Alexander Abian

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CONTINUITY BASED ON CONVERGENCE AND LOWER COMPLETE DISTRIBUTIVE LATTICES INSTEAD OF DIRECTED SETS

ALEXANDER ABIAN, Iowa, U.S.A.

Let f be a function from a topological space T into a topological space T'. The function f is defined to be continuous at a point x if and only if for every neighborhood U of $f(x)$ there exists a neighborhood V of x such that $f[V] \subseteq U$. This definition of continuity is universally accepted and is so devised as to reflect our intuitive feeling for continuity. Indeed, from an intuitive approach, we would like to require that f be continuous at x if and only if whenever "a set S of points is close to x" then "the set $f[S]$ of the images of these points is close to $f(x)$". Now, if in the above "S is close to x" is interpreted as "x is an element of the closure of S" then the definition of continuity reflects the intuitive approach faithfully. However, it is much more natural and quite intuitive to interpret "closeness" in terms of the notion of "limit". But then the usage of "limit" in a straightforward way would be inappropriate because if "x is a limit point of S" and f a continuous function then it is not necessarily the case that "$f(x)$ is a limit point of $f[S]$". In other words, because a continuous function "does not necessarily preserve limit points". For instance, a constant function is continuous, however, in general, it does not preserve limit points, since in no topological space is p a limit point of the singleton $\{p\}$.

Thus, to interpret "close to x" in terms of "limit" in such a way that it would be suitable to characterize continuity the usage of "limit" in a roundabout way seems to be necessary. As shown below, this can be accomplished by interpreting the notion "close to x" in terms of "limit" in connection with the concept of convergence of an indexed family of points to x. Accordingly, f turns out to be continuous (naturally, in the universally accepted sense) if and only if "f preserves convergence". Also, with this interpretation of the notion "close to x", the test for continuity of f at x is initiated at x in the sense that we choose points converging to x and test the images for convergence to $f(x)$. This procedure for testing continuity is much more direct than the one immediately suggested by the definition of continuity. Indeed, according to the definition of continuity in order to ascertain the continuity of f at x, we start by choosing an arbitrary neighborhood of $f(x)$ and consequently, we initiate the test of continuity starting with $f(x)$ instead of x as our intuition would lead us to do.

Before the introduction of a general notion of convergence in topological spaces, we give the following example by way of motivation.

Let us consider the relation R given by

$$(1) \qquad R = \{(a, a), (b, a), (b, c), (c, a), (c, b)\}$$

where distinct letters represent distinct elements.

We observe that R is quite arbitrary inasmuch as it is neither reflexive. nor symmetric, nor antisymmetric, nor transitive. Nevertheless, based on R. we may introduce the notion "*beyond*". Thus, we say that with respect to R.

$$(2) \qquad \text{"}y \text{ is beyond } x\text{"} \textit{ if and only if } (x, y) \in R$$

Next, we show that the notion "beyond" as introduced along the above-mentioned lines, is sufficient for the definition of convergence with respect to R, in a topological space.

Let T be a topological space, where

$$(3) \qquad T = \{\alpha, \beta, \gamma, \delta\}$$

and where the list of the open sets of T is given by

$$(4) \qquad \emptyset, \{\beta\}, \{\alpha, \beta\}, \{\beta, \gamma, \delta\}, \{\alpha, \beta, \gamma. \delta\}$$

It would be quite reasonable to say that in topological space T the indexed family of points, say,

$$(5) \qquad \{\alpha_b, \beta_a, \gamma_c\} \textit{ converges with respect to } R \textit{ to } \delta$$

if and only if for every neighborhood V of δ there exists an index n such that every element of T whose index in (5) is beyond n (naturally with respect to R) is an element of V.

We note that (1) to (4) imply that (5) holds. Indeed, the only neighborhoods of δ are $\{\beta, \gamma, \delta\}$ and $\{\alpha, \beta, \gamma, \delta\}$. For $\{\beta, \gamma, \delta\}$ we may choose the index b since. as (1) and (5) show every element (in this case β and γ) of T whose index in (5) is beyond b is an element of $\{\beta, \gamma, \delta\}$. For $\{\alpha, \beta, \gamma, \delta\}$ we may again choose the index b or a or c. We may choose a, since every element (in this case only β) of T whose index in (5) is beyond a is an element of $\{\alpha. \beta, \gamma. \delta\}$.

On the other hand, however,

$$(6) \qquad \{\alpha_a, \beta_b, \gamma_c\} \textit{ does not converge with respect to } R \textit{ to } \gamma$$

since there exists no index n such that every element of T whose index in (6) is beyond n (naturally with respect to R) is an element of the neighborhood $\{\beta, \gamma, \delta\}$ of γ.

Again, based on (1) to (4), it can be verified that in topological space T given by (3) and (4), we have:

$\{\beta_a, \alpha_b, \alpha_c\}$ *converges with respect to* R *to* γ

$\{\delta_a, \gamma_c, \alpha_b\}$ *does not converge with respect to* R *to* β

$\{\beta_a, \beta_b, \beta_c\}$ *converges with respect to* R *to* δ

$\{\delta_a, \beta_b, \beta_c\}$ *does not converge with respect to* R *to* α.

Motivated by the above, we introduce:

Definition 1. *Let R be a relation and T a topological space. An indexed family $(c_i)_{i \in I}$ of points of T is said to converge with respect to R to a point x of T if and only if for every neighborhood V of x there exists an index n such that*

$$c_i \in V \quad \text{for every} \quad i \quad \text{with} \quad (n, i) \in R. \tag{7}$$

Remark 1. We observe that the above Definition is especially significant provided R is a relation from the index set I into I. However, it is not necessary to make this stipulation in the statement of the Definition.

Again, let R be a relation and f a function from a set D into a topological space T'. Moreover, let $(c_i)_{i \in I}$ be an indexed family of points of D. Then, as expected, we say that the indexed family $((f(c_i))_{i \in I}$ of points of T' converges with respect to R to a point y of T' if and only if for every neighborhood V' of y there exists an index n such that

$$f(c_i) \in V' \quad \text{for every} \quad i \quad \text{with} \quad (n, i) \in R \tag{8}$$

Naturally. Remark 1 also applies to the above.

Remark 2. We observe that if continuity of functions in topological spaces can be described based on convergence with respect to a relation R, then the more restrictive R is, the better. A reason for this is the fact that the more restrictive R is the "fewer" convergent families there will be. and, therefore the "easier" the test for continuity will be. In this connection. and, as shown below, for the purpose of describing continuity of functions in topological spaces, *lower complete distributive lattices* constitute most suitable relations.

Let us recall that a partially ordered set is called a *lattice* if and only if every two element subset of its domain has a supremum and an infimum. Moreover, a lattice is called *distributive* if and only if in it the usual distributive laws are valid [1, p. 12].

Definition 2. *A lattice L is called lower complete if and only if every nonempty subset of the domain of L has an infimum.*

Based on Definitions 1 and 2, we prove the following:

Theorem. *Let f be a function from a topological space T into a topological space T'. Then f is continuous at a point x of T if and only if for every indexed family $(c_i)_{i\in I}$ of points of T and every lower complete distributive lattice L, if $(c_i)_{i\in I}$ converges with respect to L to the point x then $(f(c_i))_{i\in I}$ converges with respect to L to $f(x)$.*

Proof. Let f be continuous at x and let $(c_i)_{i\in I}$ converge with respect to L to x. But then, for every neighborhood V' of $f(x)$ there exists a neighborhood V of x such that

$$f[V] \subseteq V'. \tag{9}$$

However, since $(c_i)_{i\in I}$ converges to x, we see that (7) holds for some n with R replaced by L. But then from this, by virtue of (9), we see that (8) holds with R replaced by L. Thus, indeed $(f(c_i))_{i\in I}$ converges with respect to L to $f(x)$.

To prove the converse, we show that f is continuous at x provided f satisfies the hypothesis that for every indexed family $(c_i)_{i\in I}$ of points of T and every lower complete distributive lattice L, if $(c_i)_{i\in I}$ converges with respect to L to x then $(f(x_i))_{i\in I}$ converges with respect to L to $f(x)$. Let $V(x)$ represent the set of all the neighborhoods of x. Clearly, $(V(x), \supseteq)$ is a lower complete distributive lattice, since the intersection of every two neighborhoods of x is their supremum with respect to $\supseteq$ and the union of every nonempty set of neighborhoods of x is the infimum of that set with respect to $\supseteq$ and obviously the required distributive laws are valid. Now, assume on the contrary that f is not continuous at x. Thus, there exists a neighborhood V' of $f(x)$ such that for every neighborhood V of x it is the case that $f[V] \nsubseteq V'$. Thus, in every neighborhood V of x there exists a point c_V such that $f(c_V) \notin V'$. From the axiom of Choice it follows that an indexed family $(c_V)_{V\in V(x)}$ of such points of T exists. Clearly, $(c_V)_{V\in V(x)}$ converges with respect to $(V(x), \supseteq)$ to x. Because, for every neighborhood V of x there exists an element of $V(x)$, namely, V such that $c_W \in V$ for every W with $V \supseteq W$. However, since $f(c_V) \notin V'$ for every $V \in V(x)$, we see that $(f(c_V))_{V\in V(x)}$ does not converge with respect to $(V(x), \supseteq)$ to $f(x)$. But this contradicts the hypothesis mentioned above. Thus, our assumption is false and f is continuous at x, as desired.

Let us observe that obviously the above Theorem and its proof remain valid if in them "*lower complete distributive lattice*" is replaced by "*relation R*", where R is arbitrary. In other words, we have:

Corollary. *Let f be a function from a topological space T into a topological space T'. Then f is continuous at a point x of T if and only if for every indexed* *family* $(c_i)_{i\in I}$ *of points of* T *and every relation* R, *if* $(c_i)_{i\in I}$ *converges with respect to* R *to* x *then* $(f(c_i))_{i\in I}$ *converges with respect to* R *to* $f(x)$.

Comparing the above Corollary with the Theorem, we see that in Topology, continuity of functions can be described based on convergence with respect to a great variety of relations.

Thus, the Corollary remains valid if in it R is replaced by either of: "Reflexive relation", or, "Symmetric relation", or, "Antisymmetric relation", or "Transitive relation", or, "Reflexive and transitive relation", or, "Transitive and antisymmetric relation", or, "Partial order relation", or, "Lattice", or, "Distributive lattice".

Remark 3. Let us call a relation Q *oriented* if and only if for every element x and y of the domain of Q there exists an element z such that $(x, z) \in Q$ and $(y, z) \in Q$. Accordingly, a *directed set* [2, p. 65] is an oriented relation which is also reflexive and transitive.

Next, let us consider the following relations which are listed according to the order of increasing restrictions:

(i) Oriented relation.
(ii) Oriented and transitive relation.
(iii) Directed set.
(iv) Lattice.
(v) Lower complete lattice.
(vi) Lower complete distributive lattice.
(vii) Simply ordered relation.
(viii) Well ordered relation.

Again, we observe that in view of the Theorem and the Corollary, the latter remains valid if in it R is replaced by either of the relations (i) to (vi). Moreover, as explained in Remark 2, for ascertaining the continuity of a function, Lower complete distributive lattices are much more preferable to any of the relations (i) to (v). Clearly, there are "fewer" Lower complete distributive lattices than any of the relations (i) to (v). Thus, the task of ascertaining the continuity of a function based on convergence with respect to Lower complete distributive lattices is much "easier" than with respect to Directed sets as is customarily done [2, p. 86].

Remark 4. Example 1 below shows that, in general, continuity of a function cannot be described based on convergence with respect to simply ordered or well ordered relations. In other words, the Corollary does not remain valid if in it R is replaced by "Simply ordered relation" or "Well ordered relation".

Since every simply ordered set has a cofinal well ordered subset, in order to exhibit the failure of the Corollary for relations such as (vii) and (viii) it is enough to consider the latter case.

Example 1. Consider the Tychonoff plank, i.e., the set $(\omega_1 + 1) \times (\omega + 1)$

topologized as described in [2, p. 132]. Let T be the plank with the lines $x = \omega_1$ and $y = \omega$ removed but with the corner point (ω_1, ω) reinstated. Let f be a function from T into the real numbers defined by:

$$f((\omega_1, \omega)) = 1 \quad \textit{and} \quad f(x) = 0 \quad \textit{otherwise}.$$

It is easy to verify that f is not continuous at (ω_1, ω). On the other hand, it is also easy to verify that for no well ordered relation W does there exist an indexed family of points of T (except for the obvious constant family) which converges with respect to W to (ω_1, ω). Thus, for every indexed family $(c_i)_{i \in I}$ of points of T and every well ordered relation W, if $(c_i)_{i \in I}$ converges with respect to W to (ω_1, ω) then $(f(c_i))_{i \in I}$ converges with respect to W to $f((\omega_1, \omega))$. However, as mentioned above, f is not continuous at (ω_1, ω).

Thus, indeed in the Corollary neither (vii) nor (viii) can be replaced for R.

Remark 5. As explained in Remark 4, continuity cannot be described, in general, based on convergence with respect to well ordered relations. Nevertheless, continuity can sometimes be so described even in some pathological topological spaces. For instance, in Example 2 below, a topological space T is given such that every fundamental system of neighborhoods of every point of T has a nondenumerable diverse (i.e., pairwise noncomparable) subset with respect to $\subseteq$. Nevertheless, continuity of functions defined on T can be described based on convergence with respect to well ordered relations.

Example 2. Let T be the set ω_1 (the first nondenumerable ordinal) topologized by the cofinite topology. It can be verified that every fundamental system of neighborhoods of every point of T is nondenumerable and has a nondenumerable diverse subset with respect to $\subseteq$. But then it is easy to verify that if f is a function from T into a topological space T' then f is continuous at a point x of T provided whenever an indexed family $(c_i)_{i \in I}$ of points of T converges with respect to a well ordered relation W to x then $(f(x_i))_{i \in I}$ converges with respect to W to $f(x)$.

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Iowa State University
Ames, Iowa