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LEBESGUE DENSITY THEOREM IN TOPOLOGICAL SPACES

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In this paper we shall prove the Lebesgue density theorem in topological spaces. This theorem can be easily deduced from Theorem 6 of paper [5] by the special choice $f(E) = E$. But this choice does not satisfy all assumptions required in [5]. The paper contains a theorem and three of its corollaries. The results of the present paper are applied in paper [4].

Let K be any system of subsets of a topological space X, H be the system of all subsets of X, m be a set function defined on H. Let S be a σ-algebra, $S \subset H$. All further notions will refer to fixed K, H, m.

Let T be a directed set. A system $\{E_t\}_{t \in T}$ converges to a point x if for any neighbourhood U of x there is such a t_0, that for every $t \geqq t_0$ we have $x \in E_t \subset U$. Let K be any system of subsets of X, m be a set function defined on H, positive and finite on K. For $x \in X$ and $M \in H$ we define

$$\bar{D}_M(x) = \sup \left\{ \lim_{t \in T} \frac{m(E_t \cap M)}{m(E_t)} : E_t \in K, \{E_t\} \text{ converges to } x \right\}.$$

$$\underline{D}_M(x) = \inf \left\{ \lim_{t \in T} \frac{m(E_t \cap M)}{m(E_t)} : E_t \in K, \{E_t\} \text{ converges to } x \right\},$$

If $\underline{D}_M(x) = \bar{D}_M(x)$, we say that M has in x the density $D_M(x) = \underline{D}_M(x) = \bar{D}_M(x)$.

A system of closed subsets of X covers a set $A \subset X$ in the Vitali sense if for any $x \in A$ and any neighbourhood U of x there is $E \in K$ such that $x \in E \subset U$.

Let a system K cover the space X in the Vitali sense. We say that the Vitali Theorem holds for a set function m (with K) if for any set $A \in H$ and any system $L \subset K$ covering A in the Vitali sense there is a sequence $\{E_i\}$ of pairwise disjoint sets from L such that $m(A - \bigcup_{i=1}^{\infty} E_i) = 0$.

A set $M \in S$ is m-regular if for any $\delta > 0$ there are an open set U and a closed set F such that $U \supset M \supset F$ $U, F \in S$ and $m(U - F) < \delta$.

A set $M \in H$ is m-measurable if for every $E \in H$ we have $m(E) = m(M \cap \cap E) + m(E - M)$.

After these preliminaries we can formulate our main result.

Theorem. *Let X be a topological space, K be a system of closed subsets covering X in the Vitali sense. Let m be an outer measure defined on the system H of all subsets of X, positive and finite on K and let for m the Vitali theorem hold (with K). Let S be a σ-algebra, $K \subset S$ and let for any $E \in S$ and $E_i \in K$ $(i = 1, 2, \ldots)$, $E_i \cap E_j = \emptyset$ $(i \neq j)$ be $m(E) \geqq \sum_{i=1}^{\infty} m(E \cap E_i)$.*

Then for any m-measurable and m-regular set $M \in S$ there exists m-almost everywhere in X the density $D_M(x)$ and the equality $D_M = \chi_M$ holds m-almost everywhere (χ_M is the characteristic function of the set M).

Proof. Let M be any m-regular, m-measurable set, $\delta > 0$ be a positive number. By an assumption there exists a closed set $E \subset M$, such that

$$(1) \qquad m(M - E) < \delta.$$

Put $G_t = \{s : \overline{D}_{M-E}(x) > t\}$. First we shall prove that

$$(2) \qquad m(G_t) \leqq \frac{1}{t} m(M - E).$$

Put $L = \{F \in K : m(F \cap (M - E))/m(F) > t\}$. Clearly L covers the set G_t in the Vitali sense, hence there is a sequence $\{E_n\}$ of pairwise disjoint sets from L such that $m(G_t - \bigcup_{n=1}^{\infty} E_n) = 0$. From this it follows that

$$m(G_t) \leqq \sum_{n=1}^{\infty} m(E_n) \leqq \frac{1}{t} \sum_{n=1}^{\infty} m(E_n \cap (M - E)) \leqq \frac{1}{t} m(M - E),$$

hence follows (2).

Let H_t be the set of all $x \notin M$ for which $\overline{D}_M(x) > t$. Since E is a closed set, it is clear that $H_t \subset G_t$. Hence by (1) and (2)

$$m(H_t) \leqq m(G_t) \leqq \frac{1}{t} m(M - E) < \frac{\delta}{t}.$$

From the last inequality it follows that for any $t > 0$ we have $m(H_t) = 0$, i. e.

$$(3) \qquad D_M(x) = \overline{D}_M(x) = 0 \text{ for } m\text{-almost every } x \notin M.$$

Since the set $X - M$ is also m-regular, we have by (3)

$$(4) \qquad D_{X-M}(x) = 0 \text{ for } m\text{-almost every } x \in M.$$

Since M is m-measurable, there exists $D_M(x)$ m-almost everywhere in X and

(5) $$D_M(x) = 1 \text{ for } m\text{-almost every } x \in M.$$

From (3) and (5) the assertion of the Theorem follows.

Corollary 1. *Let X be a locally compact Hausdorff topological space, K be a system of compact sets covering X in the Vitali sense. Let m be a Carathéodory outer measure defined on the system H of all subsets of X* (i. e. an outer measure for which $m(E \cup F) = m(E) + m(F)$ whenever there are open disjoint sets U, V such that $\bar{E} \subset U$, $\bar{F} \subset V$). *Let m be finite and positive on K and let for m the Vitali theorem hold.*

Then for any Baire set M of finite measure there exists m-almost everywhere in X the density $D_M(x)$ and m-almost everywhere in X we have $D_M = \chi_M$.

Proof. Since m is a Carathéodory outer measure, K is a system of compact sets and X is a Hausdorff space, m satisfies all the assumptions of Theorem (with $S = H$). Let M be any Baire set of finite measure. The proof will be complete if we prove that M is m-measurable and m-regular. In paper [3] it is proved, that all compact G_δ sets (and hence also all Bair sets) are m-measurable. Since m is finite on the system of all compact sets (K covers X in the Vitali sense!), m is a Baire measure on the system of all Baire sets, hence by the known results from [1], m is a regular measure.

Corollary 2. *Let X be a locally compact Hausdorff topological space, K be a system of compact G_δ sets covering X in the Vitali sense. Let m be a measure on the system S of all Baire sets, finite and positive on K. Denote by m^* the outer measure induced by m. Let for m^* the Vitali theorem with K hold.*

Then for any bounded Baire set M there exists m^*-almost everywhere in X the density $D_M(x)$ and m^*-almost everywhere in X we have $D_M = \chi_M$.

Proof. Since M is bounded, there exist Baire sets C, U such that $M \subset C \subset U$, C is compact and U open. Let L be the system of all $E \in K$ for which $E \subset U$, let T be the σ-ring of all Baire subsets of U, H be the least hereditary σ-ring over T, m^* be the outer measure on H induced by m. The topological space U, the systems L, T, H and the outer measure m^* satisfy the assumptions of the Theorem. The set M is m^*-measurable and m^*-regular. Hence for m^*-almost all $x \in U$ we have $D_M(x) = \chi_M(x)$. (Since U is open, the density defined by the help L is on U equal to that defined by the help of K.) For $x \notin U$ we have $D_M(x) = 0$.

Corollary 3. *Let X be a σ-compact Hausdorff topological space, K be a system of compact subsets of X covering X in the Vitali sense, m be a regular Borel measure defined on the σ-algebra S of all Borel subsets of X, positive on K, m^* be the outer measure induced by m and let the Vitali theorem hold for m^*.*

Then for every Borel set M of positive measure there exists m^*-almost everywhere the density D_M and m^*-almost everywhere in X, $D_M = \chi_M$.

Proof. If in Theorem we take m^* besides m, then all the assumptions of Theorem are satisfies with the same meaning of K, S, H.

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