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Matematický časopis, Vol. 23 (1973), No. 2, 103--105

Persistent URL: http://dml.cz/dmlcz/126830

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THE EMBEDDING OF SOME LATTICES INTO LATTICES OF ALL SUBGROUPS OF FREE GROUPS

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Philip M. Whitman showed ([3], Theorem 2) that any lattice is isomorphic to a sublattice of the lattice of all subgroups of some group. B. Jónsson [2] posed the question whether the class of all lattices which are isomorphic to lattices of subgroups of commutative groups can be characterized by means of identities. In this note it is proved that any distributive lattice is isomorphic to a sublattice of all subgroups of a commutative group. Moreover, an estimate for the cardinality of the group is given.

Let K be the class of all groups or the class of all commutative groups. Let X be a set. A group $G \in K$ generated by X is said to be a free K-group over X if for any group $D \in K$ and any mapping $f : X \to D$ there is exactly one homomorphism $g : G \to D$ so that $gi = f$ where $i : X \to G$ is the injection.

Theorem 1. *Any distributive lattice L is isomorphic to a sublattice of the lattice of all subgroups of a free K-group G. Moreover, the cardinality of the set of free generators of G is not greater than $2^{\mathfrak{m}}$ where $\mathfrak{m}$ is the cardinality of L.*

Theorem 2. *Let L be a lattice satisfying the following two conditions:*

(i) *If $a \in L$ is join-reducible then $a = b \cup c$ where b, c are join-irreducible.*

(ii) *If $a, b, c, d, e \in L$, $a = b \cup c = d \cup e$, $a \neq b$, $a \neq c$, $a \neq d$, $a \neq e$ then either $b = d$, $c = e$ or $b = e$, $c = d$.*

Then L is isomorphic to a sublattice of the lattice of all subgroups of the free K-group over L.

Proof of Theorem 1. The lattice L is isomorphic to a ring R of sets over a set M (G. Birkhoff [1], Theorem 25.2). Let $F(M)$ denote the free K-group over M. If $S \in R$, then $S \subset M$ and the free K-group $F(S)$ over S is a subgroup of the group $F(M)$. Denote by $P(F(M))$ the lattice of all subgroups of the group $F(M)$. To complete the proof we need the following

Lemma 1. *The mapping $f : R \to P(F(M))$ defined by $f(S) = F(S)$ is one-to-one and preserves the operations of join and meet.*

Proof. Let $A, B \in R$, $A \neq B$. If $a \in A$, $a \notin B$, then $a \in F(A)$, $a \notin F(B)$ and so f is one-to-one. We shall write the elements of $F(M)$, except the unity, in the form $x_1^{k_1}x_2^{k_2} \dots x_n^{k_n}$, where $x_j \in M$ and k_j are non-zero integers. The unity will be denoted by λ. Let $A, B \subset M$. We shall now show that $F(A) \wedge F(B) = F(A \cap B)$. Let $z \in F(A) \wedge F(B)$, this means $z \in F(A)$, $z \in F(B)$ and let us have two expressions $z = x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}$ and $z = y_1^{h_1} y_2^{h_2} \dots y_m^{h_m}$, where $x_1, x_2, \dots, x_n \in A$, $y_1, y_2, \dots, y_m \in B$. Both of them contain the same generators from M, hence $x_1, x_2, \dots, x_n \in A \cap B$, which means $z \in F(A \cap B)$. The converse inclusion is obvious. Next we shall show $F(A) \vee F(B) = F(A \cup B)$. Both of these sets contain exactly the elements of the form $x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}$, where each $x_j \in A \cup B$ and the element λ, which proves the lemma.

Thus the ring R of sets is isomorphic to the lattice of all free groups generated by the elements of the ring R. Since M can be considered as the set of all dual prime ideals of the lattice L, its cardinality is not greater than $2^{\mathfrak{m}}$. Now Theorem 1 is proved.

Any lattice satisfying (ii) is obviously distributive.

Lemma 2. *Let L be a lattice satisfying the conditions* (i), (ii). *Then from $a \| b$, $c \neq a$, $c \neq b$ it follows that $c < a$, $c < b$ or $c > a$, $c > b$.*

Proof. Suppose $b \| c$ and consider $a \cup b$, $b \cup c$. The inequality $a \cup b < b \cup c$ would imply $b \cup c = (a \cup b) \cup c$, $a \cup b \| b \cup c$ would imply $(a \cup b) \cup c = (a \cup b) \cup (b \cup c)$. Both of them as well as the equality $a \cup b = b \cup c$ contradict (ii). Similar arguments yield a contradiction with the assumption $a \| c$. Hence c must be comparable with both a and b, which proves the lemma.

Lemma 3. *Let L be a lattice satisfying the conditions* (i), (ii). *If $a = b \cup c$, $a \neq b$, $a \neq c$, then there is no element e with $a \| e$.*

Proof. If $a \| e$, then $a \cup e = b \cup c \cup e$, which contradicts (i) and (ii).

Proof of Theorem 2. We shall construct a suitable ring of sets over L. We define the set T as follows: $S \in T$ if and only if $S = (a] \cup (b]$ for some irreducible $a, b \in L$, where

$$(a] = \{x \mid x \in L,\ x \leqq a,\ x \text{ is irreducible}\}.$$

Applying Lemma 2 we conclude that T is a ring of sets. We shall show that T is isomorphic to L. Define a mapping $g : L \to T$ by the conditions

$$g(a) = \begin{cases} (a] \text{ if } a \text{ is join-irreducible} \\ (b] \cup (c] \text{ if } a \text{ is join-reducible and } a = b \cup c,\ a \neq b,\ a \neq c. \end{cases}$$

We shall prove g to be a one-to-one mapping. Let be $a, b \in L$, $a \neq b$.

First, suppose $a < b$. If both a and b are irreducible, then $g(a) = (a] \neq (b] = g(b)$. If both a and b are reducible then $a = c \cup d$, $b = e \cup t$, where c, d, e, t are irreducible. We shall show $a < e$. The inequality $e < a$ would imply $b = e \cup t \leqq a \cup t$ and on the other hand $a < b$, $t < b$ imply $a \cup t \leqq b$, hence $a \cup t = b$, which contradicts (i) and (ii). The equality $a = e$ is not possible because a is reducible and e is irreducible. The incomparability $a \| e$

contradicts Lemma 3 hence $a < e$. Now $e \notin (c] \cup (d]$ and $g(a) = (c] \cup (d] \neq$ $\neq (e] \cup (t] = g(b)$. If a is irreducible and b is reducible, then similar arguments as before prove $e \notin (a]$ or $a = e$, hence $g(a) = (a] \neq (e] \cup (t] = g(b)$. If a is reducible and b is irreducible, then $b \in (b]$, $b \notin (c] \cup (d]$, which implies $g(a) =$ $= (c] \cup (d] \neq (b] = g(b)$.

Further, suppose $a \parallel b$. Then, because of (i) and (ii), a, b are irreducible. It follows that $g(a) = (a] \neq (b] = g(b)$.

By this way g was shown to be a one-to-one mapping. An element $(a] \cup (b]$ has an inverse image $a \cup b$. Obviously, g is a lattice homomorphism. Hence g is an isomorphism from L onto T. To complete the proof we proceed similarly as in the proof of Theorem 1.

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Received August 12, 1968

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