H \to B$, and let there be given a (first order) connection in H given by a differentiable distribution $h \to T(H)_h \subset T(H)_h$ on H. Here $T(H)$ is the tangent bundle to H and we denote by $T(H)_h$ the vertical subspace of $T(H)_h$, i. e. $X \in T(H)_h \implies (\mathrm{d}b)X - 0_{bh}$, where 0_x denotes the zero element in $T(B)_x$. Thus we have for each $h \in H$ the direct decomposition $T(H)_h \quad T(H)^+_h \oplus T(H)_h$ and $T(H)^+_{hg} \quad R^*_g T(H)^+_h$ with $g \in G$. Denote by V_H the canonical vertical projection $V_H : T(H) \to T(H)$ given by the connection.

Suppose now that $E \quad E(B, F, G, H)$ is a fibre bundle associated to H and denote $\lambda : H \times F \to E$ and $p : E \to B$ the natural projections. For each $z \in E$ denote by $T(E)_z \subset T(E)_z$ the subspace of vertical elements, i. e. $\eta \in T(E)_z$ $(\mathrm{d}p)\,\eta \quad 0_{pz}$. Since the projection λ satisfies $\lambda(R_g \times L_{g^{-1}}) \quad \lambda$ for each $g \in G$, where $L_g : F \to F$ denotes the action on the left of the element $g \in G$, we have

$$(1) \qquad \mathrm{d}\lambda(R^*_g + L^*_{g^{-1}}) - \mathrm{d}\lambda \,.\, T(H) \oplus T(F) \to T(E)\,.$$

For a fixed $f \in F$ denote $i_f : H \to H \times F$ the natural injection $h \to (h.f)$ and let $\mu_f \quad \lambda i_f : H \to E$. Note that $\mathrm{d}i_f$ takes the element $X \in T(H)_h$ into $X + 0_f \in T(H)_h \oplus T(F)_f$. The linear mapping $\mathrm{d}\mu_f$ takes each $T(H)_h$ into $T(E)_x$ where $x \quad h(f)$ since

(2) $$\mu_f(h) \quad h(f)\,.$$

On the other hand $p\mu_f \quad b \Rightarrow \mathrm{d}p \; \mathrm{d}\mu_f \quad \mathrm{d}b$ and this means that $\mathrm{d}\mu_f$ takes also $T(H)$ into $T(E)$ whatever be $f \in F$ and that Ker $\mathrm{d}\mu_f \subset T(H)$.

Now we may define $T(E)_x \quad \mathrm{d}\mu_f T(H)_h$, where h and f are such that $h(f) \quad x$ In fact (1) gives $\mathrm{d}\mu_{g\,^1f}\,(T(H)^+_{hg}) \quad \mathrm{d}\lambda\, \mathrm{d}i_{g\,^1f}\, T(H)_{hg} \quad \mathrm{d}\lambda(R^*_{g^1} + L^*_g)\,(R^*_g\, T(H)_h + 0_{g\,^1f}) \quad \mathrm{d}\lambda(T(H)_h + 0_f) \quad \mathrm{d}\mu_f\, T(H)^+_h$. Thus the subspaces $T(E)^+_x$ and $T(E)$ are defined independently of the choice of $(h,f) \in \lambda\,^1(x)$, and we have

Lemma 1. $T(E)_x \quad T(E)^+_x \oplus T(E)_x$.

Proof. We have $T(E)^+_x \quad \mathrm{d}\mu_f(T(H)_h)$ for some pair $(h,f) \in \lambda\,^1(x)$ and hence dim $T(E)_x$ + dim $T(E)_x$ — dim $T(E)_x$. Suppose $\xi \in T(E)_x \cap T(E)_x$, i. e. $\xi \quad (\mathrm{d}\mu_f)X$ where $X \in T(H)^+_h$ and $\mathrm{d}p\xi \quad 0$. Then $0 \quad \mathrm{d}p(\mathrm{d}\mu_f)X \quad \mathrm{d}bX \to X \in T(H)^-_x$ and hence $X \quad 0_h$, ξ — 0_x.

Remark. We cannot write in general $T(E)_x \quad \mathrm{d}\mu_f\,(T(H)_h)$, since $\mathrm{d}\mu_f$ is an isomorphism if restricted to $T(H)^+_h$, but it need not be either an injection or projection of the whole $T(H)_h$. It might be interesting to find the explicite conditions under which it is.

First it is clear that $\mathrm{d}\lambda$ is always a projection. Let us find its kernel. We make use of the following well-known statement: ,,Let $\Phi \cdot M \to N$ be a differentiable projection of manifolds which is of maximal rank at $m \in M$. Then the points of M in which Φ takes the constant value $n \quad \Phi(m)$ in a sufficiently small neighbourhood of m form an imbedded submanifold $W \subset M$, $n \quad W$ The tangent space $T(W)_m \subset T(M)_m$ is exactly the kernel of $\mathrm{d}\Phi$ at m M'' Hence the kernel of $\mathrm{d}\lambda$ at (h,f) is the tangent space of a submanifold $W \subset H \times F$ consisting of pairs $(hg, g^{-1}f)$, $g \in G$. In other words, it consists of exactly all the tangent vectors to the curves $t \to (hg(t), g(t)\,^1f)$ at $t \quad 0$ where $t \to g(t)$ is a curve in G, $g(0) \quad e$ and e is the identity of G. Denote by $\iota_h : T(G)_e \to T(H)_h$ the canonical isomorphism of the Lie algebra of G onto the tangent space to the fibre in H at h. Further denote $j_f : T(G)_e \to T(F)'$ the canonical projection of the Lie algebra of G onto the tangent space at f to the orbit of G on F containing f. Note that j_f is an injection iff G acts freely on this orbit. Now using these notations we see easily that Ker $\mathrm{d}\lambda$ consist of exactly all the elements

$$\iota_h(Z) + j_f\,\mathrm{d}\sigma(Z) \quad \iota_h(Z) \quad j_f(Z)\,,$$

where Z is any element of $T(G)_e$ and $\sigma : \theta \to \theta\,^1$, i. e. $\mathrm{d}\sigma(Z) \quad Z$

These results yield the required characterization of the mapping $\mathrm{d}\mu_f$. We see namely that $\mathrm{d}\mu_f$ *is injective* iff there does not exist such $Z \in T(G)_e$ that $j_f(Z) = 0_f$, $Z \neq 0$, i. e. *iff G acts freely on the orbit containing $f \in F$*. Since $\mathrm{d}\lambda$ is always projective there are for each $\xi \in T(E)_x$ elements $X \in T(H)_h$ and $Y \in T(F)_f$ such that $\xi = \mathrm{d}\lambda(X + Y)$. All the elements in $T(H)_h \oplus T(F)_f$, which are taken into ξ, are of the form $X + \iota_h(Z) + Y - j_f(Z)$. The mapping $\mathrm{d}\mu_f$ is projective iff one can find such $Z \in T(G)_e$ that $Y = j_f(Z)$ for any $Y \in T(F)_f$. Hence $\mathrm{d}\mu_f : T(H)_h \to T(E)_x$ *is projective iff $f \in F$ is an interior point of the corresponding orbit of G on F.*

Returning to the general situation. Lemma 1 gives for each $\xi \in T(E)_x$ the unique decomposition

$$\xi = \mathrm{d}\mu_f(X) + \eta \,, \tag{3}$$

such that $\mathrm{d}p(\eta) = 0$ and $V_H X = 0$. Thus denoting by $V_E : T(E) \to T(E)$ the induced projection $V_E(\xi) = \eta$, we have

$$V_E \mathrm{d}\mu_f = \mathrm{d}\mu_f V_H \,. \tag{4}$$

We proceed now as follows. Let $H = H(B, G)$ and $\dot{H} = \dot{H}(B, \dot{G})$ be principal fibre bundles over B and let $\varphi : H \to \dot{H}$ be a homomorphism of them. Let in H and $\dot{H}$ connections be given with the corresponding vertical projections satisfying

$$V_{\dot{H}}\, \mathrm{d}\varphi = \mathrm{d}\varphi\, V_H \,. \tag{5}$$

If $E = E(B, F, G, H)$ and $\dot{E} = \dot{E}(B, \dot{F}, \dot{G}, \dot{H})$ are fibre bundles associated to H and $\dot{H}$ respectively, call a bundle morphism $\gamma : E \to \dot{E}$ (γ_0, φ)*-typed* if there exists a differentiable mapping $\gamma_0 : F \to \dot{F}$ such that for each $h \in H$, $f \in F$ the relation

$$\gamma(h(f)) = \varphi(h)\,(\gamma_0(f)) \tag{6}$$

holds. If we introduce analogously the mappings $\dot{\mu}_{\dot{f}} : \dot{H} \to \dot{E}$ for each $\dot{f} \in \dot{F}$ we get according to (2) the equivalent to (6) form

$$\gamma\mu_f(h) = \dot{\mu}_{\gamma_0(f)}(\varphi(h)) \,, \tag{7}$$

which implies

$$\mathrm{d}\gamma\, \mathrm{d}\mu_f = \mathrm{d}\dot{\mu}_{\gamma_0(f)}\, \mathrm{d}\varphi \,. \tag{8}$$

Proposition 1. *Let $\gamma : E(B, F, G, H) \to \dot{E}(B, \dot{F}, \dot{G}, \dot{H})$ be a (γ_0, φ)-typed bundle morphism and let there be given connections in H and $\dot{H}$ subject to (5). Then the associated vertical projections satisfy*

$$V_{\dot{E}}\, \mathrm{d}\gamma = \mathrm{d}\gamma\, V_E \tag{9}$$

Proof. First note that both sides of (9) take the tangent space at $h(f) \in E$ into the tangent space at $\varphi(h)\,\gamma_0(f) = \gamma(h(f))$ according to (6). We have by (3) the relations $\mathrm{d}\gamma\, V_E(\xi) = \mathrm{d}\gamma(\eta)$ and $V_{\dot{E}}\,\mathrm{d}\gamma(\xi) = V_{\dot{E}}\,\mathrm{d}\gamma(\eta) + V_{\dot{E}}\,\mathrm{d}\gamma\,\mathrm{d}\mu_f(X)$. But $p\gamma = p$ implies $\mathrm{d}\dot{p}\,\mathrm{d}\gamma(\eta) = \mathrm{d}p(\eta) = 0$ and thus $V_{\dot{E}}\,\mathrm{d}\gamma(\eta) = \mathrm{d}\gamma(\eta)$. Hence it suffices to show $V_{\dot{E}}\,\mathrm{d}\gamma\,\mathrm{d}\mu_f(X) = 0$. But by (8), (4) and (5) we have $V_{\dot{E}}\,\mathrm{d}\gamma\,\mathrm{d}\mu_f(X) = V_{\dot{E}}\,\mathrm{d}\dot{\mu}_{\gamma_0(f)}\,\mathrm{d}\varphi(X) = \mathrm{d}\dot{\mu}_{\gamma_0(f)}\,V_{\dot{H}}\,\mathrm{d}\varphi(X) = \mathrm{d}\dot{\mu}_{\gamma_0(f)}\,\mathrm{d}\varphi\,V_H(X) = 0$ and this yields the proposition.

The assumptions of this proposition can be modified in the following way. The bundle morphism $\gamma : E(B, F, G, H) \to \dot{E}(B, \dot{F}, \dot{G}, \dot{H})$ is called (γ_0, φ)-*antityped* if $\varphi : \dot{H} \to H$ is a covering extension of the bundle H (i. e. φ is a homomorphism of the principal bundles over B inducing isomorphisms $\mathrm{d}\varphi : T(\dot{H})_{\dot{h}} \to T(H)_{\varphi(\dot{h})}$) and γ_0 being given as before satisfies for each $\dot{h} \in \dot{H}, f \in F$

(6a) $$\gamma(\varphi(\dot{h})f) = \dot{h}(\gamma_0(f))$$

which is again equivalent to

(7a) $$\gamma\mu_f\varphi(\dot{h}) = \dot{\mu}_{\gamma_0(f)}(\dot{h})$$

implying

(8a) $$\mathrm{d}\gamma\,\mathrm{d}\mu_f = \mathrm{d}\dot{\mu}_{\gamma_0(f)}\,\mathrm{d}\varphi^{-1}.$$

Note that a covering extension φ of H assigns to each connection in H a connection in $\dot{H}$ according to

(5a) $$V_{\dot{H}} = \mathrm{d}\varphi^{-1}\,V_H\,\mathrm{d}\varphi\,.$$

Proposition 1a. *Let* $\gamma : E(B, F, G, H) \to \dot{E}(B, \dot{F}, \dot{G}, \dot{H})$ *be a* (γ_0, φ)-*antityped bundle morphism and let there be given connections in* H *and* $\dot{H}$ *subject to* (5a). *Then the associated vertical projections satisfy* (9).

The proof runs analogously to that of Proposition 1.

If now $\psi : B \to E$ is a local section in E then its *covariant differential* is usually defined as

(10) $$\nabla\psi = V_E(\mathrm{d}\psi)$$

or — if $Y \in T(B)_x$, where ψ is defined in a neighbourhood of $x \in B$

(11) $$\nabla_Y\psi = (\nabla\psi)\,(Y) = V_E(\mathrm{d}\psi)\,(Y)\,.$$

We give the analogous meaning to $\dot{\nabla}$ associated with $\dot{E}$ and get (9) in the form

(12) $$\dot{\nabla}_Y(\gamma\psi) = \mathrm{d}\gamma\nabla_Y\psi\,.$$

Note that if $\varphi : H \to \dot{H}$ is a reduction of the structure group $\dot{G}$ to its subgroup G (i. e. $H \subset \dot{H}$ and φ is the inclusion map) then (5) means that the connection in $\dot{H}$ is reducible to that in H. Applying Proposition 1 to $F = \dot{F}$

implying $E \quad \dot{E}$, and $\gamma \quad \mathrm{id}_E$ we refind the coincidence $\nabla \quad \nabla$ of both the induced covariant differentials. On the other hand if especially $H \quad \dot{H}$ and φ is the identity, then the condition (6) or (6a) on γ expresses the fact that the induced mappings $\gamma_x : E_x \to \dot{E}_x$ commute with the action of the associated groupoid (c. f. [1] or [2]) on E and $\dot{E}$.

This leads directly to a number of applications but first let us generalize Proposition 1 to the case of several fibre bundles where then γ may play the rôle of a „multiplication" connected with some structure (e. g. the tensor multiplication connected with the tensor product of vector bundles, cf. below)

The indices i run always from 1 to the integer $A \geqq 1$. Let there be given connections in the principal bundles $H_i(B_i, G_i)$ by means of the differentiable distributions $h_i \to T(H_i)_{h_i}^{\prime} \subset T(H_i)_{h_i}$.
Let

$$(13) \qquad V_{H_i} : T(H_i) \to T(H_i)$$

be the vertical projections related to these connections. Then $H \quad H_1 \times$ $H_A(B_1 \quad \dots \quad B_A, G_1 \times \dots \times G_A)$ is a principal fibre bundle and $(h_1, \dots h_A) \to T(H_1)_{h_1}^{\prime} \oplus \dots \oplus T(H_A)_{h_A}^{\prime}$ is a differentiable distribution which defines a connection in H. Here and in the following we identify as usually the tangent space $T(H)_{h_1, \dots h_A}$ with $T(H_1)_{h_1} \oplus \dots \oplus T(H_A)_{h_A}$. The vertical projection related to this connection satisfies

$$(14) \qquad V_H \quad V_{H_1} + \dots + V_{H_A},$$

where $V_{H_i}(X) \quad 0$ unless $X \in T(H_i)$.

If $\dot{E} \quad \dot{E}(B_1 \quad \dots \times B_A, \dot{F}, G_1 \times \dots \times G_A, H)$ is a fibre bundle associated to H, we get a projection $V_{\dot{E}}$ as above and it satisfies

$$(15) \qquad V_{\dot{E}}\, \mathrm{d}\dot{\mu}_{\dot{f}} \quad \mathrm{d}\dot{\mu}_{\dot{f}}\, (V_{H_1} + \dots + V_{H_A})$$

for each $\dot{f} \in \dot{F}$. At the same time suppose that there are fibre bundles E_i $E_i(B_i, F_i, G_i, H_i)$ associated to H_i and denote again by $\underset{i}{\mu_{f_i}} : H_i \to E_i$, $f_i \in F_i$, the associated mappings. The product $E \quad E_1 \times \dots \times E_A$ is a fibre bundle over $B_1 \times \dots \times B_A$ with the above defined associated groupoid H. A bundle morphism $\gamma : E_1 \times \dots \times E_A \to \dot{E}$ is called γ_0-*typed* if there is a differentiable mapping $\gamma_0 : F_1 \times \dots \times F_A \to \dot{F}$ such that for each $(h_1, \dots h_A) \in H$ and $(f_1, \dots f_A) \in F_1 \quad \dots \quad F_A$ the relation

$$(16) \qquad \gamma(h_1(f_1), \dots h_A(f_A)) \quad (h_1 \quad \dots \times h_A)\, \gamma_0(f_1, \dots f_A)$$

holds, which is again equivalent to

$$(17) \qquad \gamma(\underset{1}{\mu_{f_1}}(h_1), \dots \underset{A}{\mu_{f_A}}(h_A)) \quad \dot{\mu}_{\gamma_0(f_1, \dots f_A)}(h_1, \dots h_A),$$

implying

$$(18)\qquad \mathrm{d}\gamma(\mathrm{d}\underset{1}{\mu}_{f_1} + \dots + \mathrm{d}\underset{A}{\mu}_{f_A}) = \mathrm{d}\dot{\mu}_{\gamma_0(f_1,\dots,f_A)}\,.$$

Using this relation we prove analogously to Proposition 1

Proposition 2. *Let* $\gamma : E_1(B_1, F_1, G_1, H_1) \times \dots \times E_A(B_A, F_A, G_A, H_A) \to \to \dot{E}(B_1 \times \dots \times B_A, \dot{F}, G_1 \times \dots \times G_A, H_1 \times \dots \times H_A)$ *be a* γ_0*-typed bundle morphism and let there be given a connection in each* H_i *(i = 1, ..., A). Then the associated vertical projections satisfy*

$$(19)\qquad V_{\dot{E}}\,\mathrm{d}\gamma = \mathrm{d}\gamma(V_{E_1} + \dots + V_{E_A})\,.$$

Thus we have generalized the situation of the preceding propositions only for $\varphi = \mathrm{id}_H$ since the more general case is then obtained by combining Proposition 2 with Proposition 1 or 1a.

If now $\psi_i : B \to E_i$ are local sections in E_i, then $\gamma(\psi_1 \times \dots \times \psi_A)$ is a local section in $E_1 \times \dots \times E_A$ and (19) takes again the more usual form

$$(20)\qquad (\dot{\nabla}_{Y_1 + \dots + Y_A})\,\gamma(\psi_1 \times \dots \times \psi_A) = \mathrm{d}\gamma(\nabla_{Y_1}\psi_1 + \dots + \nabla_{Y_A}\psi_A)\,.$$

Consider the special situation with $B_1 = \dots = B_A = B$. It is now natural to investigate beside $H = H_1 \times \dots \times H_A$ the principal bundle $\square H = \square H(B, G_1 \times \dots \times G_A)$, which is the „restriction of H to the diagonal in $B \times \dots \times B$". Denoting by $\iota_H : \square H \to H$ the natural injection, we see easily that the connections in H_i define a connection in $\square H$, where the corresponding vertical projection satisfies

$$(21)\qquad \mathrm{d}\iota_H V_{\square H} = (V_{H_1} + \dots + V_{H_A})\,\mathrm{d}\iota_H\,.$$

Suppose now that there is a fibre bundle associated to $\square H$. There exists then always a fibre bundle $\dot{E}(B \times \dots \times B, \dot{F}, G_1 \times \dots \times G_A, H)$ associated to H such that the bundle associated to $\square H$ is the „restricted" bundle $\square\dot{E}$ defined analogously to $\square H$. Denote again $\iota_{\dot{E}} : \square\dot{E} \to \dot{E}$ the natural injection.

Lemma 2. *The vertical projections in* $T(\dot{E})$ *and* $T(\square\dot{E})$ *satisfy*

$$(22)\qquad \mathrm{d}\iota_{\dot{E}}\,V_{\square\dot{E}} = V_{\dot{E}}\,\mathrm{d}\iota_{\dot{E}}\,.$$

Proof. For a fixed $f \in \dot{F}$ define again the natural maps $\dot{\mu}_f : H \to \dot{E}$. The relation

$$(23)\qquad \iota_{\dot{E}}\square\dot{\mu}_f = \dot{\mu}_f\iota_H$$

follows immediately from the very definition of $\square H$ and $\square\dot{E}$. We have by (15) $V_{\dot{E}}\,\mathrm{d}\dot{\mu}_f = \mathrm{d}\dot{\mu}_f\,V_H$ and $V_{\square\dot{E}}\,\mathrm{d}(\square\dot{\mu}_f) = \mathrm{d}(\square\dot{\mu}_f)\,V_{\square H}$ which implies by (23) and (21) $\mathrm{d}\iota_{\dot{E}}\,V_{\square\dot{E}}\,\mathrm{d}(\square\dot{\mu}_f) = \mathrm{d}\dot{\mu}_f\,\mathrm{d}\iota_H\,V_{\square H} = \mathrm{d}\dot{\mu}_f\,V_H\,\mathrm{d}\iota_H = V_{\dot{E}}\,\mathrm{d}\dot{\mu}_f\,\mathrm{d}\iota_H = V_{\dot{E}}\,\mathrm{d}\iota_{\dot{E}}\,\mathrm{d}(\square\dot{\mu}_f)$. On the other hand if $\eta \in T(\square\dot{E})$ then $\mathrm{d}\iota_{\dot{E}}\,V_{\square\dot{E}}(\eta) = \mathrm{d}\iota_{\dot{E}}(\eta)$

$d\iota_{\dot E}\, V_{\dot E}(\eta)$, since $d\iota_{\dot E}$ takes clearly elements tangent to fibres in $T(\Box\dot E)$ into elements tangent to fibres in $T(\dot E)$. Combining these two results we obtain (22) since according to (3) each element $\xi \in T(\Box\dot E)$ can be decomposed into ξ $d(\Box\mu_f)(X) + \eta$, where $\eta \in T(\Box\dot E)$.

If now $\gamma : E_1 \times \ldots \times E_A \to \dot E$ is a bundle morphism there exists a unique bundle morphism $\gamma_\Box : \Box(E_1 \times \ldots \times E_A) \to \Box\dot E$ satisfying

$$\gamma\iota_E = \iota_{\dot E}\,\gamma_\Box , \tag{24}$$

where $\iota_E : \Box(E_1 \times \ldots \times E_A) \to E_1 \times \ldots \times E_A$ is defined analogously as $\iota_{\dot E}$. From the above and Lemma 2 we obtain at once the

Corollary. *If in the assumptions of Proposition 2 we have* $B_1 = \ldots = B_A - B$ *and* γ *is defined by* (24) *then*

$$d\iota_{\dot E}\, V_{\Box\dot E}\, d\gamma_\Box = d\gamma\,(V_{E_1} + \ldots + V_{E_A})\, d\iota_E \tag{25}$$

or, by means of covariant differentials,

$$\dot\nabla^{\Box}_{Y}(\gamma_\Box(\psi_1\Box \ldots \Box\psi_A)) = d\gamma_\Box\, d\iota_E^{-1}\,(\overset{(1)}{\nabla}_Y\psi_1 + \ldots + \overset{(A)}{\nabla}_Y\psi_A) , \tag{26}$$

where ψ_i *are local sections in* E_i *defined in neighbourhoods of* $x \in B$, $Y \in T(B)_x$ *and* $\psi_1\Box \ldots \Box\psi_A$ *is the natural section in* $\Box(E_1 \times \ldots \times E_A)$ *given by the sections* ψ_i. Note that (26) is well defined since $\overset{(1)}{\nabla}_Y\psi_1 + \ldots + \overset{(A)}{\nabla}_Y\psi_A \in T(E_1)_x \oplus \ldots$ $\ldots \oplus T(E_A)_x = d\iota_E(T(\Box(E_1 \times \ldots \times E_A))_y)$.

If $E(B, F, G, H)$ is a vector bundle then there is a canonical identification of $T(E)_z$ with $E_{p(z)}$ which takes $0_z \in T(E)_z$ into $0 \in E_{p(z)}$. If $\gamma : E \to \dot E$ is a vector bundle morphism, then $d\gamma\,|T(E)_z = \gamma|E_{p(z)}$. Thus in this case (12) can be written in the form

$$\dot\nabla_Y(\gamma\psi) = \gamma(\nabla_Y\psi) , \tag{27}$$

since $\nabla_Y\psi$ belongs now to E_x if $Y \in T(B)_x$ and it is the usual „linear“ covariant differential.

Suppose now that $E_1, \ldots E_A, \dot E$ appearing in (20) and (26) are vector bundles and let $\gamma : E_1 \times \ldots \times E_A \to \dot E$ be multilinear on each fibre. This means that if we fix any point $(z_1, \ldots z_A) \in E_1 \times \ldots \times E_A$, then all the mappings $\gamma_i : E_i \to \dot E$ defined by $\gamma_i(z) - \gamma(z_1, \ldots z_{i-1}, z, z_{i+1}, \ldots z_A)$ are vector bundle morphisms. From this and the above remark we obtain (20) in the form

$$(\dot\nabla_{Y_1+\ldots Y_A})\,\gamma\,(\psi_1 \times \ldots \times \psi_A) = \\ \sum_{i=1}^{A} \gamma(\psi_1 \times \ldots \times \psi_{i-1} \times \overset{(i)}{\nabla}_{Y_i}\psi_i \times \psi_{i+1} \times \ldots \times \psi_A) \tag{28}$$

and (26) in the form

$$(29)\qquad \dot{\nabla}_{Y}\,\gamma\,(\psi_1\square\dots\square\psi_A) = \sum_{i=1}^{A}\gamma\,(\psi_1\square\dots\square\psi_{i-1}\square\overset{(i)}{\nabla_Y\psi_i}\square\psi_{i+1}\square\dots\square\psi_A)\,.$$

Example 1. Let $E_1,\dots E_A$ be vector bundles and let $\dot F = F_1 \otimes \dots \otimes F_A$. Then the associated space $\dot E$ can be written as $E_1\otimes\dots\otimes E_A$ although in practice — where $B_1 = \dots = B_A$ is almost always supposed — it is the bundle $\square\dot E$ which is denoted by this symbol and called the tensor product of E_i. The fibres of $\dot E$ (and consequently of $\square\dot E$) consist of tensor products of the fibres of $E_1,\dots E_A$ at the corresponding points. Denoting $\gamma: E_1 \times \dots \times E_A \to \dot F$ the bundle morphism that assigns to $(z_1,\dots z_A)$ the element $z_1 \otimes \dots \otimes z_A$ we see that it is $\otimes$-typed, where $\otimes: F_1\times\dots\times F_A \to F_1 \otimes \dots \otimes F_A$ is the natural tensor multiplication. In this way we can write — either (28) or (29) in the form

$$(30)\qquad (\nabla_{Y_1}\times\dots\times_{Y_A})(\psi_1\otimes\dots\otimes\psi_A) = \sum_{i=1}^{A}\psi_1\times\dots\times\overset{(i)}{\nabla_{Y_i}\psi_i}\times\dots\times\psi_A\,.$$

or

$$(31)\qquad \nabla_Y(\psi_1\otimes\dots\times\psi_A) = \sum_{i=1}^{A}\psi_1\otimes\dots\times\overset{(i)}{\nabla_Y\psi_i}\otimes\dots\times\psi_A\,.$$

which are well-known formulas and show that the ∇_Y corresponding to various vector bundles behave as differentiation with respect to the tensor product. If $E_1 = \dots E_A = E$, then it shows that ∇_Y is a derivation of the algebra which is the sheaf of germs of local sections in the tensor algebra bundle over E. Here and in the following we use, as it is common, the same symbol ∇ for all covariant differentiations induced by a fixed connection in a fixed principal fibre bundle.

Note that this is in fact the only interesting example of the application of Proposition 2 to vector bundles. Namely if $\gamma: E_1\times\dots\times E_A \to \dot E$ is multilinear where $\dot E$ is now an arbitrary vector bundle over $B_1\times\dots\times B_A$, then it can be always split to a mapping $E_1\times\dots\times E_A \to E_1\otimes\dots\otimes E_A$ of the above example and a vector bundle morphism $\bar\gamma: E_1\otimes\dots\times E_A \to \dot E$, and if γ is γ_0-typed then $\bar\gamma$ is $(\bar\gamma_0, \mathrm{id})$-typed, where $\bar\gamma_0$ is given by the commutative diagram in the category of vector spaces

$$\begin{tikzcd}
F_1\times\dots\times F_A \arrow[rr] \arrow[dr, "\gamma_0"'] & & F_1\otimes\dots\times F_A \arrow[dl, "\bar\gamma_0"] \\
& \dot F &
\end{tikzcd}$$

Thus in the following since the covariant differential is really interesting mostly in the linear case, we shall give some examples of the application of the first propositions.

Example 2. Let E be a vector bundle and let $\gamma \quad \mathscr{A}_q : \overset{q}{\otimes} E \to \overset{q}{\wedge} E$ (or γ $\mathscr{S}_q \cdot \overset{q}{\times} E \to \mathbf{S}^q E$) be the natural antisymmetrization (or symmetrization) operator. Then it is clearly (γ_0, id)-typed, where $\gamma_0 : \overset{q}{\otimes} F \to \overset{q}{\wedge} F$ (or $\gamma_0 : \overset{q}{\quad} F \to \mathbf{S}^q F$) is the antisymmetrization (or symmetrization) projection of the fibre types. We obtain, combining (27) with (31), the formulas

$$(32) \qquad \nabla_Y(\psi_1 \quad \ldots \wedge \psi_A) \quad \sum_{i \ 1}^{A} \psi_1 \quad \ldots \wedge \nabla_Y \psi_i \wedge \cdots \wedge \psi_A$$

$$(33) \qquad \nabla_Y(\psi_1 \bigcirc \ \ldots \bigcirc \psi_A) \quad \sum_{i \ 1}^{A} \psi_1 \bigcirc \cdots \quad \nabla_Y \psi_i \bigcirc \cdots \bigcirc \psi_A$$

Example 3. Let $E \quad E(B, F, G, H)$ be again a vector bundle and $E^p_q(B, F^p_q$ $G, H)$ its „p-times contravariant and q-times covariant tensor power" Thus $F^p_q \quad \overset{p}{\quad} F \quad (\overset{q}{\quad} F^*)$. Let $C \quad C^a_b : E^p_q \to E^{p}_{q} {}^{1}_{1}$ be a fixed contraction bundle morphism, where the contraction acts upon the a-th contravariant and b th covariant indices. The relation (27) shows then that the contraction commutes with ∇_Y. Especially for $p \quad q \quad 1$ we derive in this way the relation

$$(34) \qquad \nabla_Y \eta, \xi \ \perp \ \eta, \nabla_Y \xi \qquad Y(\ \eta, \xi \ ) \, ,$$

where ξ and η are local sections in E and E^*, respectively, over a neighbourhood of x, where $Y \in T(B)_x$.

In general, if $E(B, F, G, H)$ and $\dot{E}(B, \dot{F}, G, H)$ are vector bundles and $\gamma_0 \cdot F \to \dot{F}$ is a homomorphism which commutes with the action of G, it induces in a natural way a (γ_0, id)-typed bundle morphism $\gamma : E \to \dot{E}$ commuting with the covariant differentiation. Conversely, if $\gamma : E \to \dot{E}$ is a (γ_0, id) typed bundle morphism, then γ_0 commutes necessarily with the action of G On the other hand in this linear case a bundle homomorphism γ is a section in $E^* \quad \dot{E}$. which is again a vector bundle associated to H.

Lemma 3. *γ is a typed bundle morphism iff $\nabla_Y \gamma \quad 0$ for any $Y \in T(B)$ and any connection in H.*

Proof. In fact, applying Proposition 2 and especially (27) and (31) to the natural pairing $(E^* \times \dot{E}) \otimes E \to \dot{E}$, we get for any local section ξ in E

$$\nabla_Y(\gamma(\xi)) \qquad (\nabla_Y \gamma)\,(\xi) + \gamma(\nabla_Y \xi)\,.$$

But the left hand side is equal to the second term of the right hand side if γ is typed. On the other hand $\nabla_Y \gamma = 0$ implies $\nabla_Y(\gamma(\xi)) \quad \gamma(\nabla_Y \xi)$ for any local section ξ, which implies (7) as one can see from the proof of Propositio 1

We apply this fact to the following

Example 4. Let $E(B, F, G, H)$ be a vector bundle and $\gamma_0 : F \to F^*$ an isomorphism which commutes with G. Then the corresponding $\gamma : E \quad E^*$

defines a metric in E which is autoparalell under any connection in H. If G operates effectively on F, $\dim F = n$, then the existence of such a γ_0 implies, of course, that G is isomorphic with a subgroup of $O(n)$. Conversely, any autoparalell metric in E can be obtained from such a γ_0 commuting with the action of G.

Example 5. Let $H = H(B, SO(n))$ be a principal bundle over B with the structure group $SO(n)$, $n = 2\nu$, $\nu \geqslant 2$ (especially let H be the principal bundle of oriented orthonormal frames of a Riemenn manifold). The group $SO(n)$ operates naturally on C^n and also on the Clifford algebra $\bullet\, C^n$ of C^n. In this way we obtain the vector bundles $E(B, C^n, SO(n), H)$ associated to H (especially the complexified tangent bundle to B), and $\bullet\, E(B, \bullet\, C^n, SO(n), H)$ (especially what one may call the Clifford bundle of the oriented Riemann structure on B), where the fibres of $\bullet\, E$ are the Clifford algebras of the corresponding fibres of E endowed with the quadratic form induced by H. Let there be given a connection in H. The injection $E \to \bullet\, E$ is clearly typed and thus Proposition 1 shows that the restriction of the covariant differentiation to E via $\bullet\, E$ coincides with the one defined directly on E. More generally, from a situation analogous to that in Example 2 we obtain the formula

$$\nabla_Y(\psi_1 \bullet \ldots \bullet \psi_A) = \sum_{i=1}^{A} \psi_1 \bullet \ldots \bullet \nabla_Y \psi_i \bullet \ldots \bullet \psi_A \,.$$

Let now $\dot{H}(B, \text{Spin}\,(n))$ be such that $\varphi : \dot{H} \to H$ is a covering extension of H, where $\text{Spin}(n)$ is the reduced Clifford group associated with the natural metric in C^n (c. f. [3]), i. e. it is the subgroup of all elements $\Lambda \in GL(2^\nu, C)$ satisfying

$$\Lambda \gamma_\alpha \Lambda^{-1} = \Lambda_\alpha^\lambda \gamma_\lambda,\ \det(\Lambda_\alpha^\lambda) = 1,\ \alpha(\Lambda)\Lambda = I \,, \tag{35}$$

where $\gamma_\alpha \in GL(2^\nu, C)$ are matrices for $\alpha = 1, \ldots n$ satisfying

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2\delta_{\alpha\beta} I \,, \tag{36}$$

$I \in GL(2^\nu, C)$ is the unity matrix and α is the main antiautomorphism of the Clifford algebra $\bullet\, C^n$ (c. f. [3]), which is identified with the algebra of all $2^\nu \times 2^\nu$ complex matrices, i. e. elements of $(C^{2^\nu})^* \otimes C^{2^\nu}$. It is well known that $\text{Ker}\varphi \cap \dot{H}_x$ consists of exactly two points for each $x \in B$, since $\text{Spin}(n)$ is a covering group of $SO(n)$, the projection being given by $\Lambda \to (\Lambda_\alpha^\lambda)$ in (36). Let $E(B, C^n, SO(n), H)$ be the vector bundle above and let $\mathscr{S} = \mathscr{S}\,(B, C^{2^\nu}, \text{Spin}(n), \dot{H})$ be the vector bundle associated to $\dot{H}$ if $\text{Spin}(n)$ operates naturally (effectively) on C^{2^ν}. The elements of $\mathscr{S}$ may be called *spinors over E* and they correspond to the usual spinors on the oriented even dimensional Riemann manifold B in the mentioned above special case. In this case the elements of $\dot{H}$ are the spinframes (cf. [4], [5]). There is a unique connection in $\dot{H}$ induced by the connection in H and hence a covariant differentiation in e. g. $\mathscr{S}^* \times \mathscr{S}$. Let $\gamma_0 : \bullet\, C^n \to (C^{2^\nu})^* \otimes C^{2^\nu}$ be the natural isomorphism of algebras which takes the vectors e_α ($\alpha = 1, \dots n$) of the canonical frame in C^n into the matrices γ_α of (35). The isomorphism γ_0 satisfies

$$(37) \qquad \dot{h}(\gamma_0(\xi)) = \dot{h}\Lambda\,(\gamma_0(\varphi_0(\Lambda)^{-1}\xi))$$

for each $\xi \in \bullet\, C^n$, $\Lambda \in \mathrm{Spin}(n)$, $\dot{h} \in \dot{H}$, where $\varphi_0 : \Lambda \to (\Lambda^\nu_\alpha)$ is the covering homomorphism $\mathrm{Spin}(n) \to SO(n)$. In fact, since $\Lambda \in \mathrm{Spin}(n)$, $\eta \in (C^{2^\nu})^* \otimes C^{2^\nu}$ implies $\Lambda(\eta) = \Lambda\eta\Lambda^{-1}$, this is for $\xi \in C^n$ equivalent to (35). Further γ_0 is an isomorphism of algebras and from there we conclude that (37) holds for any $\xi \in \bullet\, C^n$. But (37) means that there is a unique (γ_0, φ)-antityped bundle isomorphism $\gamma : \bullet\, E \to \mathscr{S}^* \otimes \mathscr{S}$ which, according to Proposition 1a, takes the covariant differentiations in $\bullet\, E$ and $\mathscr{S}^* \otimes \mathscr{S}$ one into the other. In other words one can identify the Clifford algebra bundle of E with the bundle of (1,1) spinors over E — and consequently inject the tensor algebra bundle of E into the tensor algebra bundle of $\mathscr{S}$ — *including the covariant differentiations on them* induced from any connection in H. We can express this also by saying, that the covariant differentiation in the bundle of spinors over E is a „correct" extension of the differentiation on E, a result again well-known — at least in the special case of spinors on B.

It is now clear how one could immediately obtain other results regarding the behaviour of the covariant differential of spintensors by applying Proposition 1a to other antityped bundle morphisms.

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