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THE FUBINI THEOREM AND CONVOLUTION OF VECTOR-VALUED MEASURES

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Let X be a Banach algebra. Let G be a compact Hausdorff topological semigroup. Denote $\mathscr{B}(G)$ the σ-algebra of Borel sets in G. If $m : \mathscr{B}(G) \to X$ and $n : \mathscr{B}(G) \to X$ are regular Borel measures both with finite variation, then their convolution is a regular Borel measure on $\mathscr{B}(G)$, with finite variation, with values in X which can be defined in two equivalent ways.

In the first definition, for each Borel subset D of G, $m * n(D)$ is defined to be $m \otimes n(E)$, where E is the Borel subset $\{(s, t) : st \in D\}$ of $G \times G$ and $m \otimes n$ is the unique regular Borel measure on $\mathscr{B}(G \times G)$, with finite variation, with values in X such that

$$\int_{G\times G} g \, \mathrm{d}(m \otimes n) = \int_G \{ \int_G g(s, t) \, \mathrm{d}m(s)\} \, \mathrm{d}n(t)$$

for all continuous functions g on $G \times G$.

In the second definition, $m * n$ is taken to be the unique regular Borel measure on $\mathscr{B}(G)$, with finite variation, with values in X satisfying

$$\int_G \mathrm{f} \, d(m * n) = \int_G \{ \int_G f(st) \, \mathrm{d}m(s)\} \, \mathrm{d}n(t)$$

for all continuous functions f on G [cf. 5].

We wish to prove that both definitions are equivalent, similarly as in a complex case [cf. 3 and 9]. Also the first definition makes it possible, in case G is a group, to give $m * n$ explicitly by the formula

$$m * n(D) = \int_G m(Dt^{-1}) \, \mathrm{d}n(t) = \int_G n(s^{-1}D) \, \mathrm{d}m(s)$$

for each D in $\mathscr{B}(G)$. For this and other purposes the Fubini theorem for vector-valued measures is needed. Thus we establish a theorem of this kind convenient for our purposes.

1. Vector-valued measures in product spaces

Let X, Y and Z be Banach spaces. Let a bilinear continuous mapping of $X \times Y$ into Z be given, denoted by juxtaposition, $z = xy$, $x \in X$, $y \in Y$, $z \in Z$ ($|xy| \leqslant |x|\,|y|$). Let S and T be compact Hausdorff topological spaces. Denote by $\mathscr{B}(S)$, $\mathscr{B}(T)$ the σ-algebra of Borel sets in S, T, respectively. For our purposes it is convenient to introduce a vector-valued measure in the product space $S \times T$ by means of dominated operators introduced by Dinculeanu [cf. 4, p. 379] and we use the terminology from his book. By $C(S)$ is meant, as usual, the Banach space of all continuous functions $f: S \to C$ (C = real line or complex plane) equipped with the standard supremum norm. Following Dinculeanu [4, p. 379] we say that a linear operator $U: C(S) \to X$ is dominated if there is a regular positive Borel measure a such that

$$|U(f)| \leqq \int_S |f| \, \mathrm{d}a$$

for every f in $C(S)$. According to [4, p. 380] there is an isomorphism $U \leftrightarrow m$ between the set of the dominated linear operators $U: C(S) \to X$ and the set of the regular Borel measures $m: \mathscr{B}(S) \to X$ with finite variation $\mu = |m|$, given by the equality

$$U(f) = \int_S f \, \mathrm{d}m, \text{ for every } f \in C(S).$$

The measure $\mu = |m|$ is a least positive regular measure a dominating U.

Let $m: \mathscr{B}(S) \to X$ and $n: \mathscr{B}(T) \to Y$ be regular Borel measures with finite variation, $\mu = |m|$, $\nu = |n|$, respectively. Then the mappings

$$U(f) = \int_S f \, \mathrm{d}m, \quad f \in C(S),$$

$$V(g) = \int_T g \, \mathrm{d}n, \quad g \in C(T)$$

are the dominated operators from $C(S)$ into X, $C(T)$ into Y, respectively. Take now h in $C(S \times T)$. Then for every $s \in S$, the mapping $t \to h(s, t)$ is a continuous function on T. Further the mapping from S into Z, given by the relation

$$s \to \int_T h(s, t) \, \mathrm{d}n(t)$$

is continuous. We have

$$\Big| \int_S \Big\{ \int_T h(s, t) \, \mathrm{d}n(t) \Big\} \mathrm{d}m(s) \Big| \leqslant \int_S \Big\{ \int_T |h(s, t)| \, \mathrm{d}|n|\,(t) \Big\} \mathrm{d}|m|\,(s).$$

It is easy to see that the mapping given by

$$h \to \int_S \{\int_T h(s,t)\, \mathrm{d}|n|\,(t)\}\, \mathrm{d}|m|\,(s),\ h \in C(S \times T),$$

is a positive linear functional on $C(S \times T)$ and thus the mapping W, given by the formula

$$W(h) = \int_S \{\int_T h(s,t)\, \mathrm{d}n(t)\}\, \mathrm{d}m(s), \quad h \in C(S \times T),$$

is a dominated linear operator on $C(S \times T)$ into Z [4, p. 392]. Therefore there exists a regular Borel measure $l : \mathscr{B}(S \times T) \to Z$ with finite variation $\varrho = |l|$ such that

$$W(h) = \int_{S \times T} h\, \mathrm{d}l, \text{ for every } h \in C(S \times T).$$

We denote the measure l by $l = m \otimes n$. Similarly $|m| \otimes |n|$ is a unique positive regular Borel measure on $\mathscr{B}(S \times T)$ such that

$$\int_S \{\int_T h(s,t)\, \mathrm{d}|n|(t)\}\, \mathrm{d}|m|(s) = \int_{S \times T} h\, \mathrm{d}|m| \otimes |n|$$

for every $h \in C(S \times T)$.

Since we have

$$|W(h)| \leqq \int_{S \times T} |h(s,t)|\, \mathrm{d}|m| \otimes |n|(s,t)$$

and $|m \otimes n|$ is a least positive regular Borel measure b such that

$$|W(h)| \leqq \int_{S \times T} |h(s,t)|\, \mathrm{d}b(s,t),$$

we obtain $\varrho = |m \otimes n| \leqq |m| \otimes |n|$. Clearly

$$\int_{S \times T} h\, \mathrm{d}m \otimes n = \int_S \{\int_T h(s,t)\, \mathrm{d}n(t)\}\, \mathrm{d}m(s)$$

for every function $h \in C(S \times T)$.

We remark that $|m| \otimes |n|$, $|m \otimes n|$ and $m \otimes n$ are defined on the σ-algebra $\mathscr{B}(S \times T)$ which contains the product σ-algebra $\mathscr{B}(S) \times \mathscr{B}(T)$. The inclusion $\mathscr{B}(S) \times \mathscr{B}(T) \subset \mathscr{B}(S \times T)$ may be proper if neither S nor T is metrisable [cf. 2]. Therefore $|m| \times |n|$ as defined in [1] or $m \times n$ as defined in [6] need not be a Borel measure [cf. 7]. Thus $|m| \otimes |n|$ is the unique regular Borel extension of $|m| \times |n|$ and $m \otimes n$ is the unique regular Borel extension of $m \times n$.

Since every function in $C(S \times T)$ can be uniformly aproximated by function wich are finite sums of type

$$(s,t) \to \sum f_i(s) g_i(t)$$

with $f_i \in C(S)$ and $g_i \in C(T)$, all functions in $C(S \times T)$ are $m \times n$-integrable [4, p. 138] and we may write

$$\int_{S \times T} h \, \mathrm{d}m \otimes n = \int_{S \times T} h \, \mathrm{d}m \times n = \int_S \{ \int_T h(s,t) \, \mathrm{d}n(t) \} \, \mathrm{d}m(s)$$

for every $h \in C(S \times T)$.

2. The Fubini theorem

We take the measures m and n as in Section 1. The proof of the Fubini theorem is based on some lemmas.

Lemma 1. *Let $\mu = |m|$. For every function $f \in \mathscr{L}^1(S, \mu)$ there exists a sequence (f_n) of the functions in $C(S)$ converging to f in mean and μ-almost everywhere.*

Proof. The space $C(S)$ is dense in $\mathscr{L}^1(S, \mu)$ [4, p. 325]. So for every natural number n there exists a sequence (h_n) in $C(S)$ such that

$$\int_S |h_n - f| \, \mathrm{d}\mu < \frac{1}{n}.$$

Thus the sequence (h_n) converges to f in mean. According to [4, p. 130] the sequence (h_n) contains a subsequence (f_n) converging μ-almost everywhere and in mean to f.

Lemma 2. *Let Z be a set of $\mu \otimes \nu$-measure 0 in $S \times T$. Then for μ-almost $s \in S$ we have $\nu(Z_s) = 0$, i.e. there exists a set P of μ-measure 0 such that $\nu(Z_s) = 0$ for $s \notin P$.*

Proof. We have, using the Fubini theorem for positive Borel measures [8, p. 153]

$$0 = \mu \otimes \nu(Z) = \int c_Z \, \mathrm{d}\mu \otimes \nu = \int_S \{ \int_T c_Z(s,t) \, \mathrm{d}\nu(t) \} \, \mathrm{d}\mu(s) =$$

$$= \int_S \{ \int_T c_{Z_s}(t) \, \mathrm{d}\nu(t) \} \, \mathrm{d}\mu(s) = \int_S \nu(Z_s) \, \mathrm{d}\mu(s),$$

where c_Z denotes the characteristic function of the set Z.

Theorem 1 (Fubini). *Let f be a scalar function on $S \times T$. Let $f \in \mathscr{L}^1(S \times T, \mu \otimes \nu)$, $\mu = |m|$, $\nu = |n|$. Then*

f is $m \otimes n$-integrable;

for $\mu = |m|$-almost all s, the map f_s: $t \to f(s,t)$, is in $\mathscr{L}^1(T, \nu)$;

the map given by

$$s \to \int_T f_s \, \mathrm{d}n$$

for μ-almost all s (and defined arbitrarily for other s) is in $\mathscr{L}^1_Y(S, \mu)$ and we have

$$\int_{S \times T} f \, \mathrm{d}(m \otimes n) = \int_S \{ \int_T f(s, t) \, \mathrm{d}n(t) \} \, \mathrm{d}m(s) \, .$$

Proof. The fact that f is $m \otimes n$-integrable follows [4, p. 132] from the inequality $|m \otimes n| \leqq |m| \otimes |n| = \mu \otimes \nu$.

By Lemma 1 there exists a sequence (f_n) in $C(S \times T)$ converging to f $\mu \otimes \nu$--almost everywhere and in mean, i.e.

$$\lim_{n \to \infty} \int_{S \times T} |f(s, t) - f_n(s, t)| \, \mathrm{d}\mu \otimes \nu(s, t) = 0 .$$

From there we have

$$\lim_{n \to \infty} \int_{S \times T} |f(s, t) - f_n(s, t)| \, \mathrm{d}|m \otimes n|(s, t) = 0 ,$$

therefore

$$\lim_{n \to \infty} | \int_{S \times T} (f(s, t) - f_n(s, t)) \, \mathrm{d}m \otimes n(s, t)| = 0 ,$$

that is

$$\lim_{n \to \infty} \int_{S \times T} f_n(s, t) \, \mathrm{d}m \otimes n(s, t) = \int_{S \times T} f(s, t) \, \mathrm{d}m \otimes n(s, t) \, .$$

Let Z be a set of $\mu \otimes \nu$-measure 0 in $S \times T$ such that (f_n) converges to f outside Z and P denote a set of μ-measure 0 in S (Lemma 2) such that for $s \notin P$ we have

$$\nu(Z_s) = 0 .$$

If $s \notin P$, it follows that $(f_{n,s})$ converges pointwise to f_s on the complement of Z_s.

For each n the map g_n: $s \to f_{n,s}$ is a map of S into $C(T) \subset \mathscr{L}^1(T, \nu)$. The sequence (g_n) is Cauchy in $\mathscr{L}^1_{\mathscr{L}^1(\nu)}(S, \mu)$. In fact, we have

$$N_1(g_n - g_m) = \int_S |g_n - g_m|_{\mathscr{L}_1(\nu)} \, \mathrm{d}\mu = \int_S |g_n(s) - g_m(s)|_{\mathscr{L}_1(\nu)} \, \mathrm{d}\mu(s) =$$

$$= \int_S \int_T |f_n(s, t) - f_m(s, t)| \, \mathrm{d}\nu(t) \, \mathrm{d}\mu(s) = \int_{S \times T} |f_n - f_m| \, \mathrm{d}\mu \otimes \nu \to 0 ,$$

as $m, n \to \infty$. Since the space $\mathscr{L}^1_{\mathscr{L}^1(\nu)}(S, \mu)$ is complete there is a function

$g : S \to \mathscr{L}^1(T, \nu)$ such that (g_n) (taking subsequences if necessary) converges to g μ-almost everywhere and in mean, i.e.

$$\lim_{n\to\infty} \int_S |g_n - g|_{\mathscr{L}^1(\nu)} \, \mathrm{d}\mu = \lim_{n\to\infty} \int_S |g_n(s) - g(s)|_{\mathscr{L}^1(\nu)} \, \mathrm{d}\mu(s) = 0 .$$

This means that there is a set Q of μ-measure 0 in S such that for $s \notin Q$, the sequence $(g_n(s)) = (f_{n,s})$ is Cauchy in $\mathscr{L}^1(T, \nu)$, i.e.

$$\int_T |g_n(s) - g_m(s)| \, \mathrm{d}\nu = \int_T |f_{n,s} - f_{m,s}| \, \mathrm{d}\nu \to 0 ,$$

as $m, n \to \infty$ for $s \notin Q$.

If $s \notin P \cup Q$, we know that $(f_{n,s}(t))$ converges to $f_s(t)$ for ν-almost all $t \in T$. Hence by [4, p. 133] we conclude that $f_s \in \mathscr{L}^1(T, \nu) \subset \mathscr{L}^1(T, n)$ and that $(f_{n,s})$ is $\mathscr{L}^1(T,\nu)$-convergent to f_s, so that

$$\left| \int_T f_{n,s} \, \mathrm{d}n - \int_T f_s \, \mathrm{d}n \right| \leqq \int_T |f_{n,s} - f_s| \, \mathrm{d}\nu \to 0 ,$$

as $m, n \to \infty$, for all $s \notin P \cup Q$, i.e. $\int_T f_{n,s} \, \mathrm{d}n$ converges to $\int_T f_s \, \mathrm{d}n$ for $s \notin P \cup Q$.

Finally, we note that the map h_n,

$$h_n(s) = \int_T f_{n,s} \, \mathrm{d}n,$$

is a continuous function from S into Y, $h_n \in C_Y(S) \subset \mathscr{L}^1_Y(S, \mu)$. Furthermore, (h_n) is Cauchy in $\mathscr{L}^1_Y(S, \mu)$,

$$\int_S |h_n - h_m| \, \mathrm{d}\mu = \int_S |h_n(s) - h_m(s)| \, \mathrm{d}\mu(s) =$$

$$= \int_S \left| \int_T f_{n,s} \, \mathrm{d}n - \int_T f_{m,s} \, \mathrm{d}n \right| \mathrm{d}\mu(s) \leqq \int_S \int_T |f_{n,s} - f_{m,s}| \, \mathrm{d}\nu \, \mathrm{d}\mu(s) \to 0 ,$$

as $m, n \to \infty$, and since for $s \notin P \cup Q$ $h_n(s)$ converges to

$$h(s) = \int_T f_s \, \mathrm{d}n ,$$

(h_n) is $\mathscr{L}^1_Y(S, \mu)$-convergent to h, and h is in $\mathscr{L}^1_Y(S, \mu)$.

For $n \to \infty$ we have

$$\left| \int_S \int_T f_{n,s} \, \mathrm{d}n \, \mathrm{d}m(s) - \int_S \int_T f_s \, \mathrm{d}n \, \mathrm{d}m(s) \right| \leqq \int_S \left| \int_T f_{n,s} \, \mathrm{d}n - \int_T f_s \, \mathrm{d}n \right| \mathrm{d}\mu(s) \to 0 ,$$

i.e.

$$\lim_{n\to\infty} \int_S \int_T f_n(s, t) \, \mathrm{d}n(t) \, \mathrm{d}m(s) = \int_S \int_T f(s, t) \, \mathrm{d}n(t) \, \mathrm{d}m(s) ,$$

but

$$\lim_{n\to\infty} \int_S \int_T f_n(s, t)\, dn(t)\, dm(s) = \lim_{n\to\infty} \int_{S\times T} f_n(s, t)\, dm \otimes n(s, t) =$$

$$= \int_{S\times T} f(s, t)\, dm \otimes n(s, t) ,$$

i.e.

$$\int_{S\times T} f(s, t)\, d(m \otimes n) = \int_S \{\int_T f(s, t)\, dn(t)\}\, dm(s) .$$

Corollary. *Let Q be a Borel set in $S \times T$. Then we have*

$$\int_{S\times T} c_Q\, d(m \otimes n) = \int_S \int_T c_{Q_s}\, dn dm(s) = \int_S \int_T c_{Q_s}(t)\, dn(t)\, dm(s) .$$

3. Images of measures and the convolution formula

Let T and S be compact Hausdorff spaces and suppose that $p : T \to S$ is a continuous function. Let X be a Banach space and $m : \mathscr{B}(T) \to X$ a regular Borel measure with finite variation μ on T. For every $A \in \mathscr{B}(S)$ we put

$$n(A) = m(p^{-1}(A))$$

and

$$\nu(A) = \mu(p^{-1}(A)) .$$

Since $p^{-1}(A) \in \mathscr{B}(T)$ for every $A \in \mathscr{B}(S)$, n and ν are well defined, n has finite variation, $|n| \leqq \nu$, and n is regular [4, p. 402—403]. The regular Borel measure $n : \mathscr{B}(S) \to X$ is called the image of m by the function p and is denoted $p(m)$ [4]. Then ν is denoted $p(\mu)$ and the inequality $|n| \leqq \nu$ is now written $|p(m)| \leqq \leqq p(|m|)$. Since μ is bounded, $p(\mu)$ is bounded.

Let now $S = G$ be a compact Hausdorff topological semigroup, and $T = = G \times G$. Let $m : \mathscr{B}(G) \to X$ and $n : \mathscr{B}(G) \to Y$ be two regular Borel measures with finite variation μ and ν, respectively. Let $\mu \overset{1}{*} \nu$ and $m \overset{1}{*} n$ denote the measures, which are the images of $\mu \otimes \nu$, $m \otimes n$, respectively by the semigroup operation $p(s, t) = st$,

$$\mu \overset{1}{*} \nu = p(\mu \otimes \nu), \quad m \overset{1}{*} n = p(m \otimes n) .$$

Let $f \in C(S)$. Then $f \in \mathscr{L}^1(G, \mu \overset{1}{*} \nu)$ and $f \circ p \in \mathscr{L}^1(G \times G, \mu \otimes \nu)$ [4, p. 404] and we have

$$\int_{G\times G} f \circ p\, dm \otimes n = \int_G f\, dp(m \otimes n) ,$$

in other words

$$\int_{G\times G} f(st)\,\mathrm{d}m \otimes n(s,t) = \int_G f\,\mathrm{d}m^1_* n\,.$$

Since the last equality holds for every function $f \in C(G)$, we have

$$\int_G f\,\mathrm{d}m * n = \int_{G\times G} f(st)\,\mathrm{d}m \otimes n(s,t) = \int_G f\,\mathrm{d}m^1_* n$$

for every $f \in C(G)$. However this means that

$$m * n = m^1_* n$$

on $\mathscr{B}(G)$ [4, p. 326].

If G is a group, then the convolution formula is an easy consequence of Corollary of Theorem 1.

Theorem 2. *Let G be a compact Hausdorff group, m and n regular Borel measures on $\mathscr{B}(G)$ with finite variation and with values in X and Y, respectively. Then, for each Borel subset D of G*

(1) $$t \to m(Dt^{-1})$$

is an n-integrable function on G and we have

(2) $$m * n(D) = \int_G m(Dt^{-1})\,\mathrm{d}n(t)\,.$$

Proof. We have, putting $E = p^{-1}(D)$,

$$\int_G c_E(s,t)\,\mathrm{d}m(s) = m(Dt^{-1})\,,$$

and

$$m * n(D) = m \otimes n(E) = \int_{G\times G} c_E\,\mathrm{d}m \otimes n = \int_G \left\{ \int_G c_E(s,t)\,\mathrm{d}m(s) \right\} \mathrm{d}n(t)\,,$$

using the fact that if $g \in \mathscr{L}^1(G, \mu * \nu)$, then $g \circ p \in \mathscr{L}^1(G \times G, m \otimes n)$ and we have

$$\int_{G\times G} g \circ p\,\mathrm{d}m \otimes n = \int_G g\,\mathrm{d}m * n$$

[cf. 4. p. 404], in particular for $g = c_D$.

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