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A MINIMAXIMIN FORMULA AND ITS APPLICATION TO DOUBLY STOCHASTIC MATRICES

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1. Introduction. In this note, we intend to prove the formula

$$\min_{\substack{0\neq z=(z_i)\in R_n\\ \sum_{i\in N} z_i=0}} \ \max_{\substack{M\subset N\\ \sum_{j\in M} z_j\neq 0}} \ \min_{\substack{i\in M\\ k\notin M}} \frac{z_i - z_k}{\sum_{j\in M} z_j} = 2\left(1 - \cos\frac{\pi}{n}\right) \tag{1}$$

where R_n denotes the real n-dimensional space of column vectors and $N = \{1, \ldots, n\}$.

Then we shall show that an inequality for eigenvalues of doubly stochastic matrices proved in [1] by another method follows easily from (1).

2. Proof of (1). Define

$$X_0 = \{z = (z_i) \in R_n \mid z \neq \mathbf{0}, \sum_{i\in N} z_i = 0\}$$

and for $z = (z_i) \in X_0$ let

$$m(z) = \max_{\substack{M\subset N\\ \sum_{j\in M} z_j\neq 0}} \ \min_{\substack{i\in M\\ k\notin M}} \frac{z_i - z_k}{\sum_{j\in M} z_j}. \tag{2}$$

Lemma 2.1. *Let* $z = (z_i) \in X_0$ *satisfy*

$$z_1 \geqq z_2 \geqq \ldots \geqq z_n. \tag{3}$$

Then

$$\sum_{j=1}^{s} z_j > 0 \qquad for \quad s = 1, \ldots, n-1$$

and

$$m(z) = \max_{s=1,\ldots,n-1} \frac{z_s - z_{s+1}}{\sum_{j=1}^{s} z_j}. \tag{4}$$

Proof. Denote, for $s = 1, \ldots, n-1$, $M_s = \{1, 2, \ldots, s\}$. Let $z = (z_i) \in X_0$ satisfy (3). Then clearly

(5) $$z_1 > 0, \quad z_n < 0.$$

Assume first that for some $t \in M_{n-1}$,

$$\sum_{j \in M_t} z_j \leqq 0.$$

Then $z_t < 0$ and $\sum_{j=t+1}^{n} z_j < 0$ so that

$$0 = \sum_{i \in N} z_i = \sum_{j \in M_t} z_j + \sum_{j=t+1}^{n} z_j < 0,$$

a contradiction.

Let now $s \in M_{n-1}$. Then clearly

$$\min_{\substack{i \in M_s \\ k \notin M_s}} \frac{z_i - z_k}{\sum_{j \in M_s} z_j} = \frac{z_s - z_{s+1}}{\sum_{j \in M_s} z_j}.$$

Thus,

(6) $$m(\boldsymbol{z}) \geqq \max_{s \in M_{n-1}} \frac{z_s - z_{s+1}}{\sum_{j \in M_s} z_j},$$

and also

(7) $$m(\boldsymbol{z}) > 0,$$

since the coordinates z_i are not all equal.

Let now M_0 be that (non-void proper) subset of N for which the maximum in (2) is attained:

$$m(\boldsymbol{z}) = \min_{\substack{i \in M_0 \\ k \notin M_0}} \frac{z_i - z_k}{\sum_{j \in M_0} z_j}.$$

Since $\bar{M}_0 = N \setminus M_0$ also satisfies

$$m(\boldsymbol{z}) = \min_{\substack{i \in \bar{M}_0 \\ k \notin \bar{M}_0}} \frac{z_i - z_k}{\sum_{j \in \bar{M}_0} z_j},$$

we can assume that $1 \in M_0$. Let us show that $M_0 = M_{p-1}$, where p is the least index in $\bar{M}_0$.

By (7),

$$0 < m(\boldsymbol{z}) \leqq \frac{z_1 - z_p}{\sum_{j \in M_0} z_j}$$

so that

$$\sum_{j \in M_0} z_j > 0 .$$

Suppose that there is an element $q \in M_0$ such that $q > p$. Then

$$0 < m(\boldsymbol{z}) \leqq \frac{z_q - z_p}{\sum_{j \in M_0} z_j}$$

so that

$$z_q > z_p ,$$

a contradiction with (3). Thus $M_0 = M_{p-1}$ and the proof is complete.

Lemma 2.2. The $n \times n$ matrix

$$\boldsymbol{A}_n = \begin{pmatrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{pmatrix}$$

has eigenvalues $2(1 - \cos(k\pi/n))$, $k = 0, 1, \ldots, n - 1$, corresponding to the eigenvectors $\boldsymbol{u}_k = (\cos(k\pi/2n), \cos(3k\pi/2n), \ldots, \cos((2n-1)k\pi/2n))^T$, $k = 0, \ldots, n-1$.

Proof. It follows by direct computation that

$$\boldsymbol{A}_n \boldsymbol{u}_k = 2(1 - \cos(k\pi/n))\boldsymbol{u}_k .$$

We are now able to prove the main result (1), which can be written as

$$\min_{\boldsymbol{z} \in X_0} m(\boldsymbol{z}) = 2\left(1 - \cos \frac{\pi}{n}\right).$$

Let $\boldsymbol{z} \in X_0$. Since for any permutation matrix $\boldsymbol{P}$

$$m(\boldsymbol{P}\boldsymbol{z}) = m(\boldsymbol{z}),$$

we can assume that $\boldsymbol{z} = (z_j)$ satisfies (3). According to Lemma 2.1,

$$\begin{aligned} z_1 - z_2 &\leqq m(\boldsymbol{z}) z_1, \\ z_2 - z_3 &\leqq m(\boldsymbol{z})(z_1 + z_2), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ z_{n-1} - z_n &\leqq m(\boldsymbol{z})(z_1 + \ldots + z_{n-1}), \\ 0 &= m(\boldsymbol{z})(z_1 + \ldots + z_n). \end{aligned}$$

Let us multiply the first inequality by $z_1 - z_2$, the second by $z_2 - z_3$ etc., the last by z_n and add. By Abel's summation formula,

$$\sum_{i=1}^{n-1} (z_i - z_{i+1})^2 \leqq m(\boldsymbol{z}) \sum_{i=1}^{n} z_1^2 . \tag{8}$$

Denoting by $(\boldsymbol{y}, \boldsymbol{z})$ the inner product $\sum_{i=1}^{n} y_i z_i$ of the vectors $\boldsymbol{y} = (y_i)$, $\boldsymbol{z} = (z_i)$, (8) can be written as

$$m(\boldsymbol{z}) \geqq \frac{(\boldsymbol{A}_n \boldsymbol{z}, \boldsymbol{z})}{(\boldsymbol{z}, \boldsymbol{z})},$$

where $\boldsymbol{A}_n$ is the matrix from Lemma 2, 2. Thus,

$$\min_{z \in X_0} m(\boldsymbol{z}) \geqq \min_{z \in X_0} \frac{(\boldsymbol{A}_n \boldsymbol{z}, \boldsymbol{z})}{(\boldsymbol{z}, \boldsymbol{z})} .$$

Since $\boldsymbol{e} = (1, \ldots, 1)^T$ is the eigenvector $\boldsymbol{u}_0$ of $\boldsymbol{A}_n$ corresponding to the smallest eigenvalue, the right-hand side is, according to the well-known Courant—Fischer principle, equal to the second smallest eigenvalue of the matrix $\boldsymbol{A}_n$, which is by Lemma 2,2 equal to $2(1 - \cos(\pi/n))$. Thus,

$$\min_{z \in X_0} m(\boldsymbol{z}) \geqq 2\left(1 - \cos\frac{\pi}{n}\right). \tag{9}$$

An easy computation shows that for the vector $\boldsymbol{u}_1$ from Lemma 2,2,

$$m(\boldsymbol{u}_1) = 2\left(1 - \cos\frac{\pi}{n}\right),$$

and, since $\boldsymbol{u}_1 \in X_0$, equality in (9) holds. The proof is complete.

3. An application. Let us recall that an $n \times n$ matrix $\boldsymbol{A} = (a_{ik})$ is doubly stochastic iff $\boldsymbol{A} \geqq \boldsymbol{0}$ and $\sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} a_{ki} = 1$ for all $i \in N$. For such a matrix $\boldsymbol{A}$, the so called measure of irreducibility $\mu(\boldsymbol{A})$ was defined in [1] by

$$\mu(\boldsymbol{A}) = \min_{\emptyset \neq M \neq N} \sum_{\substack{i \in M \\ k \notin M}} a_{ik} .$$

It was proved in [1] that if $\boldsymbol{A}$ is symmetric, doubly stochastic with eigenvalues $1 = \lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n$, then

$$\lambda_1 - \lambda_2 \geqq 2(1 - \cos(\pi/n))\mu(\boldsymbol{A}) \tag{10}$$

and, as an easy consequence, that if $\boldsymbol{A}$ is doubly stochastic, then any eigenvalue $\lambda \neq 1$ satisfies

(11) $$|1-\lambda| \geqq 2(1-\cos(\pi/n))\mu(\boldsymbol{A}) .$$

We shall show that (10), and thus (11), follows easily from the formula (1). The same idea was used in [2] for obtaining similar results for general non-negative matrices.

Let $\boldsymbol{A}$ be a symmetric doubly stochastic matrix with eigenvalues $1 = \lambda_1 \geqq \geqq \lambda_2 \geqq \ldots \geqq \lambda_n$.

Then $\boldsymbol{e} = (1, \ldots, 1)^T$ is an eigenvector of $\boldsymbol{A}$ corresponding to the eigenvalue $\lambda_1 = 1$. Let $\boldsymbol{z} = (z_1, \ldots, z_n)^T$ be an eigenvector corresponding to λ_2. If $\lambda_1 = \lambda_2$, we choose $\boldsymbol{z}$ orthogonal to $\boldsymbol{e}$. The well-known orthogonality property ensures then $\boldsymbol{z} \in X_0$. Let M_0 be that subset of N for which the maximum in (2) is attained:

$$m(\boldsymbol{z}) = \min_{\substack{i \in M_0 \\ k \notin M_0}} \frac{z_i - z_k}{\sum\limits_{j \in M_0} z_j}$$

(thus $\sum\limits_{j \in M_0} z_j \neq 0$).

Without loss of generality, we can assume that $M_0 = \{1, \ldots, m\}$, where $1 \leqq m \leqq n-1$.

Let $\boldsymbol{z}_1 = (z_1, \ldots, z_m)^T$, $\boldsymbol{z}_2 = (z_{m+1}, \ldots, z_n)^T$. We can write, in the partitioned form,

$$\boldsymbol{e} = \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}, \quad \boldsymbol{z} = \begin{pmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{11}, & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{12}^T, & \boldsymbol{A}_{22} \end{pmatrix},$$

where $\boldsymbol{e}_1$ has m rows and $\boldsymbol{A}_{11}$ is $m \times m$. Since

$$\boldsymbol{A}\boldsymbol{e} = \boldsymbol{e},$$

$$\boldsymbol{A}\boldsymbol{z} = \lambda_2 \boldsymbol{z},$$

we obtain

$$\boldsymbol{A}_{11}\boldsymbol{e}_1 + \boldsymbol{A}_{12}\boldsymbol{e}_2 = \boldsymbol{e}_1,$$

$$\boldsymbol{A}_{11}\boldsymbol{z}_1 + \boldsymbol{A}_{12}\boldsymbol{z}_2 = \lambda_1 \boldsymbol{z}_1 .$$

If we multiply the first equality from the left by $\boldsymbol{z}_1^T$, the second by $\boldsymbol{e}_1^T$ and subtract, we have, by symmetry of $\boldsymbol{A}_{11}$,

$$\boldsymbol{z}_1^T \boldsymbol{A}_{12} \boldsymbol{e}_2 - \boldsymbol{e}_1^T \boldsymbol{A}_{12} \boldsymbol{z}_2 = (1-\lambda_2) \boldsymbol{e}_1^T \boldsymbol{z}_1 .$$

This can be written in the form

$$\sum_{\substack{i \in M_0 \\ k \notin M_0}} a_{ik}(z_i - z_k) = (1-\lambda_2) \sum_{j \in M_0} z_j .$$

Thus, by (1) and the definition of $\mu(\boldsymbol{A})$,

$$\lambda_1 - \lambda_2 = 1 - \lambda_2 = \sum_{\substack{i \in M_0 \\ k \notin M_0}} a_{ik} \frac{z_i - z_k}{\sum\limits_{j \in M_0} z_j} \geqq \left(\sum_{\substack{i \in M_0 \\ k \notin M_0}} a_{ik} \right) m(\boldsymbol{z}) \geqq$$

$$\geqq 2 \left(1 - \cos \frac{\pi}{n} \right) \sum_{\substack{i \in M_0 \\ k \notin M_0}} a_{ik} \geqq 2 \left(1 - \cos \frac{\pi}{n} \right) \mu\,(\boldsymbol{A})\,.$$

The proof of (10) is complete.

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