Matematický časopis 23 (1973), No. 2

CHARACTERISTIC TYPES OF CONVERGENCE FOR CERTAIN CLASSES OF DARBOUX-BAIRE 1 FUNCTIONS

JAROSLAV SMÍTAL, Bratislava

In the sequel all functions are real-valued functions defined on a real interval I.

Bruckner and Leonard [3] posed the following problem: If $\{f_n\}_{n=1}^{\infty}$ is a sequence of $\mathscr{D}\mathscr{B}_1$ (= Darboux Baire 1) functions converging pointwise to a limit f what additional restrictions on the convergence are necessary and sufficient to guarantee that f also be a $\mathscr{D}\mathscr{B}_1$ function, i. e. what is the "characteristic" type of convergence for Darboux Baire 1 functions?

It is known that the uniform convergence of $\mathscr{D}\mathscr{B}_1$ functions preserves the Darboux continuity (see [2]). L. Mišík [5] has shown a necessary and sufficient condition to guarantee that the pointwise limit of a sequence of continuous functions be a Darboux function. In [6] a condition is given which is necessary and sufficient to guarantee that the uniform limit of Darboux functions be a Darboux function.

In the present paper there is given a solution of the problem of Bruckner and Leonard mentioned above. Moreover, there are given necessary and sufficient conditions to guarantee that the pointwise limit of a sequence of functions in $\mathscr{A}$ also be in $\mathscr{A}$ where $\mathscr{A}$ is the class $\mathscr{D}\mathscr{B}_1$, the class $\mathscr{M}_2$ of Zahorski [7] or the class of approximately continuous functions, respectively.

The relevant kind of convergence for functions in Baire class α for fixed α has been obtained by Gagaeff [4]:

Theorem. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Baire α functions converging pointwise to a function f; then f is a Baire α function if and only if for each $\varepsilon > 0$ there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of sets of the additive class α and a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $I = \bigcup_{n=1}^{\infty} A_n$ and $|f(x) - f_{n_k}(x)| < \varepsilon$ for each $x \in A_k$.*

We begin with the following.

Definition. Let I be an interval and let $\{A_n\}_{n=1}^{\infty}$ be a countable system of subsets of I.

The system $\{A_n\}_{n=1}^{\infty}$ has the property $\mathbf{P}_1$ if each A_n is of the type F_σ and if for each $x_0 \in I$, and for each unilateral neighbourhood $O(x_0)$ of x_0 there is some k such that $x_0 \in A_k$ and $O(x_0) \cap A_k - \{x_0\} \neq \emptyset$.

The system $\{A_n\}_{n=1}^{\infty}$ has the property $\mathbf{P}_2$ if each A_n is of the type F_σ, and if for each $x_0 \in I$ and each unilateral neighbourhood $O(x_0)$ of x_0 there is some k such that $x_0 \in A_k$ and the set $O(x_0) \cap A_k$ has the positive Lebesgue measure ($|O(x_0) \cap A_k| > 0$).

The system $\{A_n\}_{n=1}^{\infty}$ has the property $\mathbf{P}_3$ if for each $\eta > 0$ *and each* $x_0 \in I$ there exists a neighbourhood $O(x_0)$ of x_0 with this property: For each neighbourhood interval J of x_0 which is contained in $O(x_0)$ there is some k such that $x_0 \in A_k$ *and*

$$|J \cap A_k| \, / \, |J| > 1 - \eta \, .$$

Now the following theorem gives a characterization of the sequences of $\mathscr{D}\mathscr{B}_1$ functions whose limits are $\mathscr{D}\mathscr{B}_1$ functions.

Theorem 1. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $\mathscr{D}\mathscr{B}_1$ converging pointwise to a function f. Then f is in $\mathscr{D}\mathscr{B}_1$ if and only if, for each $\varepsilon > 0$ there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of sets with the property $\mathbf{P}_1$ and a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $|f(x) - f_{n_k}(x)| < \varepsilon$ and $|f_{n_k}(x) - f_{n_k}(y)| < \varepsilon$, for each $x, y \in A_k$.*

Proof: Assume that the assumptions of the theorem are satisfied. We show that $f \in \mathscr{D}\mathscr{B}_1$. From the above quoted theorem of B. Gagaeff it follows that $f \in \mathscr{B}_1$. To show that $f \in \mathscr{D}$ we use the criterion of Zahorski [7]: A Baire 1 function g is in $\mathscr{D}$ if and only if each of the sets $[g > \lambda]$, $[g < \lambda]$ is bilaterally dense in itself, for each λ. We show that each set $[f > \lambda]$ is bilaterally dense in itself (the proof for $[f < \lambda]$ is similar). Let $f(x_0) > \lambda$. There is some $\varepsilon > 0$ such that $f(x_0) > \lambda + 3\varepsilon$. Let $O(x_0)$ be a unilateral neighbourhood of x_0. Since the sequence $\{A_n\}_{n=1}^{\infty}$ has the property $\mathbf{P}_1$, there is some k and a point $z \neq x_0$ such that $z \in O(x_0) \cap A_k$. For such a z we have $|f(x_0) - f(z)| \leq \leq |f(x_0) - f_{n_k}(x_0)| + |f_{n_k}(x_0) - f_{n_k}(z)| + |f_{n_k}(z) - f(z)| < 3\varepsilon$, hence clearly $f(z) > \lambda$. Thus $f \in \mathscr{D}\mathscr{B}_1$.

Conversely, let $f \in \mathscr{D}\mathscr{B}_1$ and let $\varepsilon > 0$. Put $B_k = \{x; |f(x) - f_k(x)| < \varepsilon/3\}$, and $B^l = \{x; l\,\varepsilon/6 < f(x) < (l + 2)\varepsilon/6\}$, for each positive integer k, and each integer l. Let $C_k^l = B_k \cap B^l$. We show that the system $\{C_k^l\}$ has the property $\mathbf{P}_1$. Since each of the sets B_k, B^l is F_σ the set C_k^l is also F_σ. Now let $x_0 \in I$ and let $O(x_0)$ be a unilateral neighbourhood of x_0. There is some p such that $x_0 \in B^p$. Since $f \in \mathscr{D}\mathscr{B}_1$, the set B^p is bilaterally dense in itself, hence there exists a point $z \in O(x_0) \cap B^p$, $z \neq x_0$. There is also an integer q such that both $z \in B_q$ and $x_0 \in B_q$, hence $x_0 \in C_q^p$ and $z \in C_q^p \cap O(x_0) - \{x_0\} \neq \emptyset$ and hence the system $\{C_k^l\}$ has the property $\mathbf{P}_1$. Finally let $\{A_n\}_{n=1}^{\infty}$ be an enumeration of the sets C_k^l and let $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers such that $n_k = m$ if $A_k = C_m^l$. The sequences $\{A_n\}_{n=1}^{\infty}$ and $\{n_k\}_{k=1}^{\infty}$ have all the desired properties, since for each $x, y \in C_k^l$ we have $|f(x) - f_k(x)| < \varepsilon/3 < \varepsilon$ and $|f_k(x) - f_k(y)| \leq |f_k(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_k(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. The theorem is proved.

Theorem 2. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in the class $\mathcal{M}_2$ converging pointwise to a function f. Then $f \in \mathcal{M}_2$ if and only if for each $\varepsilon > 0$ there is a sequence $\{A_n\}_{n=1}^{\infty}$ of sets with the property $\mathbf{P}_2$ and a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $|f_{n_k}(x) - f(x)| < \varepsilon$ and $|f_{n_k}(x) - f_{n_k}(y)| < \varepsilon$, for each $x, y \in A_k$.*

The proof of the theorem is omitted. It is similar to that of Theorem 1.

Theorem 3. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of approximately continuous functions converging pointwise to a function f. Then f is approximately continuous if and only if for each $\varepsilon > 0$ there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of sets with the property $\mathbf{P}_3$ and a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $|f_{n_k}(x) - f(x)| < \varepsilon$ and $|f_{n_k}(x) - f_{n_k}(y)| < \varepsilon$, for each $x, y \in A_k$.*

Proof: Assume that the sequence $\{f_n\}_{n=1}^{\infty}$ satisfies the conditions of the theorem. We show that f is approximately continuous. Let $x_0 \in I$ and $\varepsilon > 0$. It suffices to show that the set $\{x; |f(x) - f(x_0)| < 3\varepsilon\}$ has the Lebesgue density 1 at x_0. Let $\eta > 0$; let $O(x_0)$ be a neighbourhood of x_0 whose existence s guaranteed by the property $\mathbf{P}_3$. Let J_0 be a subinterval of $O(x_0)$, which contains x_0, and let m be a number such that

$$|A_m \cap J_0| \,/\, |J_0| > 1 - \eta\,;$$

For such m we have $\{x; |f(x) - f(x_0)| < 3\varepsilon\} \supset A_m$; clearly, for each $x \in A_m$, $|f(x) - f(x_0)| \leq |f(x) - f_{n_m}(x)| + |f_{n_m}(x) - f_{n_m}(x_0)| + |f_{n_m}(x_0) - f(x_0)| < 3\varepsilon$. Thus for each interval $J_0 \subset O(x_0)$ such that $x_0 \in J_0$, we have

$$|J_0 \cap \{x;\ |f(x) - f(x_0)| < 3\varepsilon\}| \,/\, |J_0| \geq |J_0 \cap A_m|/|J_0| > 1 - \eta$$

and hence f is approximately continuous.

Conversely, let f be approximately continuous. Let $\varepsilon > 0$. Similarly as in the proof of Theorem 1 form the sets $B_k = \{x; |f(x) - f_k(x)| < \varepsilon/3\}$, $B^l = \{x; l\,\varepsilon/6 < f(x) < (l + 2)\,\varepsilon/6\}$, for each positive integer k, and each integer l, and put $C_k^l = B_k \cap B^l$. Clearly, for $x, y \in C_k^l$, $|f_k(x) - f(x)| < \varepsilon$ and $|f_k(x) - f_k(y)| < \varepsilon$. Hence to prove the theorem it suffices to show that the system $\{C_k^l\}$ has the property $\mathbf{P}_3$. Let $\eta > 0$ and $x_0 \in I$. Assume that $x_0 \in B^s$. There exists a positive number $\varepsilon_1 < \varepsilon$ such that the set $\{x; |f(x_0) - f(x)| < \varepsilon_1\}$ is a subset of the set B^s. Since f is approximately continous, the set $\{x; |f(x) - f(x_0)| < \varepsilon_1\}$ has the Lebesgue density 1 at the point x_0. Thus there is a neighbourhood $O(x_0)$ of x_0 such that for each subinterval J_0 of $O(x_0)$ which contains x_0 we have

$$(1) \qquad |J_0 \cap \{x;\ |f(x) - f(x_0)| < \varepsilon_1\}| \,/\, |J_0| > 1 - \eta/2\,.$$

Since the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f on the interval J_0 of finite measure it converges also in measure to f, hence there is some m such that

(2) $$|J_0 \cap \{x; |f(x) - f_m(x)| < \varepsilon/3\}| \,/\, |J_0| > 1 - \eta/2 .$$

But the set

$$J_0 \cap \{x; |f(x) - f_m(x)| < \varepsilon/3\} \cap \{x; |f(x) - f(x_0)| < \varepsilon_1\}$$

is a subset of $J_0 \cap C_m^s$, hence

$$|J_0 \cap C_m^s|/|J_0| > 1 - \eta$$

(see (1), (2)), q. e. d.

Remark. Similar characterizations as in Theorems 1—3 are possible also for functions in the classes $\mathscr{M}_3$ and $\mathscr{M}_4$ of Zahorski [7]. However the corresponding properties **P** are very complicated.

REFERENCES

[1] BRUCKNER, A. M.,—CEDER, J. G.: Darboux Continuity. Jahresber. Dtsch. Math.-Ver. *67*, 1965, 93—117.

[2] BRUCKNER, A. M.—CEDER, J. G.—WEISS, M.: Uniform Limits of Darboux Functions. Colloq. math. *15*. 1966, 65—77.

[3] BRUCKNER, A. M.—LEONARD, J.: Derivatives. Amer. Math. Monthly *73*, 1966, 24—56.

[4] GAGAEFF, B.: Sur les suites convergentes de functions measurables B. Fundam. math. *17*, 1932, 182—188.

[5] MIŠÍK, L.: Über die Eigenschaft von Darboux und einiger Klassen von Funktionen. Rev. math. pures et appl. *11*, 1966, 411—430.

[6] SMÍTAL, J.: Some Characterizations of Darboux Continuity of Real Functions. Mat. časop. *22*, 1972, 59—70.

[7] ZAHORSKI, Z.: Sur la premiere dérivée. Trans. Amer. Math. Soc. *69*, 1950. 1—54.

Received January 3, 1971

Katedra algebry a teórie čísel
Prírodovedeckej fakulty Univerzity Komenského
Bratislava