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THE REGULAR IDEAL IN A SEMIGROUP*

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1. Introduction and definitions: We define here in this note the regular ideal, $M(S)$, of a semigroup S with zero, as N. H. McCoy defined it for an associative ring [5] and prove some radical-like properties of $M(S)$ similar to those of the Schwarz (nilpotent) radical, the Clifford radical, the Ševrin radical and the McCoy radical [1, 7, 8]. We also prove that the mapping which takes an ideal A of a semigroup S to $M(A)$, is a lattice endomorphism in the lattice of all ideals of S (Theorem 2.6) and find a necessary and sufficient condition for a semigroup to be bound to its Schwarz radical (Theorem 3.9).

An element b of a semigroup S is called (von Neumann) regular if there exists an element b' in S such that $b = bb'b$. A zero in a semigroup with a zero is clearly regular. We assume throughout this note that S is a semigroup with a zero. An ideal (two sided) A of S is called regular if every element of A is regular. A regular ideal of S is itself a regular semigroup (actually a regular subsemigroup of S). A regular ideal B of S is called a maximal regular ideal of S if there is no regular ideal of S containing B properly. Clearly the family of regular ideals of S is non-empty. The union of all regular ideals of S is the unique maximal regular ideal of S and it is equal to $\{a \in S : J(a)$ is regular$\}$, where $J(a) = a \cup Sa \cup aS \cup SaS$, is the principal ideal of S, generated by a. We denote this unique maximal regular ideal of S by $M(S)$ and call it the regular ideal of S. It may be noted that any right (left or two sided) ideal of $M(S)$ is itself a right (left or two sided) ideal of S.

2. Radical-like properties of $M(S)$

Lemma 2.1. *Let A and B be any two ideals of S such that $A \subset B$. Then $M(A) \subseteq M(B)$, where $M(A)$ or $M(B)$, respectively is the regular ideal of the semigroup A (or B respectively).*

Proof. For an x in $M(A)$, $J(x)_A$ is a regular ideal in the semigroup A,

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where $J(x)_A$ denotes the principal ideal generated by x, in A. For any $y \in J(x)_B$, we have $y = b_1xb_2$, with $b_i \in B$ or b_i is an empty word. As $J(x)_A$ is regular, there exists x' in $J(x)_A$ such that $x = x\,x'\,x$ and $y = (b_1x)x'(xb_2) \in J(x)_A$. This means y is regular in A and hence in B, and thus $J(x)_B$ is regular in B, whence $x \in M(B)$.

Corollary 2.2. $M(M(S)) = M(S)$.

Lemma 2.3. *For any two ideals A and B of S,*

$$M(A \cap B) = M(A) \cap M(B).$$

Proof. By applying Lemma 2.1, we get $M(A \cap B) \subseteq M(A) \cap M(B)$. Now let $x \in M(A) \cap M(B)$. This means $J(x)_A$ and $J(x)_B$ are regular in semigroups A and B, respectively. For any $y \in J(x)_{A \cap B}$, there exists $y_1 \in A$, $y_2 \in B$ such that $y = y\,y_1\,y$ and $y = y\,y_2\,y$. On setting $y' = y_1\,y\,y_2$, we have $y = y\,y'\,y$ and $y' \in A \cap B$. Thus $J(x)_{A \cap B}$ is regular in $A \cap B$, placing x in $M(A \cap B)$. Thus $M(A \cap B) = M(A) \cap M(B)$.

Lemma 2.4. $M(A \cup B) = M(A) \cup M(B)$ *for any two ideals A and B of S.*

Proof. $M(A) \cup M(B) \subset M(A \cup B)$ follows from Lemma 2.1 and an x in $M(A \cup B)$ implies $J(x)_{A \cup B}$ is regular in $A \cup B$. Suppose $x \in A$; for any $y \in J(x)_A$ there exists y_1 in $A \cup B$ such that $y = y\,y_1\,y = y\,y'\,y$, where $y' = y_1\,y\,y_1 \in A$. Therefore, $J(x)_A$ is regular in A, whence $x \in M(A)$. Similarly $x \in B \Rightarrow x \in M(B)$. Hence $M(A \cup B) \subseteq M(A) \cup M(B)$. This completes the proof.

Lemma 2.5. *Let I be a regular ideal of an ideal A of S. Then I is a regular ideal of S.*

The proof is immediate.

From the foregoing lemmas, we have

Theorem 2.6. *The mapping which assigns to each ideal A of a semigroup S the regular ideal $M(A)$ is a lattice-endomorphism of the lattice of all the ideals of S.*

Theorem 2.7. *For any ideal A of S,* $M(A) = A \cap M(S)$.

Proof. $M(A) \subseteq A \cap M(S)$ is immediate in view of Lemma 2.5. Further, for a regular element b of S we can find a b_1 with $b = b\,b_1\,b$ and $b_1 = b_1\,b\,b_1$, whence $A \cap M(S) \subseteq M(A)$.

Theorem 2.8. $M(\bar{S}) = \{\bar{o}\}$, *where $\bar{S} = S/M(S)$ is the Rees factor semigroup* [2] *of S modulo the regular ideal $M(S)$.*

Proof: Let, if possible, $\bar{I}$ be a non-zero regular ideal of $\bar{S}$. Then $A = M(S) \cup B$, where $B = \bar{I} - \{\bar{o}\}$ is a regular ideal of S, containing $M(S)$ properly, which contradicts the maximality of $M(S)$. Therefore, $\bar{I} = \{\bar{o}\}$ and $M(\bar{S}) = \{\bar{o}\}$.

Proposition 2.9. *Let S be a semigroup with a restricted right cancellation (that is, $ab = cb$ and $b \neq o$ implies $a = c$, where a, b, c are in S). Then $M(S) = \{o\}$ or $M(S) = S$.*

Proof: Suppose $M(S) \neq \{o\}$ and choose a nonzero b in it. Then $b = b b' b$ for some b' in $M(S)$. For any x in S, we have $xb = (xbb')b$; by applying a restricted right cancellation, we get $x = x b b'$, which is in $M(S)$. Thus $M(S) = S$.

Remark 2.10. In the above proposition, the restricted right cancellation is an essential part of the hypothesis; for instance, if we take the semigroup $S = \{o, x, y\}$ with $x^2 = xy = yx = o$ and $y^2 = y$. Here the restricted right cancellation does not hold and as a result of that $M(S) = \{o, y\}$, which is obviously neither zero nor the whole of S.

3. The Schwarz (nilpotent) radical

Definition 3.1. *Let S be a semigroup with a zero. The union of all nilpotent ideals of S is called the Schwarz (nilpotent) radical of S and it is denoted by $R(S)$ [6].*

Definition 3.2. *Let A be any ideal of S. Then by its annihilator A^*, we mean the set consisting of those elements x of S for which $xA = Ax = \{o\}$. Clearly A^* is also an ideal of S.*

Lemma 3.3. *Let A be any ideal of S. Then $R(A) = A \cap R(S)$, where $R(A)$ is the union of all the nilpotent ideals of the semigroup A.*

This is Theorem 4.1 of Luh [4].

Theorem 3.4. *If $M(S)$ is the regular ideal and $R(S)$ is the nilpotent radical of a semigroup S with zero, then $M(S) \cap R(S) = \{o\}$, $R(S) \subseteq M(S)^*$, $M(S) \subseteq R(S)^*$, $M(S) \cap M(S)^* = \{o\}$, $M(S) = M(R(S)^*)$ and $R(S) = R(M(S)^*)$.*

Proof. As the nilpotent radical $R(S)$ is nil, it has no nonzero idempotents. For an x in $M(S) \cap R(S)$, we have x' in S such that $x = x x' x$, but then xx' is idempotent and is in $R(S)$, therefore it must be zero; whence $M(S) \cap R(S) = \{o\}$. This, in turn, gives $M(S) . R(S) = \{o\}$, which yields $M(S) \subseteq R(S)^*$ and $R(S) \subseteq M(S)^*$. That $M(S) \cap M(S)^* = \{o\}$ is immediate; and from $M(S) \subseteq R(S)^*$, we get $M(S) \subseteq M(R(S)^*)$ by applying Lemma 2.1 and Corollary 2.2. The reverse inclusion also holds by Lemma 2.5. Again $R(S) \subseteq M(S)^*$ gives $R(S) \subseteq R(M(S)^*)$ and the opposite inclusion follows from Lemma 3.3.

Corollary 3.5. *Let S be a principal ideal semigroup (that is a semigroup each of whose ideal is a principal ideal). Then either $M(S) = \{o\}$ or $R(S) = \{o\}$.*

Proof. First we show that the ideals of S are totally ordered, and for this, it suffices to show that for any a and b in S, either $J(a) \subseteq J(b)$ or $J(b) \subseteq J(a)$. By hypothesis, there exists some c in S such that $J(a) \cup J(b) = J(c)$, whence the assertion. By Theorem 3.4, we have $M(S) \cap R(S) = \{o\}$, from which the corollary follows.

Remark 3.6. One cannot omit from the hypothesis of the above corollary that S is a principal ideal semigroup. For instance, the semigroup in Remark 2.10 is not a principal ideal semigroup and consequently $M(S) = \{o, y\}$, $R(S) = \{o, x\}$ and none is contained in the other.

Proposition 3.7. *In a semigroup S any one-sided annihilator of $M = M(S)$ is two-sided.*

Proof. Let $Mx = \{o\}$ for some x in S. We will show that $xM = \{o\}$. Clearly $(xM)^2 = \{o\}$, so that xM is a nilpotent right ideal of S and hence $xM \subseteq R(S)$; but $R(S) \subseteq M(S)^* = M^*$ by Theorem 3.4; therefore $xM \subseteq M^*$ whence $xM^2 = \{o\}$, but $M^2 = M$, so $xM = \{o\}$. Similarly $yM = \{o\} \Rightarrow My = \{o\}$.

Definition 3.8. *A semigroup S is said to be bound to its Schwarz (nilpotent) radical $R(S)$ if $R(S)^* \subseteq R(S)$.*

This concept is defined by Hall [3] for a ring and its Jacobson radical. We now prove a result similar to Theorem 6 of [5].

Theorem 3.9. *Let S be a semigroup such that $\bar{S} = S/R(S)$ is regular. Then S is bound to $R(S)$ if and only if $M(S) = \{o\}$.*

Proof. Let S be bound to $R(S)$. By definition $R(S)^* \subseteq R(S)$. Also Theorem 3.4 gives $M(S) \subseteq (R(S)^*$ and $M(S) \cap R(S) = \{o\}$. These coupled together yield $M(S) = \{o\}$. In this part we do not make use of the regularity of $\bar{S}$, at all. On the other hand, let $M(S) = \{o\}$ and $\bar{S} = S/R(S)$ be regular. Now $x \in R(S) \cap (R(S)^*)^2 \Rightarrow x = ab$ for some $a, b \in R(S)^*$. If $a \in R(S)$, then $x = ab = o$; if $a \notin R(S)$, then, since $\bar{S}$ is regular, we have $a = aa'a$ for some a' in $\bar{S}$. Thus $x = ab = aa'x \in R(S)^*. R(S) = \{o\}$. Therefore, $R(S) \cap (R(S)^*)^2 = \{o\}$. We prove now that $(R(S)^*)^2 = \{o\}$; as for any nonzero y in $(R(S)^*)^2$, $y \notin R(S)$, whence y is regular in $\bar{S}$ and hence in S, so that $(R(S)^*)^2$ is a regular ideal of S and hence it must be contained in $M(S)$, which is equal to $\{o\}$. Thus $(R(S)^*)^2 = \{o\}$; that is $R(S)^*$ is a nilpotent ideal of S and therefore, $R(S)^* \subseteq R(S)$, which means that S is bound to $R(S)$. This completes the proof of the Theorem.

Remark 3.10. For the second part in Theorem 3.10 the regularity of $\bar{S}$ is an essential requirement. For instance, if we consider the multiplicative semigroup S of non-negative integers, we have $M(S) = \{o\} = R(S)$, $\bar{S}$ is not regular and as a result, S is not bound to $R(S)$, as $R(S)^* = S \nsubseteq R(S)$.

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