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A NOTE ON THE GENERALIZATION OF A SUMMATION FORMULA FOR APPELL'S FUNCTION F_2

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Recently, Bhatt [(2)] has given a summation formula for Appell's function F_2; in a generalization of this work, I have established here a summation formula for the Kampé de Fériet function, which is as follows:

$$\sum_{m=0}^{n} \frac{(1+A)_m}{m!} F_{1,1}^{2,1}\left[\begin{matrix} -\alpha+\beta, \beta: & -m, -m; \\ -\alpha+\beta-N: & 1+A, 1+A; \end{matrix} \; x, y\right]$$

$$= \frac{(2+\alpha-\beta)_N (1+A)_{n+1} (x-y)^{-1}}{(1+\alpha-\beta)_N n! \, (\beta-1)} \times$$

$$\times \left\{ F_{1,1}^{2,1}\left[\begin{matrix} -\alpha+\beta-1, \beta-1: & -n, -n-1; \\ -\alpha+\beta-N-1 & : 1+A, 1+A; \end{matrix} \; x, y\right] - \leftrightharpoons \right\}$$

where "$\leftrightharpoons$" shows the presence of a similar term with x and y interchanged.

1. Introduction

Kampé de Fériet [(1), p. 150] has defined a generalized hypergeometric function of two variables which is represented in a modified notation given by Srivastava and Saran [(4), p. 435] as

$$(1.1) \quad F_{\nu,\varrho}^{\lambda,\mu}\left[\begin{matrix} |\alpha|_\lambda : |\beta, \beta'|_\mu; \\ |\lambda|_\nu : |\delta, \delta'|_\varrho; \end{matrix} \; x, y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\Pi(\alpha_\lambda)_{m+n} \, \Pi[(\beta_\mu)_m (\beta'_\mu)_n] \, x^m y^n}{m! \, n! \, \Pi(\gamma_\nu)_{m+n} \, \Pi[(\delta_\varrho)_m (\delta'_\varrho)_n]},$$

where $\Pi(a_p)_s$ stands for the product $(a_1)_s(a_2)_s \ldots (a_p)_s$; for the absolute con vergence of the series $|x| < 1$, $|y| < 1$, $\lambda + \mu \leq \nu + \varrho + 1$.

In the present note, I have given a summation formula for Kampé de Fériet's function which generalizes a well-known result given by Bhatt [(2)]. The formula may prove to be useful as the exact solution of various problems in the theory of quantum mechanics and heat conduction that have been extensively given in terms of Kampé de Fériet's function.

2. Summation Formula

We have from a well-known result [(5), p. 119, (4.5) for $m = n = 0$, $\xi = 1$]:

$$\int_0^\infty e^{-u} u^{\beta-1} L_N^{(\alpha)}(u)\, {}_lF_p\begin{bmatrix} b_l \\ d_p \end{bmatrix}; xu\,\Big]\, {}_lF_p\begin{bmatrix} b'_l \\ d'_p \end{bmatrix}; yu\,\Big]\, du = \tag{2.1}$$

$$= \frac{(1+\alpha-\beta)_N \Gamma(\beta)}{N!} F^{2,l}_{1,p}\begin{bmatrix} -\alpha+\beta, \beta : |b, b'|_l; & x, y \\ -\alpha+\beta-N : |d, d'|_p; & \end{bmatrix},$$

where $|x| < 1$, $|y| < 1$, $Re(\beta) > 0$; a_p represents the sequence $a_1, a_2, \ldots, a_p$ and $L_n^{(\alpha)}(x) = \dfrac{(1+\alpha)_n}{n!}\, {}_1F_1\,(-n;\, 1+\alpha;\, x)$ is the generalized Laguerre polynomial.

A special case of (2.1) is

$$\left\{\frac{(1+A)_m}{m!}\right\}^2 F^{2,1}_{1,1}\begin{bmatrix} -\alpha+\beta, \beta : -m, -m; & x, y \\ -\alpha+\beta-N : 1+A, 1+A; & \end{bmatrix} \tag{2.2}$$

$$= \frac{N!}{(1+\alpha-\beta)_N \Gamma(\beta)} \int_0^\infty e^{-u} u^{\beta-1} L_N^{(\alpha)}(u)\, L_m^{(A)}(xu)\, L_m^{(A)}(yu)\, du.$$

Therefore

$$\sum_{m=0}^{n} \frac{(1+A)_m}{m!} F^{2,1}_{1,1}\begin{bmatrix} -\alpha+\beta, \beta : -m, -m; & x, y \\ -\alpha+\beta-N : 1+A, 1+A; & \end{bmatrix}$$

$$= \frac{N!\Gamma(1+A)}{(1+\alpha-\beta)_N \Gamma(\beta)} \sum_{m=0}^{n} \frac{m!}{\Gamma(m+A+1)} \int_0^\infty e^{-u} u^{\beta-1} L_N^{(\alpha)}(u) L_N^{(A)}(xu)\, L_m^{(A)}(yu)\, du.$$

The interchange of the order of summation and integration is easily justified and we have the expression on the right-hand side as

$$= \frac{N!\Gamma(1+A)}{(1+\alpha-\beta)_N \Gamma(\beta)} \int_0^\infty e^{-u} u^{\beta-1} L_N^{(\alpha)}(u) \left\{ \sum_{m=0}^{n} \frac{m!}{\Gamma(m+A+1)} L_m^{(A)}(xu)\, L_m^{(A)}(yu) \right\} du.$$

Making use of the well-known Christoffel-Darboux formula [(3), p. 188, (9)]:

$$\sum_{m=0}^{n} \frac{m!}{\Gamma(m+\alpha+1)} L_m^{(\alpha)}(x)\, L_m^{(\alpha)}(y)$$

$$= \frac{(n+1)!}{\Gamma(n+\alpha+1)} \frac{1}{x-y} [L_n^{(\alpha)}(x) L_{n+1}^{(\alpha)}(y) - L_{n+1}^{(\alpha)}(x) L_n^{(\alpha)}(y)]\,,$$

the expression on the R.H.S. reduces to

$$\frac{(n+1)!\, \Gamma(1+A) N!\, (x-y)^{-1}}{(1+\alpha-\beta)_N\, \Gamma(\beta)\, \Gamma(n+A+1)} \times$$

$$\int_0^\infty e^{-u} u^{\beta-2} L_N^{(\alpha)}(u)[L_n^{(A)}(xu) L_{n+1}^{(A)}(yu) - L_{n+1}^{(A)}(xu) L_n^{(A)}(yu)]\, du, \quad Re(\beta) > 1\,.$$

Now separating the R.H.S. as the difference of two integrals, then evaluating these integrals with the help of (2.1), we obtain

$$(2.3) \qquad \sum_{m=0}^{n} \frac{(1+A)_m}{m!} F_{1,1}^{2,1}\left[\begin{matrix}-\alpha+\beta, \beta \quad : -m, -m; \\ -\alpha+\beta-N : 1+A, 1+A;\end{matrix} \quad x, y\right]$$

$$= \frac{(2+\alpha-\beta)_N (1+A)_{n+1} (x-y)^{-1}}{(1+\alpha-\beta)_N\, n!\, (\beta-1)} \times$$

$$\times \left\{ F_{1,1}^{2,1}\left[\begin{matrix}-\alpha+\beta-1, \beta-1 : -n, -n-1; \\ -\alpha+\beta-N-1 \quad : 1+A, 1+A;\end{matrix} \quad x, y\right] - \rightleftharpoons \right\}$$

where the abbreviation $\leftrightharpoons$ is employed to show the presence of a second term that originates from the first by interchanging x and y.

This is the required summation formula.

3. Particular case

In (2.3), taking $A = N = \alpha = 0$ etc., we obtain an interesting result on a summation formula for Appell's function F_2 due to Bhatt [(2), p. 88, (4)]:

$$\sum_{n=0}^{m} F_2(a, -n, -n; 1, 1; x, y)$$

$$= \frac{(m+1)(x-y)^{-1}}{a-1} [F_2(a-1, -m, -m-1; 1, 1; x, y) - \rightleftharpoons]\,.$$

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