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MAXIMAL GRAPHS WITH GIVEN CONNECTIVITY AND EDGE-CONNECTIVITY

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The graphs considered in this paper are undirected, finite, without loops and multiple edges.

We shall describe constructively maximal graphs with given vertex-connectivity or edge-connectivity, respectively. In addition, we prove some estimation of the number of edges, the maximal degrees and minimal degrees of graphs with a given vertex-(edge)-connectivity, respectively.

Let G, Q be graphs. Then we denote by $V(G)$ the vertex set of G, by $E(G)$ the edge set of G, by $\varkappa(G)$ the (vertex-) connectivity of G, by $\lambda(G)$ the edge-connectivity of G, by $\delta(G)$ the minimum degree of a vertex of G, by $\Delta(G)$ the maximum degree of a vertex of G, by $\bar{G}$ the complement of G, by $|M|$ the cardinal number of a set M, by K_p the complete graph with p vertices, by $G + Q$ the join of graphs G and Q and by $G \cup Q$ the union of graphs G and Q. In addition, we denote by $G + h$, where $h \in E(\bar{G})$, the graph that arose from G by adding the edge h; by $G - A$, where $A \subset E(G)$, the graph arisen from G by deleting the set A of edges; by $G - B$, where $B \subset V(G)$, the graph arisen from G by deleting every vertex $v \in B$ and all edges that are incident with it. If $A = \{x\}$, $B = \{u\}$, then we write $G - A = G - x$ and $G - B = G - u$, respectively. The symbol $G \simeq Q$ denotes that the graphs G, Q are isomorphic. Definitions of notions not included here can be found in [3].

Minimal and maximal graphs with a given property P have been studied by many authors. For example in [1], [2], [4] the so-called $\varkappa$-critical graphs are studied (i.e. such graphs G that $\varkappa(G) > \varkappa(G - v)$, for every $v \in V(G)$), $\varkappa$-edge-critical graphs (i.e. such graphs G that $\varkappa(G) > \varkappa(G - x)$, for every edge x of G); λ-critical graphs and λ-edge-critical graphs. We shall study a dual question to this one.

Definition 1. *Let G be a not complete graph and n a nonnegative integer. Then G is called $\varkappa_n$-maximal, if $\varkappa(G) = n$ and $\varkappa(G + x) > \varkappa(G)$ holds for every edge*

$x \in E(\bar{G})$. *Analogically* G *is called* λ_n*-maximal, if* $\lambda(G) = n$ *and* $\lambda(G + x) > \lambda(G)$ *holds for every edge* $x \in E(\bar{G})$.

Theorem 1. *Let* G *be a graph and* n, r, s *be natural numbers. Then* G *is*

a) $\varkappa_0$*-maximal if and only if* $G \simeq K_r \cup K_s$;

b) $\varkappa_n$*-maximal if and only if* $G \simeq K_n + (K_r \cup K_s)$.

Proof. One can easily verify that part a) holds.

b) Let $G \simeq K_n + (K_r \cup K_s)$. Let us denote $V(G) = A \cup B \cup C$, where A, B, C are mutually disjoint and $A = V(K_n)$, $B = V(K_r)$, $C = V(K_s)$. The graph $G - A$ is not connected, hence $\varkappa(G) \leqslant n$. It can be easily seen from the construction of G that between every two different vertices u, v there exist at least n vertex-disjoint paths. Especially, if $u \in B$, $v \in C$, then there exist exactly n vertex-disjoint paths. Thus according to Whitney's theorem (see [3], p. 48) we have $\varkappa(G) \geqslant n$ and hence $\varkappa(G) = n$. Every edge not belonging to $E(G)$ joins some vertex from B with one from C. One can verify that in the graph $G + x$, where $x \in E(\bar{G})$ between any two different vertices u, v, there exist at least $n + 1$ vertex-disjoint paths. Hence $\varkappa(G + x) \geqslant n + 1 > \varkappa(G)$. Thus the graph G is $\varkappa_n$-maximal.

Let G be $\varkappa_n$-maximal. Then there exists a set A of vertices of G such that $|A| = n$ and the graph $G - A$ consists of exactly two components that are complete graphs. Let them be K_r, K_s. From the $\varkappa$-maximality of G it follows that $G \simeq K_n + (K_r \cup K_s)$. Thus the theorem holds.

Remark 1. For every graph G we have $\varkappa(G) \leqslant \lambda(G) \leqslant \delta(G)$, see, e.g. [3], p. 43. Thus if we denote $p = |V(G)|$, $q = |E(G)|$, then we have $q \geqslant \dfrac{P \cdot \varkappa(G)}{2}$, $q \geqslant \dfrac{P \cdot \lambda(G)}{2}$ and the equalities hold in every regular graph G of degree $k = \varkappa(G)$. Now we shall prove some estimates for q, $\delta(G)$, $\Delta(G)$.

Theorem 2. *Let* G *be a graph with* p *vertices,* q *edges. Let* $\varkappa(G) = n$. *Then we have:*

a) $\delta(G) \leqslant \dfrac{p + n}{2} - 1$, *if the graph* G *is not complete.*

b) $\Delta(G) \leqslant p - 1$.

c) $q \leqslant \dfrac{(p - 1)(p - 2)}{2} + n$.

Proof. If $G \simeq K_p$, then one can easily verify these assertions. If G is not a complete graph, then it can be completed to a $\varkappa_n$-maximal graph Q by adding some edges from $E(\bar{G})$. It is clear that $\delta(G) \leqslant \delta(Q)$, $q(G) \leqslant q(Q)$

holds. According to Theorem 1 either $n = 0$ and $Q \simeq K_r \cup K_s$ or $n > 0$ and $Q \simeq K_n + (K_r \cup K_s)$, where r, s are natural. Hence $p = |V(Q)| = n + r + s$. Directly from the construction of the graph Q it follows that $\delta(Q) = \min(n + r - 1, n + s - 1)$. If p, n are given, then the maximum of these minima is obtained for $r = s$. Then $r = \frac{p - n}{2}$ and $\delta(Q) \leqslant \frac{p + n}{2} - 1$. This estimation is reached for the graphs $K_n + (K_r \cup K_r)$, where r is a natural number.

One can verify that $q(Q) = \frac{n(n-1)}{2} + \frac{r(r-1)}{2} + \frac{s(s-1)}{2} + rn + sn = r^2 + nr - rp + \frac{p^2}{2} - \frac{p}{2} = f(r)$, where $1 \leqslant r \leqslant p - n - 1$ and the number s was substituted by $p - n - r$. The maximum of the function $f(r)$, $1 \leqslant r \leqslant p - n - 1$ is equal to $\frac{(p-1)(p-2)}{2} + n$. Hence $q(G) \leqslant q(Q) \leqslant \frac{(p-1)(p-2)}{2} + n$. It is clear that $\Delta(G) \leqslant p - 1$. These two estimations are reached in graphs $K_n + (K_1 \cup K_{p-n-1})$. Hence the theorem holds.

Let the symbol $\mathscr{A}(m, r, s)$, where $m + 2 \leqslant r + s$, denote the class of graphs that arose from the graph $K_r \cup K_s$ by adding m new edges.

Theorem 3. *Let m, r, s be natural numbers. Then a graph G is:*

a) *λ_0-maximal if and only if $G \simeq K_r \cup K_s$;*

b) *λ_m-maximal if and only if it can be obtained from the graph $K_r \cup K_s$ by adding m edges, whereby either $r = 1$, $s \geqslant m + 1$, or $r \geqslant m + 2$, $s \geqslant m + 2$.*

Proof. It is clear that part a) holds.

b) It can be verified that if either $G \in \mathscr{A}(m, 1, m + s)$ or $G \in \mathscr{A}(m, m + 1 + r, m + 1 + s)$, then G is λ_m-maximal.

Let G be λ_m-maximal. Then $m = \lambda(G) \leqslant \delta(G)$ holds, see [3]. Then G contains at most two vertices of the degree m, because it is a λ_m-maximal graph. If the degree of the vertex u, $\deg(u) = m$, then every edge of the graph G is incident with vertex u, because in the reverse case the graph G would not be λ_m-maximal.

If $\deg(a) = \deg(b) = m$, for $a, b \in V(G)$, $a \neq b$, then the graph G contains the only edge (a, b). Hence G is isomorphic to the graph $K_{m+2} - x$, where x is any edge and then $G \in \mathscr{A}(m, 1, m + 1)$. If $\deg(a) = m$ and $\deg(u) > m$ for every $u \in V(G) - \{a\}$, then the graph G contains only edges (a, x) for some vertices x. Thus we can write $G \in \mathscr{A}(m, 1, m + 1 + s)$, where s is a natural number.

In the graph G there exists a set of edges Φ of the cardinality m, such that the graph $G - \Phi$ is not connected. The graph $G - \Phi$ consists of two complete components, because G is maximal. If $\deg(u) > m$, for every $u \in V(G)$, then each component of the graph $G - \Phi$ has at least $m + 2$ vertices. Hence $G \in \mathscr{A}(m, m + 1 + r,\ m + 1 + r)$, where r, s are natural. Q.E.D.

By using this theorem we prove the following inequalities.

Theorem 4. *Let G be a graph with p vertices, q edges and let $\lambda(G) = m$. Then we have:*

a) $\delta(G) \leqslant \begin{cases} m, \text{ if } m + 1 \leqslant p \leqslant 2m + 3,\ m \neq 0; \\ [p/2] - 1, \text{ if either } m = 0, \text{ or } m \neq 0,\ p \geqslant 2m + 4; \end{cases}$

b) $\Delta(G) \leqslant p - 1$;

c) $q \leqslant \dfrac{(p-1)(p-2)}{2} + m.$

Proof. It is clear that part b) holds and if $G \in \mathscr{A}(m, 1, m + 1 + r)$, then $\Delta(G) = p - 1$.

a) If $\lambda(G) = m$, then $p \geqslant m + 1$. If $m = 0$, then $\delta(G) \leqslant \left[\dfrac{p}{2}\right] - 1$. Let $m \geqslant 1$. If $G = K_p$, then $\delta(G) = m = p - 1$. If G is not complete, it can be completed to a λ_m-maximal graph Q by adding some new edges, whereby $\delta(G) \leqslant \delta(Q)$.

If $m + 2 \leqslant p \leqslant 2m + 3$, then according to Theorem 3 $G \in \mathscr{A}(m, 1, p - 1)$ so that $\delta(Q) = m$. If $p \geqslant 2m + 4$, then by Theorem 3 either $Q \in \mathscr{A}(m, 1, p - 1)$ and then $\delta(Q) = m$, or $Q \in \mathscr{A}(m, m + 1 + r, m + 1 + s)$ for $r \geqslant 1$, $s \geqslant 1$, $p = 2m + 2 + r + s$. Then it can be seen that $\delta(Q) = \min(m + r, m + s) \leqslant \left[\dfrac{p}{2}\right] - 1$. Hence the part a) holds.

c) If $\lambda(G) = m$, then after removing certain m edges, a graph with at least two components will arise. Let r be the number of vertices of some of them.

Then $1 \leqslant r \leqslant p - 1$ and moreover $q(G) \leqslant \dfrac{r(r-1)}{2} + \dfrac{(p-r)(p-r-1)}{2} + m = r^2 - rp + \dfrac{p^2 - p}{2} + m \leqslant \dfrac{(p-1)(p-2)}{2} + m$. This estimation is obtained in the graphs from $\mathscr{A}(m, 1, m + s)$, where s is natural. Q.E.D.

Corollary 1. *Let r, s be natural and n a non-negative integer. Then a graph G is $\varkappa_n$-maximal and λ_n-maximal if and only if*

a) $G \simeq K_r \cup K_s$, *for* $n = 0$

b) $G \simeq K_n + (K_1 \cup K_r)$, *for* $n > 0$

The proof follows immediately from the structure of $\varkappa_n$-, (λ_n-) maximal graphs, given in Theorem 1 (or Theorem 3, respectively).

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