INDEPENDENCE OF EQUATIONAL CLASSES

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Preliminaries. Equational classes $K_0, K_1, \ldots, K_{n-1}$ of the same type are said to be *independent* (for $i = 0, 1$ see [6]) if there exists an n-ary polynomial symbol p [5] such that the identity $p(x_0, x_1, \ldots, x_{n-1}) = x_i$ holds in K_i, $i = 0, 1, \ldots, n-1$. (We shall also say that the set $\{K_0, K_1, \ldots, K_{n-1}\}$ is independent.) $K_0 \vee K_1 \vee \ldots \vee K_{n-1}$ will denote the smallest equational class containing all K_i, and $K_0 >< K_1 >< \ldots >< K_{n-1}$ will denote the class of all algebras which are isomorphic to an algebra of the form $\mathfrak{A}_0 >< \mathfrak{A}_1 >< \ldots >< \mathfrak{A}_{n-1}$, $\mathfrak{A}_i \in K_i$, $i = 0, 1, \ldots, n-1$. A set $\{\Theta_\gamma : \gamma \in \Gamma\}$ of congruence relations on an algebra $\mathfrak{A} = \langle A; F \rangle$ is called *absolutely permutable*[1]) ([7], [9]) if for any family $(x_\gamma : \gamma \in \Gamma)$ of elements of A such that $x_\alpha \equiv x_\beta(\bigvee\{\Theta_\gamma : \gamma \in \Gamma\})$ for any $\alpha, \beta \in \Gamma$, there exists $x \in A$ with $x \equiv x_\gamma(\Theta_\gamma)$ for any $\gamma \in \Gamma$. Note that any subset of an absolutely permutable set S of congruence relations is absolutely permutable, in particular any two congruence relations of S are permutable. But the pairwise permutability of S is not sufficient to the absolute permutability of S. We shall use the symbols ω and ι for the least and the greatest congruence relations. The symbol $\cong$ will denote an isomorphism.

1. Statement of the results

Theorem 1. *Equational classes K_i, $i = 0, 1, \ldots, n-1$, are independent if and only if the following conditions* (1) *and* (2), *or* (1) *and* (2′), *or* (1) *and* (2″) *are satisfied*:

(1) *$K_0 \wedge K_1 \wedge \ldots \wedge K_{n-1}$ consists of one-element algebras only.*

(2) *For every $\mathfrak{A} \in K_0 \vee K_1 \vee \ldots \vee K_{n-1}$ the smallest congruence relations Θ_i on $\mathfrak{A}$ such that $\mathfrak{A}/\Theta_i \in K_i$, $i = 0, 1, \ldots, n-1$, are absolutely permutable.*

(2′) *Given $\mathfrak{A} \in K_0 \vee K_1 \vee \ldots \vee K_{n-1}$ and arbitrary congruence relations Φ_i on $\mathfrak{A}$ such that $\mathfrak{A}/\Phi_i \in K_i$, $i = 0, 1, \ldots, n-1$, then Φ_i ($i = 0, 1, \ldots, n-1$) are absolutely permutable.*

[1]) In [7] the term "assoziiert" is used.

(2″) *For every* $\mathfrak{A} \in K_0 \vee K_1 \vee \ldots \vee K_{n-1}$, *arbitrary congruence relations* Φ_i *on* $\mathfrak{A}$ *such that* $\mathfrak{A}/\Phi_i \in K_i$, $i = 0, 1, \ldots, n-1$, *satisfy* $\Phi_i . \wedge \{\Phi_j : j \neq i, j = 0, 1, \ldots, n-1\} = \vee \{\Phi_j : j = 0, 1, \ldots, n-1\}$ *for each* $i \in \{0, 1, \ldots, n-1\}$.

Theorem 2. *Let each* $\mathfrak{A} \in K_0 \vee K_1 \vee \ldots \vee K_{n-1}$ *have a distributive congruence lattice and let any two congruence relations on* $\mathfrak{A}$ *be permutable. Then* K_i, $i = 0, 1, \ldots, n-1$, *are independent if and only if* (1) *and one of the two following conditions hold*:

(2a) *For each* $\mathfrak{A} \in K_0 \vee K_1 \vee \ldots \vee K_{n-1}$ *the smallest congruence relations* Θ_i *on* $\mathfrak{A}$ *such that* $\mathfrak{A}/\Theta_i \in K_i$, $i = 0, 1, \ldots, n-1$, *satisfy* $\Theta_k \vee \Theta_j = \vee \{\Theta_i : i = 0, 1, \ldots, n-1\}$ *for each* $k \neq j$, $k, j \in \{0, 1, \ldots, n-1\}$.

(2′a) *For each* $\mathfrak{A} \in K_0 \vee K_1 \vee \ldots \vee K_{n-1}$ *and arbitrary congruence relations* Φ_i *on* $\mathfrak{A}$ *such that* $\mathfrak{A}/\Phi_i \in K_i$, $i = 0, 1, \ldots, n-1$, *the eguality* $\Phi_k \vee \Phi_j = \vee \{\Phi_i : i = 0, 1, \ldots, n-1\}$ *holds for each* $k \neq j$, $k, j \in \{0, 1, \ldots, n-1\}$.

Theorem 3. *Equational classes* $K_0, K_1, \ldots, K_{n-1}$ *are independent if and only if for each* $i \in \{1, 2, \ldots, n-1\}$, K_i *and* $K_0 \vee K_1 \vee \ldots / K_{i-1}$ *are independent.*

Corollary 1. *Let* $K_0, K_1, \ldots, K_{n-1}$ *be independent equational classes. Then any subset of* $\{K_0, K_1, \ldots, K_{n-1}\}$ *is independent too. In particular* K_i, K_j *are independent for any* $i \neq j$, $i, j \in \{0, \ldots, n-1\}$.

Remark 1. If each proper subset of $\{K_0, K_1, \ldots, K_{n-1}\}$ is independent then $K_0, K_1, \ldots, K_{n-1}$ need not be independent as it can be seen in Example 6, but this holds in special cases (see Theorem 4 and Example 8).

Theorem 4. *Let* $K_0, K_1, \ldots, K_{n-1}$ $(n > 2)$ *be equational classes* (*of the same type*) *and let* $k \in \{2, 3, \ldots, n-1\}$ *exist such that the following conditions are satisfied*:

(3) *Each* k *classes of the set* $\{K_0, K_1, \ldots, K_{n-1}\}$ *are independent.*

(4) *There exist* $n-k$ *classes of the set* $\{K_0, K_1, \ldots, K_{n-1}\}$ *which have only idempotent operations.*

Then $K_0, K_1, \ldots, K_{n-1}$ *are independent.*

Remark 2. The number $n-k$ of (4) in Theorem 4 cannot be lowered in general, as it can be seen in Example 7.

Theorem 5. *Let* K_i, $i = 0, 1, \ldots, n-1$, *be independent. Then* $K_0 \vee K_1 \vee \ldots \vee K_{n-1} = K_0 >< K_1 >< \ldots >< K_{n-1}$ *and each algebra* $\mathfrak{A} \in K_0 \vee K_1 \vee \ldots \vee K_{n-1}$ *has, up to isomorphism, a unique representation* $\mathfrak{A} \cong \mathfrak{A}_0 >< \mathfrak{A}_1 >< \ldots >< \mathfrak{A}_{n-1}$, $\mathfrak{A}_i \in K_i$, $i = 0, 1, \ldots, n-1$.

Remark 3. In particular case $n = 2$ the Theorem 5 yields a somewhat stronger result[2]) than [6, Theorem 1]. In [6, Theorem 1] to get the unicity, the modularity of the lattice of all congruence relations of each algebra $\mathfrak{A} \in K_0 \vee K_1$ is postulated.

Remark 4. In [6, Theorem 2] the following assertion is proved: "Let $K_0 \wedge K_1$ consist of one-element algebras only and let every $\mathfrak{A} \in K_0 \vee K_1$ have a modular congruence lattice. Then $K_0 \vee K_1 = K_0 >< K_1$ if and only if K_0 and K_1 are independent." The "only if" part of this assertion cannot be enlarged to the case of more than two equational classes (as the Example 4 shows), even if we replace modularity by distributivity (see Remark 6 in §3). One way of enlarging of this part of the assertion is given in Theorems 6 and 7.

Remark 5. If $K_i, i = 0, 1, \ldots, n-1$, are independent then using Theorem 5 and results of [8], analogously as in [6], we get that in Theorem 2 the condition "each $\mathfrak{A} \in K_0 \vee K_1 \vee \ldots \vee K_{n-1}$ has a distributive congruence lattice and any two congruence relations on $\mathfrak{A}$ are permutable" can be replaced by "each $\mathfrak{A}_i \in K_i$, $i = 0, \ldots, n-1$, has a distributive congruence lattice and any two congruence relations on $\mathfrak{A}_i$ are permutable".

Theorem 6. *Let the following conditions be satisfied:*

(5) $K_0 \vee K_1 \vee \ldots \vee K_{n-1} = K_0 >< K_1 >< \ldots >< K_{n-1}$.

(6) *For each* $i \in \{1, 2, \ldots, n-1\}$, $(K_0 \vee K_1 \vee \ldots \vee K_{i-1}) \wedge K_i$ *consists of one-element algebras only.*

(7) *Every algebra* $\mathfrak{A} \in K_0 \vee K_1 \vee \ldots \vee K_{n-1}$ *has a modular congruence lattice.*
Then $K_0, K_1, \ldots, K_{n-1}$ *are independent.*

Theorem 7. *Let the following conditions be satisfied:*

(5) $K_0 \vee K_1 \vee \ldots \vee K_{n-1} = K_0 >< K_1 >< \ldots >< K_{n-1}$.

(6') *For each* $i \neq j$, $i, j = 0, 1, \ldots, n-1$, $K_i \wedge K_j$ *consists of one-element algebras only.*

(7') *Every algebra* $\mathfrak{A} \in K_0 \vee K_1 \vee \ldots \vee K_{n-1}$ *has a distributive congruence lattice.*
Then $K_0, K_1, \ldots, K_{n-1}$ *are independent.*

2. Proofs of the theorems

We shall use the following assertions:

Lemma A ([7], [9]). *Let* $\mathfrak{A}$ *be an algebra. There exists a one-one correspondence between the non-trivial direct decompositions* $\prod(\mathfrak{A}_\gamma : \gamma \in \Gamma)$ *of the algebra* $\mathfrak{A}$ *and*

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[2]) Added Mai 25, 1972. The manuscript of this paper had been accepted for publication before the author knew that a proof of Theorem 5 is obtained (in another way) by Tah-Kai Hu and P. Kelenson, *Independence and direct factorization of universal algebras*, Math. Nachr. *51*, 1971, 83—99.

the sets $S = \{\Theta\gamma : \gamma \in \Gamma\}$ *of non-trivial congruence relations (different from* ω *and* ι*) on* $\mathfrak{A}$ *having the following properties*:

(i) $\bigwedge \{\Theta_\gamma : \gamma \in \Gamma\} = \omega$.

(ii) $\bigvee \{\Theta_\gamma : \gamma \in \Gamma\} = \iota$.

(iii) *S is absolutely permutable.*

Given the set S, the corresponding direct decomposition is

$$\mathfrak{A} \cong \Pi(\mathfrak{A}/\Theta_\gamma : \gamma \in \Gamma).$$

Lemma B [3]. *A set* $\{\Theta_0, \Theta_1, \ldots, \Theta_{n-1}\}$ *of congruence relations on an algebra* $\mathfrak{A}$ *is absolutely permutable if and only if for every* $i \in \{0, 1, \ldots, n-1\}$ *the next condition holds*:

$$\Theta_i \,.\, \bigwedge \{\Theta_j : j \neq i, j = 0, 1, \ldots, n-1\} = \bigvee \{\Theta_j : j = 0, 1, \ldots, n-1\}.$$

Proof of Theorem 1. The conditions (2″) and (2′) are equivalent by Lemma B.

Necessity. Let x_i, $i = 0, 1, \ldots, n-1$, be elements of $\mathfrak{A} \in K_0 \wedge K_1 \wedge \ldots \wedge K_{n-1}$; then $p(x_0, x_1, \ldots, x_{n-1}) = x_i$, $i = 0, 1, \ldots, n-1$, hence (1) holds. Now we shall show (2′), hence (2) too. Let $x_0, x_1, \ldots, x_{n-1}$ be elements of $\mathfrak{A} \in K_0 \vee K_1 \vee \ldots \vee K_{n-1}$. Then $[x_i]\Phi_i = p([x_0]\Phi_i, \ldots, [x_{n-1}]\Phi_i) = [p(x_0, x_1, \ldots, x_{n-1})]\Phi_i$ hence $x_i \equiv p(x_0, x_1, \ldots, x_{n-1})(\Phi_i)$, $i = 0, 1, \ldots, n-1$. It follows that $\{\Phi_i : i = 0, 1, \ldots, n-1\}$ is absolutely permutable.

Sufficiency. Let (1), (2) hold. Let $\mathfrak{F}$ be the free algebra over $K_0 \vee K_1 \vee \ldots \vee K_{n-1}$ with n generators x_i, $i = 0, 1, \ldots, n-1$. Let Θ_i, $i = 0, 1, \ldots, n-1$, be the smallest congruence relations on $\mathfrak{F}$ such that $\mathfrak{F}/\Theta_i \in K_i$, $i = 0, 1, \ldots, n-1$. Since $\mathfrak{F}/\Theta_0 \vee \ldots \vee \Theta_{n-1}$ is a homomorphic image of $\mathfrak{F}/\Theta_i$, $i = 0, 1, \ldots, n-1$, then $\mathfrak{F}/\Theta_0 \vee \ldots \vee \Theta_{n-1} \in K_i$ for all $i = 0, 1, \ldots, n-1$, hence $\Theta_0 \vee \Theta_1 \ldots \vee \Theta_{n-1} = \iota$. According to the definition of Θ_i, $\mathfrak{F}/\Theta_i$ is the free algebra over K_i with n generators $[x_0]\Theta_i, [x_1]\Theta_i, \ldots, [x_{n-1}]\Theta_i$, $i = 0, 1, \ldots, n-1$. In view of (2) and $\Theta_0 \vee \Theta_1 \vee \ldots \vee \Theta_{n-1} = \iota$, we get that for the elements $x_0, x_1, \ldots, x_{n-1} \in \mathfrak{F}$ there exists $p(x_0, x_1, \ldots, x_{n-1}) \in \mathfrak{F}$ such that $x_i \equiv p(x_0, x_1, \ldots, x_{n-1})(\Theta_i)$, $i = 0, 1, \ldots, n-1$. It follows $[x_i]\Theta_i = [p(x_0, x_1, \ldots, x_{n-1})]\Theta_i$, hence $[x_i]\Theta_i = p([x_0]\Theta_i, [x_1]\Theta_i, \ldots, [x_{n-1}]\Theta_i)$ holds in $\mathfrak{F}/\Theta_i$, $i = 0, 1, \ldots, n-1$. Because the algebra $\mathfrak{F}/\Theta_i$ is free over K_i with the generators $[x_0]\Theta_i, \ldots, [x_{n-1}]\Theta_i$, then the identity $p(a_0, a_1, \ldots, a_{n-1}) = a_i$ holds in any K_i, $i = 0, 1, \ldots, n-1$. Hence K_i, $i = 0, 1, \ldots, n-1$, are independent.

Proof of Theorem 2. By [5, Chap. V., Exercise 68] the Chinese remainder theorem holds in any $\mathfrak{A} \in K_0 \vee \ldots \vee K_{n-1}$. Hence a set $\{\Phi_0, \Phi_1, \ldots, \Phi_{n-1}\}$ of congruence relations on $\mathfrak{A} \in K_0 \vee \ldots \vee K_{n-1}$ is absolutely permutable if and only if $\Phi_k \vee \Phi_j = \bigvee \{\Phi_i : i = 0, 1, \ldots, n-1\}$ holds for any $k \neq j$, $k, j \in \{0, \ldots, n-1\}$ (see Lemma B). Now it suffices to use Theorem 1.

Proof of Theorem 3. Let $K_0, K_1, \ldots, K_{n-1}$ be independent, then there exists an n-ary polynomial symbol p such that $p(x_0, x_1, \ldots, x_{n-1}) = x_j$ in K_j, $j = 0, 1, \ldots, n-1$. Now it is sufficient to take the binary polynomial symbol $q(x_0, x_i) = p(x_0, x_0, \ldots, x_0, x_i, x_0, \ldots, x_0)$. The identity $q(x_0, x_i) = x_0$ holds in any K_j, $j = 0, \ldots, i-1$, hence it holds in $K_0 \vee K_1 \vee \ldots \vee K_{i-1}$ too, and $q(x_0, x_i) = x_i$ in K_i. Hence K_i and $K_0 \vee K_1 \vee \ldots \vee K_{i-1}$ are independent. The converse assertion will be proved by induction. For $n = 2$ it is trivial. Let it hold for an index n and let the classes $K_0, K_1, \ldots, K_n$ satisfy the conditions of Theorem 3. Because of independence of $K_0, K_1, \ldots, K_{n-1}$ there exists an n-ary polynomial symbol s such that $s(x_0, x_1, \ldots, x_{n-1}) = x_j$ in K_j, $j = 0, 1, \ldots, n-1$. Because of independence of $K_0 \vee K_1 \vee \ldots \vee K_{n-1}$ and K_n, there exists a binary polynomial symbol t such that $t(x_0, x_1) = x_0$ in $K_0 \vee K_1 \vee \ldots \vee K_{n-1}$ and $t(x_0, x_1) = x_1$ in K_n. Now it suffices to take the $(n+1)$-ary polynomial symbol $r(x_0, x_1, \ldots, x_n) = t(s(x_0, x_1, \ldots, x_{n-1}), x_n)$. In K_j, $j = 0, 1, \ldots, n-1$, $r(x_0, x_1, \ldots, x_n) = s(x_0, x_1, \ldots, x_{n-1}) = x_j$ holds. In K_n, $r(x_0, x_1, \ldots, x_n) = x_n$. Hence $K_0, K_1, \ldots, K_n$ are independent.

Proof of Theorem 4. We shall proceed by induction on n. First let $n = 3$. Then $k = 2$. Let K_0 be the class having only idempotent operations. Since K_i, $i = 0, 1, 2$, are pairwise independent hence for each $i, j \in \{0, 1, 2\}$, $i < j$, there exists a polynomial symbol p_{ij} such that $p_{ij}(x_i, x_j) = x_i$ in K_i and $p_{ij}(x_i, x_j) = x_j$ in K_j. Now it suffices to take the polynomial symbol $q(x_0, x_1, x_2) = p_{12}(p_{01}(x_0, x_1), p_{02}(x_0, x_2))$. Obviously $q(x_0, x_1, x_2) = x_i$ in K_i, $i = 0, 1, 2$ (for $q(x_0, x_1, x_2) = p_{12}(x_0, x_0) = x_0$ in K_0), hence K_i, $i = 0, 1, 2$, are independent. Assume now that the assertion of the Theorem holds for $n = m$ and let the assumptions of the Theorem be fulfilled for $n = m + 1$. Let $K_0, K_1, \ldots, K_{m-k}$ be the classes having only idempotent operations. Assume first $k < m$. In the set $\{K_0, K_1, \ldots, K_{m-1}\}$ the classes $K_0, K_1, \ldots, K_{m-k-1}$ have only idempotent operations and each k classes are independent, hence by induction assumption

(b) K_i, $i = 0, 1, \ldots, m-1$, are independent.

By the similar argument (by replacing K_{m-1} by K_m) we get that

(c) K_i, $i = 0, 1, \ldots, m-2, m$, $(i \neq m-1)$ are independent.

If $k = m$ the assertions (b), (c) are trivial, for by the assumption each k classes are independent. Using Corollary 1 and the conditions (b), (c) we get:

(d) For each $h \in \{1, 2, \ldots, k-1, k\}$ the classes $K_0, \ldots, K_{m-k}, K_{m+h-k}$ are independent.

Hence for each $h \in \{1, 2, \ldots, k\}$ there exists an $(m+2-k)$-ary polynomial symbol p_h such that $p_h(x_0, x_1, \ldots, x_{m-k}, x_{m+h-k}) = x_j$ in K_j, $j = 0, 1, \ldots, m-k, m+h-k$. Using condition (3) for $n = m+1$ we get that the classes K_{m+h-k}, $h = 1, 2, \ldots, k$, are independent, hence there exists an k-ary polynomial symbol q such that $q(x_{m+1-k}, x_{m+2-k}, \ldots, x_{m+h-k}, \ldots, x_m) = x_{m+h-k}$

in K_{m+h-k}, $h = 1, \ldots, k$. Now it suffices to take the $(m + 1)$-ary polynomial symbol $p(x_0, x_1, \ldots, x_m) = q(p_1(x_0, \ldots, x_{m-k}, x_{m+1-k}), p_2(x_0, \ldots, x_{m-k}, x_{m+2-k}), \ldots, p_h(x_0, \ldots, x_{m-k}, x_{m+h-k}), \ldots, p_k(x_0, \ldots, x_{m-k}, x_m))$. In K_j, $j = 0, 1, \ldots, m - k$, $p(x_0, \ldots, x_m) = x_j$ because of idempotent operations. In K_{m+h-k}, $h = 1, 2, \ldots, k$, $p(x_0, x_1, \ldots, x_m) = p_h(x_0, x_1, \ldots, x_{m-k}, x_{m+h-k}) = x_{m+h-k}$. Hence $K_0, K_1, \ldots, K_m$ are independent.

Proof of Theorem 5.[3]) We proceed by induction. First we shall prove the Theorem for $n = 2$. Let K_0, K_1 be independent. We shall show that:

(8) $K_0 >< K_1$ is equational class and

(9) $\mathfrak{A}_0 >< \mathfrak{A}_1 \cong \mathfrak{A} \cong \mathfrak{B}_0 >< \mathfrak{B}_1$, $\mathfrak{A}_i \in K_i$, $\mathfrak{B}_i \in K_i$, $i = 0, 1$ imply $\mathfrak{A}_i \cong \mathfrak{B}_i$, $i = 0, 1$.

Proof of (8): a) Let $\mathfrak{B} = <B;F>$ be a subalgebra of $\mathfrak{A}_0 >< \mathfrak{A}_1$, $\mathfrak{A}_i \in K_i$, $i = 0, 1$. Denote $B_0 = \{b_0$: there exists $a_1 \in \mathfrak{A}_1$, $(b_0, a_1) \in B\}$, $B_1 = \{b_1$: there exists $a_0 \in \mathfrak{A}_0$, $(a_0, b_1) \in B\}$. It is clear that $\mathfrak{B}_i = <B_i;F>$ is a subalgebra of $\mathfrak{A}_i$, $i = 0, 1$. We shall show that $B = B_0 >< B_1$. If $(b_0, b_1) \in B_0 >< B_1$, then there exist $a_i \in \mathfrak{A}_i$, $i = 0, 1$, such that $(a_0, b_1), (b_0, a_1) \in B$. This implies $(b_0, b_1) = (p(b_0, a_0), p(a_1, b_1)) = p((b_0, a_1), (a_0, b_1)) \in B$. Hence $B \supset B_0 >< B_1$. The converse inclusion is trivial.

b) To prove that $K_0 >< K_1$ is closed under epimorphic images we use the following easy assertions.

Let $h : \mathfrak{A} \to \mathfrak{A}'$ be an epimorphism of algebras and Θ_h the corresponding congruence relation on $\mathfrak{A}$ ($x \equiv y(\Theta_h)$ iff $h(x) = h(y)$). Let Φ be a congruence relation on $\mathfrak{A}$ which is permutable with Θ_h. Define the relation Φ' on $\mathfrak{A}'$ as follows. $x' \equiv y'(\Phi')$ if $x, y \in \mathfrak{A}$ exist such that $x \equiv y(\Phi)$ and $x' = h(x)$, $y' = h(y)$. Then Φ' is a congruence relation on $\mathfrak{A}'$ and the mapping $h' : \mathfrak{A}/\Phi \to \mathfrak{A}'/\Phi'$ defined by $h : [x]\Phi \mapsto [h(x)]\Phi'$ is an epimorphism. If Φ_1, Φ_2 are congruence relations on $\mathfrak{A}$, both permutable with Θ_h, such that $\Phi_1 . \Phi_2 = \iota$ then the corresponding congruence relations Φ'_1, Φ'_2 on $\mathfrak{A}'$ satisfy $\Phi'_1 . \Phi'_2 = \iota$.

Now let $h : \mathfrak{A}_0 >< \mathfrak{A}_1 \to \mathfrak{C}$, $\mathfrak{A}_i \in K_i$, $i = 0, 1$, be an epimorphism. Let Φ_0, Φ_1 be the congruence relations on $\mathfrak{A}_0 \times \mathfrak{A}_1$ corresponding to the direct decomposition $\mathfrak{A}_0 >< \mathfrak{A}_1$ (Lemma A). Φ_0 and Θ_h are permutable: Let (a_0, a_1). (b_0, b_1), $(c_0, c_1) \in \mathfrak{A}_0 >< \mathfrak{A}_1$ and $(a_0, a_1)\Phi_0(b_0, b_1)\Theta_h(c_0, c_1)$. Then $a_0 = b_0$ and $h(b_0, b_1) = h(c_0, c_1)$. Further $h(c_0, a_1) = h(p(c_0, a_0), p(c_1, a_1)) = h(p((c_0, c_1), (a_0, a_1))) = p(h(c_0, c_1), h(a_0, a_1)) = p(h(b_0, b_1), h(a_0, a_1)) = h(p((b_0, b_1),$

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[3]) One can prove Theorem 5 by the similar method as that of [6, Th. 1] for $n = 2$. To get the unicity of given representation in the proof of [6, Th. 1] it suffices to use [1, Chap. IV., Th. 13], hence the modularity of congruence lattices in [6, Th. 1] need not be postulated. In the proof of Theorem 5 by the similar way it suffices to use [2, Corollary 3.5 (vi)] to get the unicity of the given representation. We give here another proof of Theorem 5 by induction. The first step, the proof of Theorem 5 for $n = 2$, differs from that in [6, Th. 1].

$(a_0, a_1))) = h(p(b_0, a_0), p(b_1, a_1)) = h(a_0, a_1)$, hence $(a_0, a_1)\Theta_h(c_0, a_1)\Phi_0(c_0, c_1)$. Similarly, Φ_1 and Θ_h are permutable. By the above assertions h induces congruence relations Φ'_0, Φ'_1 on $\mathfrak{C}$ such that $\Phi'_0 . \Phi'_1 = \iota$ and $\mathfrak{C}/\Phi'_i$ is an epimorphic image of $\mathfrak{A}_0 >< \mathfrak{A}_1/\Phi_i \cong \mathfrak{A}_i$, hence $\mathfrak{C}/\Phi'_i \in K_i$, $i = 0, 1$. It remains to show that $\Phi'_0 \wedge \Phi'_1 = \omega$. If $c \equiv d(\Phi'_0 \wedge \Phi'_1)$ then $c = h(a_0, a_1)$, $d = h(a_0, b_1)$, $c = h(e_0, e_1)$, $d = h(f_0, e_1)$. Because $\mathfrak{C} \in K_0 \vee K_1$ and $p(x, x) = x$ holds in K_0 and in K_1 too, we get $c = p(c, c) = p(h(a_0, a_1), h(e_0, e_1)) = h(p((a_0, a_1), (e_0, e_1))) = h(p(a_0, e_0), p(a_1, e_1)) = h(a_0, e_1)$. By the same argument we get $d = h(a_0, e_1)$ hence $c = d$. By [1, Chap. VI., Th. 22], (8) holds. Now we prove (9): Let $\mathfrak{B}_0 >< \mathfrak{B}_1 \cong \mathfrak{A} \cong \mathfrak{A}_0 >< \mathfrak{A}_1$, $\mathfrak{A}_i, \mathfrak{B}_i \in K_i$, $i = 0, 1$. There exists an isomorphism $i\colon \mathfrak{A}_0 >< \mathfrak{A}_1 \to \mathfrak{B}_0 >< \mathfrak{B}_1$. We have to show $\mathfrak{A}_i \cong \mathfrak{B}_i$, $i = 0, 1$. We shall prove $\mathfrak{A}_0 \cong \mathfrak{B}_0$. First we show:

(a$_1$) $i(x, y) = (x_1, y_1)$ and $i(x, y_2) = (x_2, z_1)$ imply $x_1 = x_2$.

(a$_2$) $i(x, y) = (x_1, y_1)$ and $i(x_2, y) = (x_3, y_2)$ imply $y_1 = y_2$.

From the assumption of (a$_1$) we get $(x_2, y_1) = (p(x_2, x_1), p(z_1, y_1)) = p((x_2, z_1), (x_1, y_1)) = ip((x, y_2), (x, y)) = i(p(x, x), p(y_2, y)) = i(x, y) = (x_1, y_1)$. Hence $x_2 = x_1$. The proof of (a$_2$) is similar. Now we shall define a mapping $t : \mathfrak{A}_0 \to \mathfrak{B}_0$ as follows: Let $t(x)$ be an element of $\mathfrak{B}_0$ such that for an $y \in \mathfrak{A}_1$, $i(x, y) = (t(x), y_1)$. We assert that t is an isomorphism. t is surjective, because if $x_1 \in \mathfrak{B}_0$, $y_1 \in \mathfrak{B}_1$ and if we denote $(x, y) = i^{-1}(x_1, y_1)$, then $i(x, y) = (x_1, y_1)$, hence $x_1 = t(x)$. t is injective, for if $t(x) = t(x_1)$ then for $y \in \mathfrak{A}_1$ we get $i(x, y) = (t(x), y_1)$, $i(x_1, y) = (t(x_1), y_2)$. By (a$_2$), $y_1 = y_2$. Hence $i(x, y) = i(x_1, y)$. This implies $(x, y) = (x_1, y)$, hence $x = x_1$. t is a homomorphism: Let f be an n-ary operation, $x_1, \ldots, x_n \in \mathfrak{A}_0$, $y_1, \ldots, y_n \in \mathfrak{A}_1$. Let $i(x_k, y_k) = (t(x_k), y_k^0)$. Then $(f(t(x_1), \ldots, t(x_n)), f(y_1^0, \ldots, y_n^0)) = f((t(x_1), y_1^0), \ldots, (t(x_n), y_n^0)) = f(i(x_1, y_1), \ldots, i(x_n, y_n)) = if((x_1, y_1), \ldots, (x_n, y_n)) = i(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n))$. This implies $tf(x_1, \ldots, x_n) = f(t(x_1), \ldots, t(x_n))$. Hence $\mathfrak{A}_0 \cong \mathfrak{B}_0$. $\mathfrak{A}_1 \cong \mathfrak{B}_1$ can be proved analogously.

Now let Theorem 5 hold for $n = k$ and let $K_0, K_1, \ldots, K_k$ be independent. Using Theorem 3 we get that $K_0 \vee K_1 \vee \ldots \vee K_{k-1}$ and K_k are independent, hence $(K_0 \vee K_1 \vee \ldots \vee K_{k-1}) \vee K_k = (K_0 \vee K_1 \vee \ldots \vee K_{k-1}) >< K_k$ and any algebra $\mathfrak{A} \in K_0 \vee K_1 \vee \ldots \vee K_k$ has, up to isomorphism, a unique representation $\mathfrak{A} \cong \mathfrak{B} >< \mathfrak{A}_k$ where $\mathfrak{B} \in K_0 \vee \ldots \vee K_{k-1}$, $\mathfrak{A}_k \in K_k$. By Corollary 1, $K_0, K_1, \ldots, K_{k-1}$ are independent, too, and using the induction assumption we get $K_0 \vee \ldots \vee K_{k-1} = K_0 >< \ldots >< K_{k-1}$ and $\mathfrak{B}$ has, up to isomorphism, a unique representation $\mathfrak{B} \cong \mathfrak{A}_0 >< \ldots >< \mathfrak{A}_{k-1}$, $\mathfrak{A}_i \in K_i$, $i = 0, 1, \ldots, k-1$. Hence $K_0 \vee K_1 \vee \ldots \vee K_{k-1} \vee K_k = K_0 >< K_1 >< \ldots >< K_{k-1} >< K_k$ and $\mathfrak{A} \cong \mathfrak{B} >< \mathfrak{A}_k \cong \mathfrak{A}_0 >< \mathfrak{A}_1 >< \ldots >< \mathfrak{A}_{k-1} >< \mathfrak{A}_k$ where the representation is unique up to isomorphism. This completes the proof.

Proof of Theorem 6. We shall proceed by induction. For $n = 2$ this theorem holds by [6, Theorem 2]. Let the theorem hold for $n = k$ and let the conditions (5), (6), (7) be satisfied for $n = k + 1$. We assert that (5) holds for $n = k$, too. Indeed, let $\mathfrak{A} \in K_0 \vee \ldots \vee K_{k-1} \subset K_0 \vee \ldots \vee K_{k-1} \vee K_k = K_0 \times \ldots \times K_{k-1} \times K_k$, then $\mathfrak{A} \cong \mathfrak{A}_0 \times \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_k$, $\mathfrak{A}_i \in K_i$, $i = 0, 1, \ldots, k$. Hence $\mathfrak{A}_k \in K_0 \vee \ldots \vee K_{k-1}$ because $\mathfrak{A}_k$ is a homomorphic image of $\mathfrak{A}$. With respect to (6), $\mathfrak{A}_k$ is one-element algebra. Hence $\mathfrak{A} \cong \mathfrak{A}_0 \times \ldots \times \mathfrak{A}_{k-1}$. By the induction assumption $K_0, K_1, \ldots, K_{k-1}$ are independent. Now the two classes K_k and $K_0 \vee \ldots \vee K_{k-1}$ satisfy the assumptions of Theorem 6 for $n = 2$ and this implies that K_k and $K_0 \vee \ldots \vee K_{k-1}$ are independent. By Theorem 3, $K_0, K_1, \ldots, K_{k-1}, K_k$ are independent, too.

Proof of Theorem 7. We shall use Theorem 1. The condition (1) is obviously fulfilled. Now we shall prove the condition (2). Let $\mathfrak{A} \in K_0 \vee \ldots \vee K_{n-1}$. Then (by Lemma A and (5)) $\mathfrak{A} \cong \mathfrak{A}/\Phi_0 \times \ldots \times \mathfrak{A}/\Phi_{n-1}$, where $\mathfrak{A}/\Phi_i \in K_i$, $i = 0, 1, \ldots, n-1$, and $\{\Phi_0, \Phi_1, \ldots, \Phi_{n-1}\}$ is absolutely permutable. From Lemma A and Lemma B we get $\Phi_h \vee \Phi_j = \bigvee \{\Phi_i : i = 0, 1, \ldots, n-1\} = \iota$ for any $h \neq j$, $h, j = 0, 1, \ldots, n-1$. Let Θ_i, $i = 0, 1, \ldots, n-1$, be the least congruence relations on $\mathfrak{A}$ such that $\mathfrak{A}/\Theta_i \in K_i$, $i = 0, 1, \ldots, n-1$. With respect to (6') we get $\Theta_i \vee \Theta_j = \iota$ for any $i \neq j$, $i, j = 0, 1, \ldots, n-1$. Using (7') we get:

$\Theta_j \vee (\bigwedge \{\Phi_i : i \neq j, i, j = 0, 1, \ldots, n-1\}) =$

$= \bigwedge \{(\Theta_j \vee \Phi_i) : i \neq j, i = 0, 1, \ldots, n-1\} = \iota$. Then $\Phi_j = \Phi_j \wedge \iota =$

$= \Phi_j \wedge [\Theta_j \vee \bigwedge \{\Phi_i : i \neq j, i = 0, 1, \ldots, n-1\}] = \Theta_j \vee \omega = \Theta_j$

for each $j = 0, 1, \ldots, n-1$. Hence the set $\{\Theta_0, \Theta_1, \ldots, \Theta_{n-1}\}$ is absolutely permutable and $K_0, K_1, \ldots, K_{n-1}$ are independent.

3. Examples

The first two examples will give independent equational classes K_i, $i = 0, 1, \ldots, n-1$, (of the same type) such that not every algebra $\mathfrak{A}$ of $K_0 \vee K_1 \vee \ldots \vee K_{n-1}$ has a modular congruence lattice.

Example 1. Let K_i, $i = 0, 1, \ldots, n-1$, consist of all algebras $\mathfrak{A}_i = \langle A_i; f \rangle$, where f is an n-ary operation and $f(x_0, \ldots, x_{n-1}) = x_i$ in K_i, $i = 0, 1, \ldots, n-1$. Then K_i, $i = 0, 1, \ldots, n-1$, are independent (for it is sufficient to take $p(x_0, \ldots, x_{n-1}) = f(x_0, \ldots, x_{n-1})$) hence $K_0 \ldots K_{n-1} = K_0 \times \ldots \times K_{n-1}$. Any equivalence relation Ψ on $\mathfrak{A}_i (i \in \{0, 1, \ldots, n-1\})$ is a congruence relation on $\mathfrak{A}_i \in K_i$, because $x_j \equiv y_j(\Psi)$, $j = 0, 1, \ldots, n-1$, imply $f(x_0, x_1, \ldots, x_{n-1}) = x_i \equiv y_i = f(y_0, y_1, \ldots, y_{n-1})\ (\Psi)$. Hence by [1] congruence lattices on the algebras of K_i are not modular if card $A_i > 3$.

Example 2. Let K_0 consist of all groups $\mathcal{G} = \langle G; f_0, f_1 \rangle$ where $f_0(x, y)$

$= xy$, $f_1(x, y) = xy^{-1}$. Let K_1 consist of all skew-lattices (Schiefverbände [4]) $\mathfrak{S} = <S; f_0, f_1>$ where $f_0(x, y) = x \wedge y$, $f_1(x, y) = x \vee y$. In K_1 the identity $(x \wedge y) \vee y = y$ holds. K_0 and K_1 are independent for it suffices to set $p(x, y) = f_1(f_0(x, y), y)$. There are skew-lattices such that any equivalence relation on them is a congruence relation. For example the algebra $\mathcal{M} = <M; \wedge, \vee>$ where $x \wedge y = x$, $x \vee y = y$ for any x, y of the set M is such a skew-lattice, hence by [1] the congruence lattice on $\mathcal{M}$ is not modular if card $M > 3$.

Example 3. Let K_{p_i} (where p_i, $i = 0, 1, \ldots, n-1$, are distinct primes) denote the equational classes of Abelian groups satisfying $p_i x = 0$, $i = 0, 1, \ldots, n-1$. Denote $m = p_0 p_1 \ldots p_{n-1}$ and $q_i = m/p_i$. Let t_i, $i = 0, 1, \ldots, n-1$, be integers satisfying $q_i t_i \equiv 1 (p_i)$. Then it suffices to set $p = q_0 t_0 x_0 + q_1 t_1 x_1 + \ldots + q_{n-1} t_{n-1} x_{n-1}$ because $q_j \equiv 0 (p_i)$ for $i \neq j$, $i = 0, 1, \ldots, n-1$. It follows that K_{p_i}, $i = 0, 1, \ldots, n-1$, are independent hence $K_{p_0} \vee \ldots \vee K_{p_{n-1}} = K_{p_0} >< \ldots >< K_{p_{n-1}}$. The same result can be obtained if we replace $K_{p_i} (i = 0, 1, \ldots, n-1)$ by the class of all rings of the characteristic p_i.

Example 4. We give an example of equational classes K_0, K_1, K_2 with the following properties:

(a) $K_0 \wedge K_1 \wedge K_2$ consists of one-element algebras only.

(b) Every algebra $\mathfrak{A} \in K_0 \vee K_1 \vee K_2$ has a modular congruence lattice.

(c) $K_0 \vee K_1 \vee K_2 = K_0 >< K_1 >< K_2$.

(d) K_0, K_1, K_2 are not independent.

Let C_0, C_1, C_2 are the classes K_{p_i} of Exercise 3 where $p_i = 3, 5, 7$, respectively. Then $K_0 = C_0 >< C_1$, $K_1 = C_1 >< C_2$, $K_2 = C_0 >< C_2$ and $K_0 \vee K_1 \vee K_2 = C_0 >< C_1 >< C_2$ are equational classes. The condition (a) can be easily verified. Since the algebras of the class $K_0 \vee K_1 \vee K_2$ are groups, the condition (b) is satisfied. Finally, let $\mathfrak{A}_i \in C_i$ $(i = 0, 1, 2)$ be groups having more than one element. Then the algebra $\mathfrak{A}_0 >< \mathfrak{A}_1 >< \mathfrak{A}_2$ has more than one representation as a direct product of algebras of K_i $(i = 0, 1, 2)$. Hence K_0, K_1, K_2 cannot be independent by Theorem 5.

Remark 6. There are equational classes K_0, K_1, K_2 satisfying conditions (a), (c), (d) of Example 4 and the next condition:

(b′) Every algebra $\mathfrak{A} \in K_0 \vee K_1 \vee K_2$ has a distributive congruence lattice. Such an example can be constructed by the same way as in Example 4 by replacing classes C_0, C_1, C_2 by the following classes: C_0', C_1', C_2' are classes of algebras $\mathfrak{A} = <A; \wedge, \vee, f>$ where $<A; \wedge, \vee>$ are lattices and $f(x_0, x_1, x_2) = x_i$ in C_i', $i = 0, 1, 2$.

Example 5. As an application of Theorem 1 we shall show that the following classes K_0, K_1 are not independent. Let K_0, K_1 be equational classes of algebras $<A; f_0, f_1, f_2>$, where in K_0 $<A; f_0, f_1, f_2>$ are lattices with the least element (the operation f_2), $f_0(x, y) = x \wedge y$, $f_1(x, y) = x \vee y$. In K_1, $<A; f_0, f_1, f_2>$ are Boolean rings, $f_0(x, y) = x \cdot y$, $f_1(x, y) = x + y$, (f_2 represents the zero element). Let $\mathfrak{A}$ be the two-element lattice with the elements o, i and $\mathfrak{B}$ the two-element Boolean ring with the elements $0, 1$. The subset $C = \{(o, 0), (i, 0), (i, 1)\}$ of the direct product $\mathfrak{A} >< \mathfrak{B}$ forms a subalgebra of $\mathfrak{A} >< \mathfrak{B}$, hence $\mathfrak{C} \in K_0 \vee K_1$. Consider the equivalence relations on $C : (a, b) \equiv (c, d)(\Theta_0)$ iff $a = c$, and $(a, b) \equiv (c, d)\,(\Theta_1)$ iff $b = d$. Then Θ_0, Θ_1 are congruence relations on $\mathfrak{C}$ and $\mathfrak{C}/\Theta_i \in K_i$. Nevertheless Θ_0 and Θ_1 are not permutable, hence K_0, K_1 are not independent (by Theorem 1). Moreover $\mathfrak{C}$ cannot be represented as a direct product $\mathfrak{C}_0 >< \mathfrak{C}_1$, $\mathfrak{C}_i \in K_i$, hence $K_0 \vee K_1 \neq K_0 >< K_1$. (Note that the same result can be obtained with K_0 as the class of distributive lattices with the least element.)

Example 6. We shall give an example of classes K_0, K_1, K_2 such that for any couple (i, j), $i \neq j$, $i, j = 0, 1, 2$, K_i and K_j are independent but K_i, $i = 0, 1, 2$, are not independent. Let K_i, $i = 0, 1, 2$, be equational classes of algebras $<A_i; f_1, f_2, f_3>$, where in $K_0 : f_1(x, y) = x, f_2(x, y) = x, f_3(x, y) = f_3(u, v)$, in $K_1 : f_1(x, y) = y, f_2(x, y) = f_2(u, v), f_3(x, y) = x$, in $K_2 : f_1(x, y) = f_1(u, v), f_2(x, y) = y, f_3(x, y) = y$. Consider the algebras $\mathfrak{A}_i = <A_i; f_1, f_2, f_3> \in K_i$, $i = 0, 1, 2$, where $A_i = \{0, 1\}$ and $f_3(x, y) = 0$ in $\mathfrak{A}_0$, $f_2(x, y) = 0$ in $\mathfrak{A}_1$, $f_1(x, y) = 0$ in $\mathfrak{A}_2$. Obviously the set $A_0 >< A_1 >< A_2 - \{(1,1,1)\}$ forms a subalgebra of $\mathfrak{A}_0 >< \mathfrak{A}_1 >< \mathfrak{A}_2$ but cannot be decomposed into a direct product $\mathfrak{B}_0 >< \mathfrak{B}_1 >< \mathfrak{B}_2$, where $\mathfrak{B}_i \in K_i$, $i = 0, 1, 2$. To show the independence of every couple K_i, K_j, $i \neq j$, $i, j = 0, 1, 2$, it suffices to take $p(x, y) = f_1(x, y)$ (for K_0, K_1), $p(x, y) = f_2(x, y)$ (for K_0, K_2), $p(x, y) = f_3(x, y)$ (for K_1, K_2).

Example 7. This example shows that the number $n - k$ of Theorem 4 cannot be lowered. It suffices to join to the classes K_i, $i = 0, 1, 2$, of Example 6 the class K_3 of algebras $<A; f_1, f_2, f_3>$ where $<A; f_1, f_2>$ are lattices ($f_1(x, y) = x \wedge y$, $f_2(x, y) = x \vee y$) and $f_3(x, y) = x + y$ where $+$ satisfies the following identities: $x + x = x$, $x \wedge (x + y) = y$, $x + (x \vee y) = x$. Hence in K_3 there are idempotent operations only. For each $i \in \{0, 1, 2\}$, K_i and K_3 are independent: The corresponding polynomial symbols $p(x, y)$ are $f_1(f_2(x, y), y)$, $f_1(x, f_3(x, y))$ and $f_3(x, f_2(x, y))$, respectively. Every triple K_i, K_j, K_3 is independent for each $i \neq j$, $i, j = 0, 1, 2$, by Theorem 4. But K_0, K_1, K_2, K_3 are not independent because K_0, K_1, K_2 are not independent (see Corollary 1).

Example 8. In the paper [6] it is shown that the equational class K_0 of all groups $\mathfrak{G} = <G; f_0, f_1>$, where $f_0(x, y) = xy$, $f_1(x, y) = xy^{-1}$ and the class K_1 of all algebras $<L; f_0, f_1>$ where $\mathfrak{L}$ is a lattice, $f_0(x, y) = x \vee y$, $f_1(x, y) = x \wedge y$, are independent. In Example 3 it is shown that K_{p_i}, $i = 0, 1, \ldots, n - 1$, are independent. Hence K_1 and K_{p_i} are independent for each $i \in \{0, 1, \ldots, n - 1\}$. Because K_1 has only idempotent operations, using Theorem 4 more times we get that $K_1, K_{p_0}, K_{p_1}, \ldots, K_{p_{n-1}}$ are independent, too. (Note that the independence of these classes can also be obtained by using Theorem 3.) The same result holds if we replace the class K_1 of Example 8 by the class of all skew-lattices from Example 2 or if we replace the mentioned classes by the classes of all algebras $\langle A; f_0, f_1, f_2 \rangle$ where K_1 is the class of Brouwerian lattices ($f_0(x, y) = x \vee y, f_1(x, y) = x \wedge y, f_2(x, y) = x : y$) and K_{pi} ($i = 0, 1, \ldots, n - 1$) is the class of rings of characteristic p_i ($f_0(x, y) = x + y, f_1(x, y) = x - y, f_2(x, y) = x \,.\, y$).

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