Matematický časopis

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Matematický časopis, Vol. 17 (1967), No. 3, 240--241

Persistent URL: http://dml.cz/dmlcz/126941

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EQUIVALENCE OF A NUMBER-THEORETICAL PROBLEM WITH A PROBLEM FROM THE GRAPH THEORY

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Let $k \geqq 3$ be a natural number. Let $N = g(k)$ be the greatest natural number such that all edges of a complete graph of N vertices can be coloured by two colours in such a way that there does not arise any clique (= complete subgraph) of k vertices, all edges of which are coloured by the same colour. (A clique, all edges of which are coloured by the same colour, is called monochromatic.)

Problem 1. *Determine $g(k)$ for an arbitrary $k \geqq 3$.*

From [1] and [2] it follows that

$$2^{\frac{k}{2}} \leqq g(k) \leqq \binom{2k-2}{k-1} - 1. \tag{1}$$

In [2] the following exact values are given: $g(3) = 5$, $g(4) = 17$.

We shall now formulate a problem from the theory of numbers and show it to be equivalent to the above problem.

Let $k \geqq 3$ be a natural number. Let us denote by $u(k)$ the greatest natural number (resp. infinity) for which there can be found $u(k)$ natural numbers such that in arbitrary k-tuple of them there exists at least one pair of coprime numbers and at least one pair of not coprime numbers.

Problem 2. *Determine $u(k)$ for an arbitrary $k \geqq 3$.*

Theorem. *For an arbitrary $k \geqq 3$ we have*

$$u(k) = g(k).$$

Proof. First we prove that $u(k) \leqq g(k)$. Let us suppose that the numbers $x_1, x_2, \dots, x_{u(k)}$ have the property that every k-tuple of them contains at least one pair of coprime and at least one pair of not coprime numbers. Let to each of these numbers correspond one point of the plane. Two of these points are joined by a red edge if the numbers corresponding to them are not coprime and by a blue edge if these numbers are coprime. In this way a complete graph of $u(k)$ vertices arises, which obviously does not contain any monochromatic clique of k vertices, and so it has at most $g(k)$ vertices. Hence $u(k) \leqq g(k)$.

We have still to prove that $u(k) \geqq g(k)$. Let G be a complete graph of $g(k)$ vertices. We colour all edges of G by two colours (red and blue) in such a way that there does not arise any monochromatic clique of k vertices (owing to the definition of $g(k)$ it can be always done). Let us assign to every red edge a prime (to different edges are assigned different primes) and to every blue edge the number 1. We assign to every vertex the number equal to the product of all numbers corresponding to the edges incident with this vertex. In this way we get $g(k)$ natural numbers. Let us take any k-tuple of them. If in this k-tuple there does not exist any pair of coprime (not coprime) numbers, then all edges of the clique of k vertices corresponding to the numbers of our k-tuple are red (blue). This is a contradiction. Hence $g(k) \leqq u(k)$; q. e. d.

Remark. From (1) and from the Theorem it follows that

$$2^{\frac{k}{2}} \leqq u(k) \leqq \binom{2k-2}{k-1} - 1 .$$

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Received June 3, 1966.

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