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THE CONVERGENCE OF SUCCESSIVE APPROXIMATIONS FOR BOUNDARY VALUE PROBLEMS OF HYPERBOLIC EQUATIONS IN THE BANACH SPACE

VLADIMÍR ĎURIKOVIČ, Bratislava

I. INTRODUCTION

Some results concerning the uniqueness of solution of boundary value problems defined for the equations

$$\frac{\partial^t u}{\partial x_1^{k_1} \dots \partial x_m^{k_m}} = f(x_1, \dots, x_m, u), \quad t = k_1 + \dots + k_m$$

and the convergence of successive approximations are studied in paper [4]. Those results were obtained under the conditions of the uniqueness of the Krasnosielski-Krein type by classical methods.

The purpose of the present paper is to show that more general conditions than the above-mentioned conditions guarantee both the existence and uniqueness of boundary value problems given for the equations

$$\frac{\partial^t u}{\partial x_1^{k_1} \dots \partial x_m^{k_m}} - f\left(x_1, \dots, x_m, u, \dots, \frac{\partial^{\gamma_1 + \dots + \gamma_m} u}{\partial x_1^{\gamma_1} \dots x_m^{\gamma_m}}, \dots\right), \quad \gamma_1 + \dots + \gamma_m < t$$

and the convergence of successive approximations. Instead of the usual method of proving convergence of successive approximations we shall apply certain general theorems concerning mapping defined on some appropriate function space in our considerations. These theorems are published in papers by M. Edelstein [2] and by W. A. Luxemburg [3].

II. TWO FIXED-POINT THEOREMS

An abstract, non-void set A on which a distance function $d(x, y)$ is defined such that for $x, y, z \in A$:

a) $d(x, y)$ is a non-negative real valued function $(0 \leq d(x, y) < +\infty)$, defined on the Cartesian product $A \times A$,

b) $d(x, y) = 0$ if and only if $x = y$,

c) $d(x, y) = d(y, x)$,

d) $d(x, y) \leq d(x, z) + d(z, y)$,

e) Every d-Cauchy sequence $\{x_k\}_{k=1}^{\infty}$ converges to a limit in A, i. e. $\lim_{k,l\to\infty} d(x_k, x_l) = 0$ implies the existence of an element $x \in A$ such that $\lim_{k\to\infty} d(x_k, x) = 0$

is called *a generalized complete metric space*. It differs from the concept of a complete metric space by the fact that not every pair of elements necessarily has a finite distance.

Now we formulate the following theorems:

Theorem 1. (Luxemburg [3]). *Let A be a generalized complete metric space and T a mapping defined on A into itself satisfying the following conditions:*

1°. *There exists a constant λ, $0 < \lambda < 1$, such that*

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in A$ with $d(x, y) < +\infty$.

2°. *For every sequence of successive approximations $x_k = Tx_{k-1}$, $k = 1, 2, \ldots$ where x_0 is an arbitrary element of A, there exists an index $N(x_0)$ such that $d(x_N, x_{N+l}) < +\infty$ for all $l = 1, 2, \ldots$.*

3°. *If x and y are two fixed points of T, i. e. $Tx = x$ and $Ty = y$, then $d(x, y) < +\infty$.*

Then the equation $Tx = x$ has one and only one solution in A and every sequence of successive approximations $\{x_k\}_{k=1}^{\infty}$ converges in the distance $d(x, y)$ to this unique solution.

Theorem 2. (Edelstein [2]). *Let A be a complete metric space and T a mapping defined on A into itself satisfying the following conditions:*

1°. *For all $x, y \in A$, $x \neq y$ we have*

$$d(Tx, Ty) < d(x, y).$$

2°. *For every sequence of successive approximations $x_k = Tx_{k-1}$, $k - 1, 2 \ldots$, where x_0 is an arbitrary element of A, there exists a subsequence which converges to a point $x \in A$.*

Then the equation $Tx = x$ has one and only one solution in A and every sequence of successive approximations $\{x_k\}_{k=1}^{\infty}$ converges in the distance $d(x, y)$ to this unique solution.

III. THE FORMULATION OF THE BOUNDARY VALUE PROBLEM

In this section we want to introduce some notations and notions used throughout the present paper.

1. Denote the set of points $X(x_1, \ldots, x_m)$, $m \geq 2$ with the coordinates $0 < x_j \leq a_j$, $0 \leq x_j \leq a_j$, $a_j > 0$ for $j = 1, \ldots, m$ by R^c, R respectively and the set of points $X_l(x_1, \ldots, x_{l-1}, 0, x_{l+1}, \ldots, x_m)$, $X_{rs}(x_1, \ldots, x_{r-1}, 0, x_{r+1}, \ldots, x_{s-1}, 0, x_{s+1}, \ldots, x_m)$ with the coordinates $0 \leq x_j \leq a_j$ for $j \neq l$, $j \neq r, s$ by R_l, R_{rs} respectively ($1 \leq l, r, s \leq m$; $r < s$).

2. Moreover we shall employ the symbol $\Sigma(\alpha, \xi)$ to denote simplex in the α-dimensional Euclidean space E^α with the $\alpha + 1$ linearly independent vertices $\Xi_0(0, \ldots, 0)$, $\Xi_1(\xi, 0, \ldots, 0)$, $\ldots$, $\Xi_{\alpha-1}(\xi, \ldots, \xi, 0)$, $\Xi_\alpha(\xi, \ldots, \xi)$, $\xi > 0$. Consequently, $\Sigma(\alpha, \xi)$ is the set of points $P \in E^\alpha$ such that

$$P = \tau_0 \Xi_0 + \ldots + \tau_\alpha \Xi_\alpha,\ \tau_0 + \ldots + \tau_\alpha = 1,\ \tau_i \geq 0,\ i = 0, 1, \ldots, \alpha.$$

If $\xi \quad 0$, then we set $\Sigma(\alpha, 0) = \Xi_0$.

3. Let $k_1, \ldots, k_m$ be fixed natural numbers ($m \geq 2$). Denote $n = \sum_{j=1}^{m} k_j$.

Then we may define the sets of indices $\Delta_0^i(\gamma)$, $\Delta_1^i(\gamma)$ and $\Delta_2^i(\gamma)$ as follows:

a) $\Delta_0^i(\nu)$ for $i - 0, 1, \ldots, n - 1$ is the set of elements $(\gamma_1 \ldots \gamma_m)$ with the integer components $\gamma_1, \ldots, \gamma_m$ for which:

$$0 \leq \gamma_j \leq k_j,\ j = 1, \ldots, m \quad \text{and} \quad \sum_{j=1}^{m} \gamma_j = i.$$

b) Analogically, $\Delta_1^i(\gamma)$ for $i = 0, 1, \ldots, n - m$ is the set of elements $(\gamma_1 \ldots \gamma_m)$ with the integer components $\gamma_1, \ldots, \gamma_m$ for which:

$$0 \leq \gamma_j \leq k_j - 1,\ j = 1, \ldots, m \quad \text{and} \quad \sum_{j=1}^{m} \gamma_j = i.$$

c) If $k_j \geq 2$ for $j = 1, \ldots, m$ then $\Delta_2^i(\gamma)$ for $i = 0, 1, \ldots, n - 2m$ is the set of elements $(\gamma_1 \ldots \gamma_m)$ with the integer components $\gamma_1, \ldots, \gamma_m$ for which:

$$0 \leq \gamma_j \leq k_j - 2,\ j = 1, \ldots, m \quad \text{and} \quad \sum_{j=1}^{m} \gamma_j = i.$$

Thus for $\varrho - 0, 1, 2$ and $i = 0, 1, \ldots, n - \nu(\varrho)$, where $\nu(\varrho)$ is an arbitrary real-valued function with $\nu(0) = 0$, $\nu(1) = m$ and $\nu(2) = 2m$ (for instance $\nu(\varrho) - \Gamma(\varrho + 1) m^{3\ 2-\ \varrho-3\ 2|}$, where $\Gamma(x)$ is the Gamma function) we can define the above-mentioned sets $\Delta_\varrho^i(\gamma)$ as follows:

$$\Delta_\varrho^i(\gamma) - \{(\gamma_1 \ldots \gamma_m) : 0 \leq \gamma_j \leq k_j - \varrho,\ \sum_{j=1}^{m} \gamma_j = i, \quad \text{where} \quad k_j \geq \Gamma(\varrho + 1) \quad \text{and}$$
$$\gamma_j \text{ are integers for } j = 1, \ldots, m\}.$$

The union of the sets $\Delta_\varrho^i(\gamma)$ for $i = 0, 1, \ldots, n - \nu(\varrho)$ with the fixed $\varrho = 0, 1, 2$ will be denoted by $\Delta_\varrho(\gamma)$, i. e. $\Delta_\varrho(\gamma) = \bigcup_{i\ 0}^{n-\nu(\varrho)} \Delta_\varrho^i(\gamma)$.

Next we shall write briefly ν instead of $\nu(\varrho)$.

Remark 1. We shall denote the set of elements $(\gamma_{l_1} \dots \gamma_{l_j})$, in which $\gamma_{l_1}, \dots, \gamma_{l_j}$ are all non-vanishing components of an arbitrary element $(\gamma_1 \dots \gamma_m)$ of the set $\Delta^i_\varrho(\gamma)$ by $\tilde{\Delta}^i_\varrho(\gamma)$ for $i = 1, \dots, n - \nu$ and $\varrho = 0, 1, 2$. Also we put $\tilde{\Delta}^0_\varrho(\gamma) = \{0\}$.

The sets $\Delta^i_\varrho(\gamma)$ and $\tilde{\Delta}^i_\varrho(\gamma)$ are mutually equivalent. There exists a one-to-one mapping φ of the set $\Delta^i_\varrho(\gamma)$ onto the set $\tilde{\Delta}^i_\varrho$ such that

$$\varphi(\gamma_1 \dots \gamma_m) = (\gamma_{l_1} \dots \gamma_{l_j}) \quad \text{if} \quad \gamma_1 + \dots + \gamma_m > 0$$

$$\varphi(\gamma_1 \dots \gamma_m) = 0 \quad \text{if} \quad \gamma_1 + \dots + \gamma_m = 0.$$

Two corresponding elements of the mapping φ will be considered equal, i. e.

$$(\gamma_1 \dots \gamma_m) = (\gamma_{l_1} \dots \gamma_{l_j}) \quad \text{if} \quad \gamma_1 + \dots + \gamma_m < 0$$

$$(\gamma_1 \dots \gamma_m) = 0 \quad \text{if} \quad \gamma_1 + \dots + \gamma_m = 0.$$

4. Next, $E^0_\varrho = R^\circ \times \{B \times \dots \times B\}$ and $E_\varrho = R \times \{B \times \dots \times B\}$, where B denotes the Banach space with the norm $\| \;\|$. The number p of the factors in the Cartesian product $\{B \times \dots \times B\}$ is given by the cardinal number of the set $\Delta_\varrho(\gamma)$, i. e. $p = \text{Card}\,[\Delta_\varrho(\gamma)] = \sum_{i=0}^{n-\nu} \text{Card}\,[\Delta^i_\varrho(\gamma)] = \sum_{i=0}^{n-\nu} \text{Card}\,[\tilde{\Delta}^i_\varrho(\gamma)]$.

5. Any vector $(\dots, u_{\gamma_1 \dots \gamma_m}, \dots)$ with the components $u_{\gamma_1 \dots \gamma_m} \in B$ for all $(\gamma_1 \dots \gamma_m) \in \Delta^i_\varrho(\gamma)$ will be denoted by $\mathbf{U}^i_\varrho$ for $i = 0, 1, \dots, n - \nu$. The number of components $u_{\gamma_1 \dots \gamma_m}$ of the vector $\mathbf{U}^i_\varrho$ is $\text{Card}\,[\Delta^i_\varrho(\gamma)] = \text{Card}\,[\tilde{\Delta}^i_\varrho(\gamma)]$. By means of Remark 1 we can write in $u_{\gamma_1 \dots \gamma_m}$ instead of the index $\gamma_1 \dots \gamma_m$ the index $\gamma_{l_1} \dots \gamma_{l_j}$, resp. 0.

Furthermore, the symbol $\| \mathbf{U}^i_\varrho \|^\vartheta$ for any real number ϑ means the vector $(\dots, \| u_{\gamma_1 \dots \gamma_m} \|^\vartheta, \dots)$ and the symbol $(\mathbf{U}, \mathbf{V})$ means the scalar product of the vectors $\mathbf{U}$ and $\mathbf{V}$.

If we denote the differential operator $\dfrac{\partial^{\delta_1 + \dots + \delta_m}}{\partial x_1^{\delta_1} \dots \partial x_m^{\delta_m}}$ for any non-negative integers δ_j, $j = 1, \dots, m$ by $D_{\delta_1 \dots \delta_m} = D^{\delta_1}_{x_1} \dots D^{\delta_m}_{x_m}$, then $\mathbf{D}^i_\varrho$ defines a vector whose components are formed by all differential operators $D_{\gamma_1 \dots \gamma_m}$ of the same order $i = 0, 1, \dots, n - \nu$, i. e. $\mathbf{D}^i_\varrho = (\dots, D_{\gamma_1 \dots \gamma_m}, \dots)$, where $(\gamma_1 \dots \gamma_m)$ runs through all elements of the set $\Delta^i_\varrho(\gamma)$ for any $i = 0, 1, \dots, n - \nu$.

Also we set $\mathbf{D}^n_\varrho = D_{k_1 \dots k_m}$ for any $\varrho = 0, 1, 2$. From Remark 1 it follows that $\mathbf{D}^0_\varrho u = D_0 u = u$.

6. Let $v(X)$ be a continuous mapping defined on R into the Banach space B and the derivatives $D_{\gamma_1 \dots \gamma_m} v(X)$ for $(\gamma_1 \dots \gamma_m) \in \Delta_\varrho(\gamma)$ be continuous mappings of R into B, too.

The set of abstract functions $v(X)$ satisfying the above-mentioned properties will be denoted by $M_\varrho(R)$.

Further, let the derivatives $D_{\delta_1 \ldots \delta_{r-1} \delta_{r+1} \ldots \delta_m} w(X_r)$ be continuous mappings defined on R_r into the Banach space B for $0 \leq \delta_j \leq k_j$, $j, r = 1, \ldots, m$, $j \neq r$. The set of all such abstract functions $w(X_r)$ will be denoted by $N(R_r)$.

Now, we may formulate the three following boundary value problems (ϱ 0, 1, 2):

$$(1_\varrho) \qquad \mathbf{D}_\varrho^n u(X) = f[X, u(X), \mathbf{D}_\varrho^1 u(X), \ldots, \mathbf{D}_\varrho^{n-\nu} u(X)] \quad \text{for} \quad X \in R^\circ$$

$$(2) \quad [D_{i_r} u(X)]_{x_r=0} = \sigma_r^{(i_r)}(X_r) \text{ for } X_r \in R_r, i_r = 0, 1, \ldots, k_r - 1, r = 1, \ldots, m$$

$$[D_{j_s} \sigma_r^{(i_r)}(X_r)]_{x_s=0} = [D_{i_r} \sigma_s^{(j_s)}(X_s)]_{x_r=0} \quad \text{for} \quad X_{rs} \in R_{rs}$$

$$r \neq s, \ i_r = 0, 1, \ldots, k_r - 1, \ j_s = 0, 1, \ldots, k_s - 1; \ r, s = 1, \ldots, m,$$

where $\sigma_r^{(i_r)}(X_r) \in N(R_r)$ and $f(X, \mathbf{U}_\varrho^0, \mathbf{U}_\varrho^1, \ldots, \mathbf{U}_\varrho^{n-\nu})$ is a continuous mapping defined on E_ϱ into B.

Under the solution of the problem (1_ϱ), (2) we understand any element $u(X) \in M_0(R)$ satisfying the conditions (1_ϱ) and (2).

Hence it follows that the problem (1_ϱ), (2) is equivalent to the following integro-differential equation:

$$(3) \qquad u(X) = G(X) +$$

$$+ \int_{\Sigma(k_1, x_1)} d\mu_1 \ldots \int_{\Sigma(k_m, x_m)} f[\Xi, u(\Xi), \mathbf{D}_\varrho^1 u(\Xi), \ldots, \mathbf{D}_\varrho^{n-\nu} u(\Xi)] \, d\mu_m$$

in R, where the point Ξ has the coordinates $(\xi_1, \ldots, \xi_m)$ and μ_j for $j = 1, \ldots, m$ denotes the Lebesgue measure defined in the Euclidean space E^{k_j}. The function $G(X)$ is given as follows:

$$G(X) = \sum_{j=1}^{m} \sum_{i_1, \ldots, i_j} \sum_{l_1, \ldots, l_j} (-1)^{j-1} \frac{x_{i_1}^{l_1} \ldots x_{i_j}^{l_j}}{l_1! \ldots l_j!} [D_{x_{i_1}}^{l_1} \ldots D_{x_{i_j}}^{l_j} u(X)]_{x_{i_1} = 0 \ldots x_{i_j} = 0},$$

where $0 \leq l_1 \leq k_{i_1} - 1, \ldots, 0 \leq l_j \leq k_{i_j} - 1$; $(i_1, \ldots, i_j)$ is an arbitrary combination of j numbers from the m natural numbers $(1, \ldots, m)$, $i_1 < \ldots < i_j$.

By the direct derivation of (3) we get

$$(4) \qquad D_{\delta_1 \ldots \delta_m} u(X) = D_{\delta_1 \ldots \delta_m} G(X) +$$

$$+ \int_{\Sigma(k_1 - \delta_1, x_1)} d\mu_1 \ldots \int_{\Sigma(k_m - \delta_m, x_m)} f[\Xi, u(\Xi), \mathbf{D}_\varrho^1 u(\Xi), \ldots, \mathbf{D}_\varrho^{n-\nu} u(\Xi)] \, d\mu_m$$

for $X \in R$ and $(\delta_1 \ldots \delta_m) \in \Delta_\varrho(\delta)$, where we take

$$\int_{\Sigma(0, x_j)} F(\Xi) d\mu_j = F(\xi_1, \ldots, \xi_{j-1}, x_j, \xi_{j+1}, \ldots, \xi_m).$$

In view of (3) we define the sequence $\{u_k(X)\}_{k=1}^{\infty}$ of successive approximations of Picard as follows:

$$(5) \qquad u_k(X) = G_0(X) +$$

$$+ \int_{\Sigma(k_1,x_1)} d\mu_1 \ldots \int_{\Sigma(k_m,x_m)} f[\Xi, u_{k-1}(\Xi), \mathbf{D}_\varrho^1 u_{k-1}(\Xi), \ldots, \mathbf{D}_\varrho^{n-\nu} u_{k-1}(\Xi)]\, d\mu_m$$

for $k = 1, 2, \ldots$ and arbitrary abstract functions $u_0(X) \in M_\varrho(R)$, $G_0(X) \in M_0(R)$ such that $G_0(X)$ satisfies the conditions (2) and moreover $D_{k_1 \ldots k_m} G_0(X) \quad 0$ in $R°$. Hence we have $u_k(X)$ satisfying the conditions (2) and belonging to $M_0(R)$.

IV. THE CONVERGENCE OF SUCCESSIVE APPROXIMATIONS

Theorem 3. *Suppose for* $\varrho = 0, 1, 2$:

i) *The transformation* $f(X, \mathbf{U}_\varrho^0, \mathbf{U}_\varrho^1, \ldots, \mathbf{U}_\varrho^{n-\nu})$ *of* $m +$ Card $[\Delta_\varrho(\gamma)]$ *variables maps the set* E_ϱ *into* B *and is continuous in all variables. Further,* $f(X, \mathbf{U}_\varrho^0, \mathbf{U}_\varrho^1, \ldots, \mathbf{U}_\varrho^{n-\nu})$ *is bounded on* E_ϱ *in the following sense:*

$$(6_1) \qquad \| f(X, \mathbf{U}_\varrho^0, \mathbf{U}_\varrho^1, \ldots, \mathbf{U}_\varrho^{n-\nu}) | \leq K_1$$

if $\varrho = 0, 1, 2$. *In the case of* $\varrho = 1, 2$ *we may use a weaker assumption:*

$$(6_2) \qquad \| f(X, \mathbf{U}_\varrho^0, \mathbf{U}_\varrho^1, \ldots, \mathbf{U}_\varrho^{n-\nu}) | \leq \omega(X),$$

where

$$\int_{\Sigma(\varrho,a_1)} d\mu_1 \ldots \int_{\Sigma(\varrho,a_m)} \omega(\Xi) d\mu_m \leq K_2 .$$

ii) *In the domain* E_ϱ^0

$$(7) \qquad | f(X, \mathbf{U}_\varrho^0, \mathbf{U}_\varrho^1, \ldots, \mathbf{U}_\varrho^{n-\nu}) - f(X, \mathbf{V}_\varrho^0, \mathbf{V}_\varrho^1, \ldots, \mathbf{V}_\varrho^{n\ \nu}) \quad \leq$$

$$\leq L/x_1^{k_1} \ldots x_m^{k_m} \sum_{i=0}^{n-\nu} (\mathbf{P}_\varrho^i, \| \mathbf{U}_\varrho^i - \mathbf{V}_\varrho^i \|), \ L > 0,$$

where $\mathbf{P}_\varrho^i = (\ldots, p_{\gamma_1 \ldots \gamma_m} \prod_{j=1}^{m} \{x_j^{\gamma_j} [L^{h(k_j - \gamma_j)}]^{-1/m}\}, \ldots)$ *for* $(\gamma_1 \ldots \gamma_m) \in \Delta_\varrho^i(\gamma)$ *and* $i = 0, 1, \ldots, n - \nu$ *(the number of the components of the vector* $\mathbf{P}_\varrho^i$ *equals* Card $[\Delta_\varrho^i(\gamma)]$). *The factors* $p_{\gamma_1 \ldots \gamma_m}$ *are positive constants and the function* $h(x)$ *is defined as*

$$h(x) = \begin{cases} 1 & if \quad x = 0 \\ 0 & if \quad x \neq 0. \end{cases}$$

iii) *In* E_ϱ^0

(8)
$$\|f(X, \mathbf{U}_\varrho^0, \mathbf{U}_\varrho^1, \ldots, \mathbf{U}_\varrho^{n-\nu}) - f(X, \mathbf{V}_\varrho^0, \mathbf{V}_\varrho^1, \ldots, \mathbf{V}_\varrho^{n-\nu})\| \leq$$
$$\leq x_1^{-k_1\beta_1} \ldots x_m^{-k_m\beta_m} \sum_{i=0}^{n-\nu} (\mathbf{Q}_\varrho^i, \| \mathbf{U}_\varrho^i - \mathbf{V}_\varrho^i \|^\alpha),$$

where $\mathbf{Q}_\varrho^i = (\ldots, q_{\gamma_1 \ldots \gamma_m} [x_1^{\gamma_1} \ldots x_m^{\gamma_m}]^\alpha, \ldots)$ *for* $(\gamma_1 \cdots \gamma_m) \in \Delta_\varrho^i(\gamma)$ *denotes the vector with the* Card $[\Delta_\varrho^i(\gamma)]$ *components for* $i = 0, 1, \ldots, n - \nu$ *and* $0 < \alpha < 1$, $\beta_j < \alpha$ *for* $j = 1, \ldots, m$. *The coefficients* $q_{\gamma_1 \ldots \gamma_m}$ *are non-negative constants one of which at least is non-vanishing.*

iiii) *The constants* L, $p_{\gamma_1 \ldots \gamma_m}$, α, β_j *satisfy the following relations:*

(9)
$$\sqrt[m]{L}(1 - \alpha) < k_j(1 - \beta_j) - (k_j - 1)(1 - \alpha)$$

(10)
$$\Big(\sum_{\Delta_\varrho(\gamma)} p_{\gamma_1, \ldots, \gamma_m}\Big) \sqrt[m]{L}(1 - \alpha) < k_j(1 - \beta_j) - (k_j - 1)(1 - \alpha)$$

for $j = 1, \ldots, m$. *Then there exists one and only one solution* $u(X)$ *from the class* $M_\varrho(R)$ *of the boundary value problem* (1_ϱ), (2) *and furthermore the Picard sequence of successive approximations* $\{u_k(X)\}_{k=1}^\infty$ *defined by* (5) *for any function* $u_0(X) \in M_\varrho(R)$ *converges uniformly in* R *in the norm of* B *to this unique solution, i. e.* $\lim_{k\to\infty} \| u_k(X) - u(X) \| = 0$ *uniformly in* R.

Proof. To prove this statement we shall use Theorem 1 on the contractive mapping. For this purpose we have to construct an appropriate generalized complete metric space A and a mapping T from A into itself, and to show that the conditions 1°, 2°, 3° of Theorem 1 are really satisfied.

In view of the definition of the solution of the problem (1_ϱ), (2) and of (7) a natural choice for A is the space A_ϱ with the support $M_\varrho(R)$ and with the distance function defined on $A_\varrho \times A_\varrho$:

(11)
$$d_\varrho(u, v) = \sup_{R^\circ} \frac{\sum_{i=0}^{n-\nu} (\mathbf{P}_\varrho^i, \| \mathbf{D}_\varrho^i u(X) - \mathbf{D}_\varrho^i v(X) \|)}{x_1^{g_\varrho \sqrt[m]{L} + k_1 - 1} \ldots x_m^{g_\varrho \sqrt[m]{L} + k_m - 1}}$$

for $\varrho = 0, 1, 2$, consequently $A_\varrho = [M_\varrho(R), d_\varrho]$. The number g_ϱ is taken such that $g_\varrho > 1$, $g_\varrho \sqrt[m]{L} > 1$ and

$$\sum_{\Delta_\varrho(\gamma)} p_{\gamma_1 \ldots \gamma_m} < g_\varrho < [1/\sqrt[m]{\mathrm{L}}]\{k_j[(1 - \beta_j)/(1 - \alpha)] - (k_j - 1)\}$$

for $j = 1, \ldots, m$, which is possible since we always have (9) and (10). Clearly this function $d_\varrho(u, v)$ satisfies the requirements a), b), c), d) for a metric give in section II. We have to show that the condition e) is also satisfied for $d_\varrho(u, \cdot)$, i. e. that the space A_ϱ is complete. To this end we apply the following obvious inequality:

$$\max_{R} \sum_{i=0}^{n-\gamma} (\mathbf{S}_\varrho^i, \| \mathbf{D}_\varrho^i u(X) - \mathbf{D}_\varrho^i v(X) \|) \leq d_\varrho(u, v), \tag{12}$$

where $\mathbf{S}_\varrho^i = (\ldots, s_{\gamma_1 \ldots \gamma_m}, \ldots)$ for $(\gamma_1 \ldots \gamma_m) \in \Delta_\varrho^i(\gamma)$ is a vector with the constant coordinates $s_{\gamma_1 \ldots \gamma_m}$ depending on $g_\varrho, L, a_j, k_j, p_{\gamma_1 \ldots \gamma_m}$ for $j = 1, \ldots, m$. From (12) it follows that d_ϱ-convergence of the sequence $\{u_k(X)\}_{k=1}^\infty$ of elements $u_k(X)$ from $M_\varrho(R)$ implies the convergence of the sequence of derivatives $\{D_{\gamma_1 \ldots \gamma_m} u_k(X)\}_{k=1}^\infty$ for all $\gamma_1 \ldots \gamma_m \in \Delta_\varrho(\gamma)$ and $\varrho = 0, 1, 2$ in the metric

$$\bar{d}(u, v) = \max_{R} \| u(X) - v(X) \| \tag{13}$$

and

$$\bar{\bar{d}}(u, v) = \sup_{R} \| u(X) - v(X) \| . \tag{14}$$

Let the above-mentioned sequence $\{u_k(X)\}_{k=1}^\infty$ be a d_ϱ-Cauchy sequence, i. e.

$$\lim_{k,l \to \infty} d_\varrho(u_k, u_l) = 0 . \tag{15}$$

Hence, with respect to (12), to every $\varepsilon > 0$ and $(\gamma_1 \ldots \gamma_m) \in \Delta_\varrho(\gamma)$ there exists a number $N_1(\varepsilon, \gamma_1, \ldots, \gamma_m)$ such that

$$\| D_{\gamma_1 \ldots \gamma_m} u_k(X) - D_{\gamma_1 \ldots \gamma_m} u_{k+s}(X) \| < \varepsilon \tag{16}$$

for $k > N_1$ and $s = 1, 2, \ldots$ in R.

Since $D_{\gamma_1 \ldots \gamma_m} u_k(X)$ is from B for each $X \in R$ there exists a function $W_{\gamma_1 \ldots \gamma_m}(X)$ with the range of definition R and with the range of the function from B such that $\lim_{k \to \infty} \| D_{\gamma_1 \ldots \gamma_m} u_k(X) - W_{\gamma_1 \ldots \gamma_m}(X) \| = 0$ in the every point $X \in R$. Using (16) we get

$$\| D_{\gamma_1 \ldots \gamma_m} u_k(X) - W_{\gamma_1 \ldots \gamma_m}(X) \| - \| D_{\gamma_1 \ldots \gamma_m} u_{k+s}(X) - W_{\gamma_1 \ldots \gamma_m}(X) \| < \varepsilon .$$

Then, if s tends to infinity $\| D_{\gamma_1 \ldots \gamma_m} u_k(X) - W_{\gamma_1 \ldots \gamma_m}(X) \| \leq \varepsilon$ for $k > N_1$ and $X \in R$. There exists a function $u(X) \in M_\varrho(R)$ such that $\lim_{k \to \infty} \bar{d}(D_{\gamma_1 \ldots \gamma_m} u_k, D_{\gamma_1 \ldots \gamma_m} u) = 0$ for any $(\gamma_1 \ldots \gamma_m) \in \Delta_\varrho(\gamma)$.

Analogically, the equality (15) ensures the existence of a continuous function $Z_{\gamma_1 \ldots \gamma_m}(X)$ on R° with the range of the function from B and such that the sequence $\left\{ \dfrac{p_{\gamma_1 \ldots \gamma_m} x_1^{\gamma_1} \ldots x_m^{\gamma_m}}{x_1^{g_\varrho \sqrt[m]{L} + k_1 - 1} \ldots x_m^{g_\varrho \sqrt[m]{L} + k_m - 1}} D_{\gamma_1 \ldots \gamma_m} u_k(X) \right\}_{k=1}^\infty$ tends to $Z_{\gamma_1 \ldots \gamma_m}$

by the metric (14) for $(\gamma_1 \dots \gamma_m) \in \Delta_\varrho(\gamma)$. Hence we may claim that

$$\left| D_{\gamma_1\dots\gamma_m} u_k(X) - \frac{x_1^{g_\varrho \sqrt[m]{L}+k_1-1} \dots x_m^{g_\varrho \sqrt[m]{L}+k_m-1}}{p_{\gamma_1\dots\gamma_m} x_1^{\gamma_1} \dots x_m^{\gamma_m}} Z_{\gamma_1\dots\gamma_m}(X) \right| <$$

$$< \varepsilon a_1^{g_\varrho \sqrt[m]{L}+k_1-1} \dots a_m^{g_\varrho \sqrt[m]{L}+k_m-1} / p_{\gamma_1\dots\gamma_m} a_1^{\gamma_1} \dots a_m^{\gamma_m}$$

in the domain R° for all $k > N_2(\varepsilon, \gamma_1, \dots, \gamma_m)$. If we denote $N_0(\varepsilon, \gamma_1, \dots, \gamma_m) =$ max (N_1, N_2), then by the inequality

$$\left| D_{\gamma_1\dots\gamma_m} u(X) - \frac{x_1^{g_\varrho \sqrt[m]{L}+k_1-1} \dots x_m^{g_\varrho \sqrt[m]{L}+k_m-1}}{p_{\gamma_1\dots\gamma_m} x_1^{\gamma_1} \dots x_m^{\gamma_m}} Z_{\gamma_1\dots\gamma_m}(X) \right| \leq$$

$$< D_{\gamma_1\dots\gamma_m} u_k(X) - \frac{x_1^{g_\varrho \sqrt[m]{L}+k_1-1} \dots x_m^{g_\varrho \sqrt[m]{L}+k_m-1}}{p_{\gamma_1\dots\gamma_m} x_1^{\gamma_1} \dots x_m^{\gamma_m}} Z_{\gamma_1\dots\gamma_m}(X) +$$

$$+ \| D_{\gamma_1\dots\gamma_m} u_k(X) - D_{\gamma_1\dots\gamma_m} u(X) \|$$

for $k > N_0$, we conclude that

$$Z_{\gamma_1\dots\gamma_m}(X) = p_{\gamma_1\dots\gamma_m} \prod_{j=1}^{m} x_j^{\gamma_j - g_\varrho \sqrt[m]{L} - k_j + 1} D_{\gamma_1\dots\gamma_m} u(X)$$

for $\gamma_1 \dots \gamma_0 \in \Delta_\varrho(\gamma)$ and $X \in R^\circ$, proving e/.

The natural choice for the mapping T is the following operator:

(17)
$$T_\varrho v(X) = G_0(X) +$$

$$+ \int_{\Sigma(k_1,x_1)} \mathrm{d}\mu_1 \dots \int_{\Sigma(k_m,x_m)} f[\Xi, \mathbf{D}_\varrho^0 v(\Xi), \mathbf{D}_\varrho^1 v(\Xi), \dots, \mathbf{D}_\varrho^{n-\nu} v(\Xi)] \, \mathrm{d}\mu_m$$

for $\varrho = 0, 1, 2$, which is easily seen to be a mapping of A_ϱ into itself. Furthermore, for $G_0(X) = G(X)$ the solution of the boundary value problem (1_ϱ), (2) in its equivalent form (3) corresponds to the fixed point of T_ϱ on the set $M_\varrho(R)$ and conversely.

In this case, the sequence of Picard approximations $\{u_k(X)\}_{k=1}^\infty$ and the sequence of iterations $\{T_\varrho^k u_0(X)\}_{k=1}^\infty = \{T_\varrho u_{k-1}(X)\}_{k=1}^\infty$ for any $u_0(X) \in M_\varrho(R)$ are mutually equivalent.

Proof of condition 1°. Let $u(X), v(X)$ be two arbitrary elements of A_ϱ with $d_\varrho(u, v) < +\infty$. Then by (17) and by the hypothesis (7) we obtain:

$$\| D_{\delta_1\dots\delta_m} T_\varrho u(X) - D_{\delta_1\dots\delta_m} T_\varrho v(X) \| \leq$$

$$\leq \int_{\Sigma(k_1-\delta_1,x_1)} \mathrm{d}\mu_1 \dots \int_{\Sigma(k_m-\delta_m,x_m)} \{\| f[\Xi, \mathbf{D}_\varrho^0 u(\Xi), \mathbf{D}_\varrho^1 u(\Xi), \dots, \mathbf{D}_\varrho^{n-\nu} u(\Xi)] -$$

$$-f[(\Xi), \mathbf{D}^0_\varrho v(\Xi), \mathbf{D}^1_\varrho v(\Xi), \ldots, \mathbf{D}^{n-\nu}_\varrho v(\Xi)] \,\|\} \,\mathrm{d}\mu_m \leq$$

$$\leq L \int\limits_{\Sigma(k_1-\delta_1,x_1)} \mathrm{d}\mu_1 \ldots \int\limits_{\Sigma(k_m-\delta_m,x_m)} \frac{\sum\limits_{i=0}^{n-\nu} (\mathbf{P}^i_\varrho, |\ \mathbf{D}^i_\varrho u(\Xi) - \mathbf{D}^i_\varrho v(\Xi)\ )}{\xi_1^{k_1} \ldots \xi_m^{k_m}} \,\mathrm{d}\mu_m \leq$$

$$\leq d_\varrho(u, v) L \int\limits_{\Sigma(k_1-\delta_1,x_1)} \mathrm{d}\mu_1 \ldots \int\limits_{\Sigma(k_m-\delta_m,x_m)} \xi_1^{g_\varrho \sqrt[m]{L}-1} \ldots \xi_m^{g_\varrho \sqrt[m]{L}-1} \,\mathrm{d}\mu_m -$$

$$= d_\varrho(u, v) L \prod_{j=1}^{m} \{x_j^{g_\varrho \sqrt[m]{L}+k_j-\delta_j\ 1} B^{-1}(k_j, \delta_j)\},$$

where

$$B(k_j, \delta_j) = \begin{cases} \prod\limits_{i=0}^{k_j-\delta_j\ 1} (g_\varrho \sqrt[m]{L} + i) & \text{if} \quad \delta_j \leq k_j - 1 \\ 1 & \text{if} \quad \delta_j = k_j \end{cases}$$

for $j = 1, \ldots, m$. Hence we conclude easily that

$$d_\varrho(T_\varrho u, T_\varrho v) \leq \lambda d_\varrho(u, v)$$

with $\lambda = (\sum\limits_{\Delta_\varrho(\delta)} p_{\delta_1 \ldots \delta_m})/g_\varrho$, which ends the proof of 1°.

The proof of condition 2° will be divided into two parts. First of all we prove the condition 2° in the case (6_1). Let $u_k(X) = T_\varrho u_{k-1}(X)$, $k - 1, 2, \ldots$, where $u_0(X)$ is an arbitrary element from A_ϱ, $\varrho = 0, 1, 2$. From (6_1) we find out that

$$x_1^{\delta_1} \ldots x_m^{\delta_m} \,\|\, D_{\delta_1 \ldots \delta_m} u_2(X) - D_{\delta_1 \ldots \delta_m} u_1(X) \,\| \leq 2K_1 x_1^{k_1} \ldots x_m^{k_m} \tag{18}$$

for $(\delta_1 \ldots \delta_m) \in \Delta_\varrho(\delta)$ and $X \in R$. Next, it follows by (18) and (8) that

$$x_1^{\delta_1} \ldots x_m^{\delta_m} \,\|\, D_{\delta_1 \ldots \delta_m} u_3(X) - D_{\delta_1 \ldots \delta_m} u_2(X) \,\| \leq x_1^{\delta_1} \ldots x_m^{\delta_m} \times$$

$$\times \int\limits_{\Sigma(k_1-\delta_1,x_1)} \mathrm{d}\mu_1 \ldots \int\limits_{\Sigma(k_m-\delta_m,x_m)} \sum_{i=0}^{n-\nu} (\mathbf{Q}^i_\varrho, \|\, \mathbf{D}^i_\varrho u_2(\Xi) - \mathbf{D}^i_\varrho u_1(\Xi)\, |^\alpha) \xi_1^{-k_1\beta_1} \ldots \xi_m^{-k_m\beta_m} \,\mathrm{d}\mu_m <$$

$$\leq (2K_1)^\alpha (\sum_{\Delta_\varrho(\delta)} q_{\delta_1 \ldots \delta_m}) \, x_1^{k_1[(\alpha-\beta_1)+1]} \ldots x_m^{k_m[(\alpha-\beta_m)+1]} .$$

By induction with respect to k we get:

$$x_1^{\delta_1} \ldots x_m^{\delta_m} \,\|\, D_{\delta_1 \ldots \delta_m} u_{k+3}(X) - D_{\delta_1 \ldots \delta_m} u_{k+2}(X) \,| \leq$$

$$\leq (2K_1)^{\alpha^{k+1}} (\sum_{\Delta_\varrho(\delta)} q_{\delta_1 \ldots \delta_m})^{1+\alpha+\ldots+\alpha^k} \prod_{j-1}^{m} x_j^{k_j[(\alpha-\beta_j)(1+\alpha+\ldots+\alpha^k)+1]}$$

for all $(\delta_1 \dots \delta_m) \in \Delta_\varrho(\delta)$, $k = 0, 1, \dots$ in the domain R°. Thus

$$(19) \quad \sum_{i\ 0}^{n\ \nu} (\mathbf{P}^i_\varrho, \| \mathbf{D}^i_\varrho u_{k+3}(X) - \mathbf{D}^i_\varrho u_{k+2}(X) \|) \leq (2K_1)^{\alpha^{k+1}} (\sum_{\Delta_\varrho(\delta)} q_{\delta_1 \dots \delta_m})^{1+\alpha+\dots+\alpha^k} \times$$

$$\times \sum_{\Delta_\varrho(\delta)} \{p_{\delta_1 \dots \delta_m} \prod_{j=1}^m [L^{h(k_j-\delta_j)}]^{-1/m}\} \prod_{j=1}^m x_j^{k_j[(\alpha-\beta_j)(1+\alpha+\dots+\alpha^k)+1]} .$$

The fact that $g_\varrho \sqrt[m]{L} + k_j - 1 < k_j(1 - \beta_j)/(1 - \alpha)$ for $j = 1, \dots, m$ ensures the existence of the number $N(g_\varrho)$ such that the inequality

$$k_j[(\alpha - \beta_j)(1 + \alpha + \dots + \alpha^k) + 1] = k_j[(1 - \beta_j)(1 + \alpha + \dots + \alpha^k) + \alpha^{k+1}]$$

$$k_j\{[(1 - \alpha^{k+1})(1 - \beta_j)/(1 - \alpha)] + \alpha^{k+1}\} > g_\varrho \sqrt[m]{L} + k_j - 1$$

holds for all $k > N(g_\varrho)$. This shows in particular that $d_\varrho(u_{k+1}, u_k) < +\infty$ for $k > N(g_\varrho) + 2$.

Finally, condition 2° follows from the property d) of the metric (11).

Now let us investigate the validity of 2° in the case (6_2), $\varrho - 1, 2$. From the assumption (6_2) and by

$$\int_{\Sigma(\varrho,x_1)} \mathrm{d}\mu_1 \dots \int_{\Sigma(\varrho,x_m)} \omega(\Xi)\mathrm{d}\mu_m \leq \int_{\Sigma(\varrho,a_1)} \mathrm{d}\mu_1 \dots \int_{\Sigma(\varrho,a_m)} \omega(\Xi)\mathrm{d}\mu_m \leq K_2$$

we have

$$(20) \quad x_1^{\delta_1} \dots x_m^{\delta_m} \| D_{\delta_1 \dots \delta_m} u(X) - D_{\delta_1 \dots \delta_m} v(X) \leq$$

$$\leq 2x_1^{\delta_1} \dots x_m^{\delta_m} \int_{\Sigma(k_1-\delta_1,x_1)} \mathrm{d}\mu_1 \dots \int_{\Sigma(k_m-\delta_m,x_m)} \omega(\Xi)\, \mathrm{d}\mu_m \leq$$

$$\leq 2K_2 x_1^{k_1-\varrho} \dots x_m^{k_m-\varrho}/(k_1 - \delta_1 - \varrho)! \dots (k_m - \delta_m - \varrho)! \leq 2K_2 x_1^{k_1-\varrho} \dots x_m^{k_m-\varrho}$$

for any $u(X), v(X) \in A_\varrho$, $(\delta_1 \dots \delta_m) \in \Delta_\varrho(\delta)$ and $X \in R$. By means of (8) and (20)

$$x_1^{\delta_1} \dots x_m^{\delta_m} \| D_{\delta_1 \dots \delta_m} u_3(X) - D_{\delta_1 \dots \delta_m} u_2(X) \| \leq (2K_2)^\alpha [\sum_{\Delta_\varrho(\delta)} q_{\delta_1 \dots \delta_m}] \times$$

$$x_1^{\delta_1} \dots x_m^{\delta_m} \int_{\Sigma(k_1-\delta_1,x_1)} \mathrm{d}\mu_1 \dots \int_{\Sigma(k_m-\delta_m,x_m)} [\xi_1^{(k_1-\varrho)\alpha-\beta_1 k_1} \dots \xi_m^{(k_m-\varrho)\alpha-\beta_m k_m}]\mathrm{d}\mu_m \leq$$

$$< (2K_2)^\alpha [\sum_{\Delta_\varrho(\delta)} q_{\delta_1 \dots \delta_m}] \prod_{j=1}^m \{[k_j(\alpha - \beta_j) + \varrho(1 - \alpha)]^{-1} x_j^{k_j(\alpha-\beta_j)+k_j-\varrho\alpha}\}$$

as $(k_j - \varrho)\alpha - \beta_j k_j = k_j(\alpha - \beta_j) - \varrho\alpha > -1$ for $j = 1, \dots, m$. Subsequently we can show that

$$(21) \quad x_1^{\delta_1} \dots x_m^{\delta_m} \| D_{\delta_1 \dots \delta_m} u_{k+3}(X) - D_{\delta_1 \dots \delta_m} u_{k+2}(X) \| \leq$$

$$< (2K_2)^{\alpha^{k+1}} \{ \sum_{\Delta_\varrho(\delta)} q_{\delta_1 \dots \delta_m} \prod_{j=1}^m [k_j(\alpha - \beta_j) + \varrho(1 - \alpha)]^{-1}\}^{1+\alpha+\dots+\alpha^k} \times$$

$$\times \prod_{j=1}^{m} x_j^{k_j(\alpha-\beta_j)(1+\alpha+\ldots+\alpha^k)+k_j-\varrho\alpha^{k+1}}$$

in R° since $k_j(\alpha - \beta_j)(1 + \alpha + \ldots + \alpha^k) - \varrho\alpha^{k+1} > k_j(\alpha - \beta_j) - \varrho\alpha > -1$ for $j = 1, \ldots, m$ and $k = 0, 1, \ldots$. The fact that

$$k_j[(1 - \beta_j)(1 + \alpha + \ldots + \alpha^k) + 1] - \varrho\alpha^{k+1} = k_j[(1 - \alpha^{k+1})(1 - \beta_j)/(1 - \alpha)] + \\ + (k_j - \varrho)\alpha^{k+1}$$

finishes the proof of condition 2° under the assumption (6_2), too.

Proof of 3°. Assume that both $u(X), v(X) \in A_\varrho$ are fixed points of T_ϱ, i. e. $u = T_\varrho u$ and $v = T_\varrho v$. Using the procedure just presented in the proof of condition 2° we obtain the estimates (18), (19) or (20), (21) respectively for the difference $x_1^{\delta_1} \ldots x_m^{\delta_m} \| D_{\delta_1 \ldots \delta_m} u(X) - D_{\delta_1 \ldots \delta_m} v(X) \|$ and from that we obtain easily $d_\varrho(u, v) < +\infty$.

After these verifications of conditions 1°, 2° and 3° of Theorem 1 the conclusion of Theorem 3 follows immediately from Theorem 1.

Before formulating the following Theorem 4 let us define a new metric space.

Let T_ϱ be the operator defined by (17) for $\varrho = 0, 1, 2$ and $T_\varrho M_\varrho(R)$ be the image of $M_\varrho(R)$ under the mapping T_ϱ, i. e. $T_\varrho M_\varrho(R) = \{u(X) = T_\varrho v(X) : v(X) \in \in M_\varrho(R)\}$. In general, the metric space $[T_\varrho M_\varrho(R), d_\varrho^*]$ with the distance function defined on $T_\varrho M_\varrho(R) \times T_\varrho M_\varrho(R)$ by:

$$(22) \qquad d_\varrho^*(u, v) = \max_R \sum_{i=0}^{n-\nu} (\mathbf{I}_\varrho^i, \| \mathbf{D}_\varrho^i u(X) - \mathbf{D}_\varrho^i v(X) \|) ,$$

where $\mathbf{I}_\varrho^i = (1, \ldots, 1)$ is a unit vector with the Card $[\Delta_\varrho^i(\gamma)]$ coordinates for $i = 0, 1, \ldots, n - \nu$ need not be a complete metric space. Then, there exists its completion in the sense of the metric d_ϱ^*, which will be denoted by $[M_\varrho^*(R), d_\varrho^*]$.

From the above definition of $[M_\varrho^*(R), d_\varrho^*]$ we get the following statements.

Remark 1. If the sequence $\{u_k(X)\}_{k=1}^\infty$ of functions $u_k(X) \in M_\varrho^*(R)$ converges in the distance (22) to $u(X)$, then $\{D_{\gamma_1 \ldots \gamma_m} u_k(X)\}_{k=1}^\infty$ converges in (13) to the function $D_{\gamma_1 \ldots \gamma_m} u(X)$ for $(\gamma_1 \ldots \gamma_m) \in \Delta_\varrho(\gamma)$ and thus $u(X) \in M_\varrho(R)$

Remark 2. From Remark 1 we have $M_\varrho^*(R) \subseteq M_\varrho(R)$.

Remark 3. If $v_k(X)$ is from $M_\varrho^*(R)$ for all $k = 1, 2, \ldots$ and the sequence $\{D_{\gamma_1 \ldots \gamma_m} v_k(X)\}_{k-1}^\infty$ converges by the metric (13) to a continuous function for all $(\gamma_1 \ldots \gamma_m) \in \Delta_\varrho(\gamma)$ such that $\lim_{k\to\infty} \bar{d}(v_k, v) = 0$, then $\lim_{k\to\infty} d_\varrho^*(v_k, v)$ 0 and $v(X) \in M_\varrho^*(R)$.

Theorem 4. *Assume for* $\varrho = 0, 1$.:

i) *The continuous operator* $f(X, \mathbf{U}_\varrho^0, \mathbf{U}_\varrho^1, \ldots, \mathbf{U}_\varrho^{n-\nu})$ *maps* E_ϱ *into* B *and*

$$(23) \qquad \| f(X, \mathbf{U}^0_\varrho, \mathbf{U}^1_\varrho, \ldots, \mathbf{U}^{n-\nu}_\varrho) \| \leq A x_1^{k_1 g_{\varrho 1}} \ldots x_m^{k_m g_{\varrho m}}, \quad A > 0$$

in E_ϱ, *where we take*

$$(23_1) \qquad g_{\varrho j} \geq 0 \quad \textit{for} \quad \varrho = 0, 1$$

or

$$(23_2) \qquad -1 < k_j g_{\varrho j} < 0 \quad \textit{for} \quad \varrho = 1$$

for $j = 1, \ldots, m$.

ii) *In* E^0_ϱ

$$(24) \qquad \| f(X, \mathbf{U}^0_\varrho, \mathbf{U}^1_\varrho, \ldots, \mathbf{U}^{n-\nu}_\varrho) - f(X, \mathbf{V}^0_\varrho, \mathbf{V}^1_\varrho, \ldots, \mathbf{V}^{n-\nu}_\varrho) \| \leq$$

$$\leq x_1^{k_1 r_1} \ldots x_m^{-k_m r_m} \sum_{i=0}^{n-\nu} (\mathbf{F}^i_{\varrho q}, \| \mathbf{U}^i_\varrho - \mathbf{V}^i_\varrho \|^q), \quad q \geq 1,$$

where $\mathbf{F}^i_{\varrho q} = (\ldots, f_{\gamma_1 \ldots \gamma_m} [x_1^{\gamma_1} \ldots x_m^{\gamma_m}]^q, \ldots)$, $(\gamma_1 \ldots \gamma_m) \in \Delta^i_\varrho$ *is a vector with the* Card $[\Delta^i_\varrho(\gamma)]$ *components. The factors* $f_{\gamma_1 \ldots \gamma_m}$ *are positive constants.*

iii) *The real numbers* $A, g_{\varrho j}, r_j, f_{\gamma_1 \ldots \gamma_m}$ *and* q *are connected by the following relations:*

$$(g_{\varrho j} + 1) q - r_j = g_{\varrho j}, \; j = 1, \ldots, m$$

and

$$(2A)^{q-1} / [C(g)]^q \sum_{\Delta_\varrho(\gamma)} f_{\gamma_1 \ldots \gamma_m} < 1,$$

where

$$C(g) = \begin{cases} \min\limits_{j = 1, \ldots, m} (k_j g_{\varrho j} + 1) & \textit{if} \quad g_{\varrho j} \geq 0, \quad j = 1, \ldots, m \\ \prod\limits_{j=1}^{m} (k_j g_{\varrho j} + 1) & \textit{if} \quad -1 < k_j g_{\varrho j} < 0, \quad j = 1, \ldots, m. \end{cases}$$

Then there exists one and only one solution $u(X)$ *from the class* $M^*_\varrho(R)$ *of the boundary value problem* (1_ϱ), (2) *and moreover the Picard sequence of successive approximations* $\{u_k(X)\}_{k=1}^{\infty}$ *defined by* (5) *for any function* $u_0(X) \in M_\varrho(R)$ *converges uniformly in* R *by the norm defined in* B *to this unique solution, i. e.* $\lim\limits_{k \to \infty} \| u_k(X) - u(X) \| = 0$ *uniformly in* R.

Proof. The proof of this result will be carried out in the same way as that of Theorem 3. Here we choose the space $A^*_\varrho = [M^*_\varrho(R), d_\varrho]$ for A, where the distance function $d_\varrho(u, v)$ is defined by:

$$(25) \qquad d_\varrho(u, v) = \sup_R \frac{\sum\limits_{i=0}^{n-\nu} (\mathbf{F}^i_{\varrho 1}, \| \mathbf{D}^i_\varrho u(X) - \mathbf{D}^i_\varrho v(X) \|)}{x_1^{k_1(g_{\varrho 1}+1)} \ldots x_m^{k_m(g_{\varrho m}+1)}}$$

on $A_\varrho^* \times A_\varrho^*$. It is easy to see that (25) fulfils the requirements a), b), c) and d) for a metric considered in the section II. In view of the assumption (23_1) and (23_2) we get immediately

$$\max_R \sum_{i-0}^{n-\nu} (\bar{\mathbf{S}}_\varrho^i, \| \mathbf{D}_\varrho^i u(X) - \mathbf{D}_\varrho^i v(X) \|) \leq d_\varrho(u, v), \tag{26}$$

where $\bar{\mathbf{S}}_\varrho^i = (\ldots, \bar{s}_{\gamma_1 \ldots \gamma_m}, \ldots)$ denotes a vector with the Card $[\Delta_\varrho^i(\gamma)]$ coordinates $\bar{s}_{\gamma_1 \ldots \gamma_m}$ depending on $k_j, g_{\varrho j}, a_j, f_{\gamma_1 \ldots \gamma_m}$. Hence it follows that d_ϱ-convergence of a sequence $\{u_k(X)\}_{k=1}^\infty$ of elements $u_k(X)$ from A_ϱ^* implies the convergence of the sequence $\{D_{\gamma_1 \ldots \gamma_m} u_k(X)\}_{k=1}^\infty$ in the sense of metric (13) for every $(\gamma_1 \ldots \gamma_m)$ $\Delta_\varrho(\gamma)$. Let $\{u_k(X)\}_{k\ 1}^\infty$ be a d_ϱ-Cauchy sequence of elements from A_ϱ^*, i. e. $\lim_{k,l\to\infty} d_\varrho(u_k, u_l) = 0$. To prove that the space A_ϱ^* is complete it is sufficient to show that there exists an element $u(X) \in M_\varrho^*(R)$ satisfying the condition

$$\lim_{k\to\infty} \bar{d}[D_{\gamma_1 \ldots \gamma_m} u_k(X),\ D_{\gamma_1 \ldots \gamma_m} u(X)] = 0$$

for $(\gamma_1 \ldots \gamma_m) \in \Delta_\varrho(\gamma)$. Then following the same procedure as in the previous theorem we obtain the desired equality: $\lim_{k\to\infty} d_\varrho(u_k, u) = 0$. The existence of the above-mentioned element $u(X)$ is guaranteed by (26) and by Remark 3.

We choose the same mapping T_ϱ as defined by (17) for $\varrho = 0, 1$. Then the Picard sequence $\{u_k(X)\}_{k=1}^\infty$ by (5) is equivalent to the sequence of iterations $\{T_\varrho^k u_0(X)\}_{k=1}^\infty$ and $\{T_\varrho u_{k-1}(X)\}_{k=1}^\infty$ for any $u_0(X) \in M_\varrho^*(R)$.

Proof of condition 1°. Let $u(X)$, $v(X)$ be arbitrary elements of A_ϱ^*. Then there exist sequences $\{u_k(X)\}_{k=1}^\infty$, $\{v_k(X)\}_{k=1}^\infty$ of elements $u_k(X)$, $v_k(X)$ from $T_\varrho M_\varrho(R)$ such that $\lim_{k\to\infty} d_\varrho^*(u_k, u) = \lim_{k\to\infty} d_\varrho^*(v_k, v) = 0$. By

$$x_1^{\delta_1} \ldots x_m^{\delta_m} \| D_{\delta_1 \ldots \delta_m} u(X) - D_{\delta_1 \ldots \delta_m} v(X) \| \leq x_1^{\delta_1} \ldots x_m^{\delta_m} [\, | D_{\delta_1 \ldots \delta_m} u_k(X) - \\ - D_{\delta_1 \ldots \delta_m} u(X) \| + \| D_{\delta_1 \ldots \delta_m} u_k(X) - D_{\delta_1 \ldots \delta_m} v_k(X) \| + \\ + \| D_{\delta_1 \ldots \delta_m} v_k(X) - D_{\delta_1 \ldots \delta_m} v(X) \|]$$

and by (23), using Remark 1 we obtain the following estimate:

$$x_1^{\delta_1} \ldots x_m^{\delta_m} \| D_{\delta_1 \ldots \delta_m} u(X) - D_{\delta_1 \ldots \delta_m} v(X) \| < \\ \leq x_1^{\delta_1} \ldots x_m^{\delta_m} \lim_{k\to\infty} \| D_{\delta_1 \ldots \delta_m} u_k(X) - D_{\delta_1 \ldots \delta_m} v_k(X) \| \leq x_1^{\delta_1} \ldots x_m^{\delta_m} \\ \times \lim_{k\to\infty} \int_{\Sigma(k_1 - \delta_1, x_1)} \mathrm{d}\mu_1 \ldots \int_{\Sigma(k_m - \delta_m, x_m)} [\| f(\Xi, \mathbf{D}_\varrho^0 u_k, \mathbf{D}_\varrho^1 u_k, \ldots, \mathbf{D}_\varrho^{n\ \nu} u_k) \ + \\ + \ | f(\Xi, \mathbf{D}_\varrho^0 v_k, \mathbf{D}_\varrho^1 v_k, \ldots, \mathbf{D}_\varrho^{n\ \nu} v_k) \|] \,\mathrm{d}\mu_m \leq 2A \prod_{j=1}^m \{x_j^{k_j g_{\varrho j} + k_j} \bar{B}^{-1}(k_j, \delta_j)\},$$

where

$$\bar{B}(k_j, \delta_j) = \begin{cases} \prod_{i=1}^{k_j-\delta_j} (k_j g_{\varrho j} + i) & \text{if} \quad \delta_j \le k_j - 1 \\ 1 & \text{if} \quad \delta_j = k_j \end{cases}$$

and $u_k(X)$, $v_k(X)$ are originals corresponding to the images $u_k(X)$, $v_k(X)$ under the mapping T_ϱ.

Thus in both cases $\varrho = 0, 1$ we have

$$(27) \qquad x_1^{\delta} \dots x_m^{\delta_m} \; D_{\delta_1\dots\delta_m} u(X) - D_{\delta_1\dots\delta_m} v(X) \mid \le [2A/C(g)] \prod_{j=1}^{m} x_j^{k_j(g_{\varrho j}+1)} .$$

If $d_\varrho(u, v) < +\infty$, then from (24) and (27) we conclude

$$x_1^{\delta_1} \dots x_m^{\delta_m} \mid D_{\delta_1\dots\delta_m} T_\varrho u(X) - D_{\delta_1\dots\delta_m} T_\varrho v(X) \| \le x_1^{\delta_1} \dots x_m^{\delta_m} \times$$

$$\times \int\limits_{\Sigma(k_1 \;\delta_1, x_1)} d\mu_1 \dots \int\limits_{\Sigma(k_m-\delta_m, x_m)} \frac{\sum_{i \; 0}^{n-v} (\mathbf{F}^i_{\varrho q}, \| \mathbf{D}^i_\varrho u(\Xi) - \mathbf{D}^i_\varrho v(\Xi) \|^q)}{\xi_1^{k_1 r_1} \dots \xi_m^{k_m r_m}} d\mu_m \le$$

$$\le x_1^{\delta_1} \dots x_m^{\delta_m} [2AC^{-1}(g)]^{q-1} \times$$

$$\int\limits_{\Sigma(k_1 \; \delta_1, x_1)} d\mu_1 \dots \int\limits_{\Sigma(k_m-\delta_m, x_m)} \{ \sum_{i-0}^{n-v} (\mathbf{F}^i_{\varrho 1}, \| \mathbf{D}^i_\varrho u - \mathbf{D}^i_\varrho v \|) / \prod_{j=1}^{m} \xi_j^{k_j(g_{\varrho j}+1)} \times$$

$$\times \prod_{j \; 1}^{m} \xi_j^{k_j[(g_{\varrho j}+1)(q-1) \; r_j \; g_{\varrho j}+1]} \} d\mu_m \le$$

$$< (2A)^{q-1} [C(g)] \;^q d_\varrho(u, v) x_1^{k_1(g_{\varrho 1}+1)} \dots x_m^{k_m(g_{\varrho m}+1)}$$

in the domain R° and for all $(\delta_1 \dots \delta_m) \in \Delta_\varrho(\delta)$. Hence and by the definition of d_ϱ the inequality $d_\varrho(T_\varrho u, T_\varrho v) \le \lambda d_\varrho(u, v)$ follows, where

$$\lambda - (2A)^{q-1} [C(g)]^{-q} \sum_{\Delta_\varrho(\delta)} f_{\delta_1\dots\delta_m} < 1, \text{ proving } 1^\circ.$$

The proofs of conditions 2° and 3° are trivial in this case, as the required estimates are already given by (27).

Consequently, we have proved that the sequence of iterations $\{T^k_\varrho u_0(X)\}_{k-1}^{\infty}$ for any element $u_0(X) \in M^*_\varrho(R)$ converges uniformly on R in the norm defined in B to the unique solution $u(X)$ of the problem (1_ϱ), (2) from the class $M^*_\varrho(R)$. For any function $u_0(X) \in M_\varrho(R)$ there exists an element $u_0(X)$ from $T_\varrho M_\varrho(R)$ such that $u_0 \quad T_\varrho u_0$, whence it follows that also the sequence $\{T^k u_0(X)\}_{k=1}^{\infty}$ converges uniformly in R to $u(X)$. This proves our Theorem 4.

Remark 4. It follows from (23) that the operator $f(X, \mathbf{U}_0^0, \mathbf{U}_0^1, \dots, \mathbf{U}_0^{n \; 1})$

is bounded in E_0. In the following theorem we show that the requirement of boundedness is not necessary.

Theorem 5. *If (for $\varrho = 0$)*

i) $f(X, \mathbf{U}_0^0, \mathbf{U}_0^1, \ldots, \mathbf{U}_0^{n-1})$ *is a continuous mapping defined on E_0 into the Banach space B,*

ii) *in the domain E_0^0*

$$\|f(X, \mathbf{U}_0^0, \mathbf{U}_0^1, \ldots, \mathbf{U}_0^{n-1})\| \leq A(X) x_1^{k_1 g_1} \ldots x_m^{k_m g_m}, \ -1 < k_j g_j < 0, \ j = 1, \ldots, m, \tag{28}$$

where the real-valued function $A(X)$ is continuous on R° and satisfies the inequality

$$A(X) \leq A_0 \prod_{j=1}^{m} x_j^{-\gamma_j g_j h(k_j - \gamma_j)}, \ A_0 > 0, \ (\gamma_1 \ldots \gamma_m) \in \Delta_0(\gamma)$$

with the function $h(x)$ of the one variable x determined by

$$h(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

iii) *further, in E_0^0*

$$\|f(X, \mathbf{U}_0^0, \mathbf{U}_0^1, \ldots, \mathbf{U}_0^{n-1}) - f(X, \mathbf{V}_0^0, \mathbf{V}_0^1, \ldots, \mathbf{V}_0^{n-1}) \leq$$

$$\leq [C(X)/x_1^{k_1 r_1} \ldots x_m^{k_m r_m}] \prod_{i=0}^{n-1} (\mathbf{H}_q^i, \| \mathbf{U}_0^i - \mathbf{V}_0^i \|^q), \ q \geq 1,$$

where $H_q^i = (\ldots, h_{\gamma_1 \ldots \gamma_m} \{\prod_{j=1}^{m} x_j^{\gamma_j [g_j h(k_j - \gamma_j) + 1]}\}^q, \ldots)$ *denotes the vector with the* Card $[\Delta_0^i(\gamma)]$ *components and $h_{\gamma_1 \ldots \gamma_m}$ are positive constants for $(\gamma_1 \ldots \gamma_m) \in \Delta_0(\gamma)$. The real-valued function $C(X)$ is continuous on R° and*

$$C(X) \leq C_0 \prod_{j=1}^{m} x_j^{-\gamma_j g_j h(k_j - \gamma_j)}, \ C_0 > 0$$

for every $(\gamma_1 \ldots \gamma_m) \in \Delta_0(\gamma)$,

iiii) *the constants A_0, C_0, g_0, r_j, q and $h_{\gamma_1 \ldots \gamma_m}$ are connected by*

$$(g_j + 1)q - r_j - g_j, \quad j = 1, \ldots, m$$

$$\{(2A_0)^q \ 1/[\prod_{j=1}^{m} (k_j g_j + 1)]^q\} C_0 \sum_{\Delta_0(\gamma)} h_{\gamma_1 \ldots \gamma_m} < 1 ,$$

then there exists one and only one solution $u(X)$ from the class $M_0^*(R)$ of the boundary value problem (1_0), (2) *and furthermore the Picard sequence of successive approximations $\{u_k(X)\}_{k=1}^{\infty}$ by* (5) *for an arbitrary function $u_0(X) \in M_0(R)$*

converges uniformly in R in the sense of the norm defined in B to this unique solution, i. e. $\lim_{k\to\infty} \| u_k(X) - u(X) \| = 0$ *uniformly in R.*

Proof. We shall apply again the result of Theorem 1 to prove the statement of Theorem 5. Since this proof is similar to that of Theorem 4 in the essential features, we shall indicate it only.

Here we choose the space $A_0^* = [M_0^*(R), \bar{d}_0]$ with the distance function $\bar{d}_0$ defines on $A_0^* \times A_0^*$:

$$\bar{d}_0(u, v) = \sup_R \frac{\sum_{i-0}^{n\ 1} (\mathbf{H}_1^i, \| \mathbf{D}_0^i u(X) - \mathbf{D}_0^i v(X) \|)}{x_1^{k_1(g_1+1)} \dots x_m^{k_m(g_m+1)}} \tag{30}$$

and the mapping T_0 defined by (17) for $\varrho \quad 0$. The space A_0^* is complete.

Let $u(X), v(X)$ be arbitrary functions of A_0^* with $\bar{d}_0(u, v) < +\infty$, then

$$\prod_{j=1}^{m} x_j^{\delta_j[g_j h(k_j - \delta_j)+1]} \| D_{\delta_1 \dots \delta_m} u(X) - D_{\delta_1 \dots \delta_m} v(X) \| \leq \tag{31}$$

$$\leq 2 \prod_{j-1}^{m} x_j^{\delta_j[g_j h(k_j \ \delta_j)+1]} \times$$

$$\times \int_{\Sigma(k_1 - \delta_1, x_1)} d\mu_1 \dots \int_{\Sigma(k_m \ \delta_m, x_m)} A(\Xi) \xi_1^{k_1 g_1} \dots \xi_m^{k_m g_m} d\mu_m \leq$$

$$\leq 2A_0 \, [\prod_{j\ 1}^{m} (k_j g_j + 1)]^{-1} x_1^{k_1(g_1+1)} \dots x_m^{k_m(g_m+1)}$$

for $X \in R$ and $(\delta_1 \dots \delta_m) \in \Delta_0(\delta)$. Hence and by (29) we get the required estimate

$$\prod_{j\ 1}^{m} x_j^{\delta_j[g_j h(k_j \ \delta_j)+1]} \| D_{\delta_1 \dots \delta_m} T_0 u(X) - D_{\delta_1 \dots \delta_m} T_0 v(X) \| \leq$$

$$\leq (2A_0)^{q-1} [\prod_{j\ 1}^{m} (k_j g_j + 1)]^{-q} C_0 x_1^{k_1(g_1+1)} \dots x_m^{k_m(g_m+1)}$$

and so we may claim that $\bar{d}_0(T_0 u, T_0 v) \leq \lambda \bar{d}_0(u, v)$, where $0 < \lambda < 1$.

The necessary estimates for the proof of conditions 2°, 3° are given in (31), proving this theorem.

In the following theorem we shall employ an extension of the classical condition of the uniqueness due to Nagumo to prove both the uniqueness and the existence of the solution of the problem (1_ϱ), (2) ($\varrho = 1, 2$) and the convergence of Picard successive approximations.

We shall use for the considerations the complete metric space $[M_\varrho^*(R), d_\varrho^*]$, which is a completion of the space $[T_\varrho M_\varrho(R), d_\varrho^*]$ for $\varrho = 1, 2$ in the sense of the distance d_ϱ^* given by (22). (See Remarks 1, 2 and 3 in this section.)

Theorem 6. *For* $\varrho = 1, 2$ *let* $f(X, \mathbf{U}_\varrho^0, \mathbf{U}_\varrho^1, \ldots, \mathbf{U}_\varrho^{n-\nu}$ *be a continuous mapping defined on* E_ϱ *into* B *and satisfying the following conditions*

$$\| f(X, \mathbf{U}_\varrho^0, \mathbf{U}_\varrho^1, \ldots, \mathbf{U}_\varrho^{n-\nu}) \| \leq K \quad in \quad E_\varrho \tag{32}$$

$$\| f(X, \mathbf{U}_\varrho^0, \mathbf{U}_\varrho^1, \ldots, \mathbf{U}_\varrho^{n-\nu}) - f(X, \mathbf{V}_\varrho^0, \mathbf{V}_\varrho^1, \ldots, \mathbf{V}_\varrho^{n-\nu}) \| \leq \tag{33}$$

$$x_1^{1-k_1} \ldots x_m^{1-k_m} \sum_{i=0}^{n-\nu} [\tilde{\mathbf{P}}_\varrho^i(X), \| \mathbf{U}_\varrho^i - \mathbf{V}_\varrho^i \|]$$

in the domain E_ϱ^0, *where* $\tilde{\mathbf{P}}_\varrho^i(X) = (\ldots, \tilde{p}_{\gamma_1 \ldots \gamma_m} x_1^{\gamma_1} \ldots x_m^{\gamma_m}, \ldots)$, $(\gamma_1 \ldots \gamma_m) \in \Delta_\varrho^i(\gamma)$ *denotes the vector with the* Card $\Delta_\varrho^i(\gamma)]$ *components. The factors* $\tilde{p}_{\gamma \ldots \gamma_m}$ *are non-negative constants one of which at least is non-vanishing, such that* $a_1 \ldots a_m \sum_{\Delta_\varrho(\gamma)} p_{\gamma_1 \ldots \gamma_m} \leq 1$. *Then there exists one and only one solution* $u(X)$ *from the class* $M_\varrho^*(R)$ *of the boundary value problem* (1_ϱ), (2) $\varrho = 1, 2$ *and furthermore the Picard sequence of successive approximations* $\{u_k(X)\}_{k=1}^\infty$ *defined by* (5) *for any function* $u_0(X) \in M_\varrho(R)$ *converges uniformly on* R *in the norm defined in* B *to this unique solution, i. e.* $\lim_{k\to\infty} \| u_k(X) - u(X) \| = 0$ *uniformly in* R.

Proof. Now we shall apply the result of Theorem 2 to prove Theorem 6. To do this, we may choose the space $\tilde{A}_\varrho^* = [M_\varrho^*(R), \tilde{d}_\varrho^*]$ metrized by

$$\tilde{d}_\varrho^*(u, v) = \sup_{R^\circ} \frac{\sum_{i=0}^{n-\nu} [\tilde{\mathbf{P}}_\varrho^i(X), \| \mathbf{D}_\varrho^i u(X) - \mathbf{D}_\varrho^i v(X) \|]}{x_1^{k_1-1} \ldots x_m^{k_m-1}} \tag{34}$$

as the space in the meaning A and the mapping T_ϱ in the meaning T $(T_\varrho \tilde{A}_\varrho^* \subseteq \tilde{A}_\varrho^*)$. Notice that by (17) and (32) we get for $X \in R$

$$\| D_{\delta_1 \ldots \delta_m} u(X) - D_{\delta_1 \ldots \delta_m} v(X) \| \leq \| D_{\delta_1 \ldots \delta_m} u_k(X) - D_{\delta_1 \ldots \delta_m} u(X) \mid +$$
$$+ \mid D_{\delta_1 \ldots \delta_m} u_k(X) - D_{\delta_1 \ldots \delta_m} v_k(X) \| + \| D_{\delta_1 \ldots \delta_m} v_k(X) - D_{\delta_1 \ldots \delta_m} v(X) \mid \leq$$
$$\leq 2K x_1^{k_1-\delta_1} \ldots x_m^{k_m-\delta_m}$$

for $(\delta_1 \ldots \delta_m) \in \Delta_\varrho(\delta)$ and any $u(X), v(X) \in M_\varrho^*(R)$ and $u_k(X)$, $v_k(X)$ from $T_\varrho M_\varrho(R)$ such that $\lim_{k\to\infty} d_\varrho^*(u_k, u) = \lim_{k\to\infty} d_\varrho^*(v_k, v) = 0$. Hence by means of (34) we have

$$\tilde{d}_\varrho^*(u, v) \leq 2K\, a_1 \ldots a_m \sum_{\Delta_\varrho(\gamma)} \tilde{p}_{\gamma_1 \ldots \gamma_m} < +\infty .$$

The completeness of $\tilde{A}_\varrho^*$ would be proved by the same procedure as in Theorem 4 and therefore we omit it.

Proof of 1 . Let $u(X)$, $v(X)$ be two distinct elements from $\tilde{A}_{\varrho}^{*}$ and

$$B_{uv}(X) - \begin{cases} \dfrac{\sum\limits_{i\ 0}^{n\ v}[\tilde{\mathbf{P}}_{\varrho}^{i}(X),\ \| \mathbf{D}_{\varrho}^{i}u(X) - \mathbf{D}_{\varrho}^{i}v(X) \|]}{x_1^{k_1\ 1} \dots x_m^{k_m\ 1}} & \text{if} \quad X \in R^{\circ} \\ 0 & \text{if} \quad X \in R - R^{\circ}. \end{cases}$$

From the inequality

$$B_{uv}(X) \leq 2K\, x_1 \dots x_m \sum_{\Delta_{\varrho}(\gamma)} \tilde{p}_{\gamma_1 \dots \gamma_m}$$

it is obvious that $\lim\limits_{R \ni X \to Y} B_{uv}(X) = 0$ for $Y \in R - R^{\circ}$. Then, the function $B_{uv}(X)$ is continuous in R for any $u, v \in M_{\varrho}^{*}(R)$. There exists a point $Z = (z_1, \dots, z_m) \in \in R$ in which the function $B_{uv}(X)$ attains its maximum, i. e. $B_{uv}(Z) = \tilde{d}_{2}^{*}(u, v)$.

Consider the following estimate:

$$(35) \qquad \sum_{i\ 0}^{n\ v}[\tilde{\mathbf{P}}_{\varrho}^{i}(X),\ \| \mathbf{D}_{\varrho}^{i}T_{\varrho}u(X) - \mathbf{D}_{\varrho}^{i}T_{\varrho}v(X) \|] \leq \sum_{\Delta_{\varrho}(\gamma)} \{\tilde{p}_{\gamma_1 \dots \gamma_m} x_1^{\gamma_1} \dots x_m^{\gamma_m} \times$$

$$\int\limits_{\Sigma(k_1\ \gamma_1, x_1)} \mathrm{d}\mu_1 \dots \int\limits_{\Sigma(k_m - \gamma_m, x_m)} \frac{\sum\limits_{i-0}^{n-v}[\tilde{\mathbf{P}}_{\varrho}^{i}(\Xi),\ \| \mathbf{D}_{\varrho}^{i}u(\Xi) - \mathbf{D}_{\varrho}^{i}v(\Xi) \|]}{x_1^{k_1 - 1} \dots x_m^{k_m - 1}}\, \mathrm{d}\mu_m \Bigg\} \leq$$

$$\leq \tilde{d}_{\varrho}^{*}(u, v)\, x_1^{k_1} \dots x_m^{k_m} \sum_{\Delta_{\varrho}(\gamma)} \tilde{p}_{\gamma_1 \dots \gamma_m}.$$

By the definition of $\tilde{d}_{\varrho}^{*}$ we obtain $\tilde{d}_{\varrho}^{*}(T_{\varrho}u, T_{\varrho}v) \leq \tilde{d}_{\varrho}^{*}(u, v)$. We have to prove that the equality cannot occur in (35), i. e. we must have $\tilde{d}_{\varrho}^{*}(T_{\varrho}u, T_{\varrho}v) < < \tilde{d}_{\varrho}^{*}(u, v)$. Assume the contrary, then there exists a point $\bar{Z} = (\bar{z}_1, \dots, \bar{z}_m)$ from R such that

$$\tilde{d}_{\varrho}^{*}(u, v) = \frac{\sum\limits_{i\ 0}^{n\ v}[\tilde{\mathbf{P}}_{\varrho}^{i}(Z),\ \| \mathbf{D}_{\varrho}^{i}u(Z) - \mathbf{D}_{\varrho}^{i}v(Z) \|]}{z_1^{k_1 - 1} \dots z_m^{k_m - 1}} =$$

$$= \frac{\sum\limits_{i-0}^{n\ v}[\tilde{\mathbf{P}}_{\varrho}^{i}(\bar{Z}),\ \| \mathbf{D}_{\varrho}^{i}T_{\varrho}u(\bar{Z}) - \mathbf{D}_{\varrho}^{i}T_{\varrho}v(\bar{Z}) \|]}{\bar{z}_1^{k_1 - 1} \dots \bar{z}_m^{k_m - 1}} = \tilde{d}_{\varrho}^{*}(T_{\varrho}u, T_{\varrho}v)$$

It follows from (17) and (33)

$$\tilde{d}_{\varrho}^{*}(u, v) = \tilde{d}_{\varrho}^{*}(T_{\varrho}u, T_{\varrho}v) \leq [\bar{z}_1^{k_1 - 1} \dots \bar{z}_m^{k_m - 1}]^{-1} \sum_{\Delta_{\varrho}(\gamma)} \{p_{\gamma_1 \dots \gamma_m} \bar{z}_1^{\gamma_1} \dots \bar{z}_m^{\gamma_m} \times$$

$$\times \int\limits_{\Sigma(k_1-\gamma_1,z_1)} d\mu_1 \dots \int\limits_{\Sigma(k_m-\gamma_m,z_m)} \frac{\sum_{i=0}^{n-\nu} [\tilde{\mathbf{P}}^i_\varrho(\Xi),\ \| \mathbf{D}^i_\varrho u(\Xi) - \mathbf{D}^i_\varrho v(\Xi) \|]}{\xi_1^{k_1-1} \dots \xi_m^{k_m\ 1}} d\mu_m \Bigg\} <$$

$$< \bar{d}^*_\varrho(u, v),$$

which is the desired contradiction.

Proof of 2°. Let $u_0(X)$ be an arbitrary element of $M^*_\varrho(R)$. Let us consider the family of iterates $\{T^k_\varrho u_0(X)\}_{k=1}^\infty = \{T_\varrho u_{k\ 1}(X)\}_{k-1}^\infty$. From (17) and from the hypothesis (32)

$$\| D_{\delta_1\dots\delta_m} T_\varrho u_k(X) - D_{\delta_1\dots\delta_m} T_\varrho u_k(Y) \| \le \| D_{\delta_1\dots\delta_m} G_0(X) - D_{\delta_1\dots\delta_m} G_0(Y) \ +$$

$$+ \| \sum_{j-1}^{m} \int\limits_{\Sigma(k_1-\delta_1,y_1)} d\mu_1 \dots \int\limits_{\Sigma(k_j\ 1-\delta_j\ 1,y_{j-1})} d\mu_{j-1} \times$$

$$\times \{ \int_{y_j}^{x_j} d\varphi_1 \int_0^{\varphi_1} d\varphi_2 \dots \int_0^{\varphi_{k_j\ \delta\ -1}} d\xi_j \} \times$$

$$\int\limits_{\Sigma(k_j+1\ \delta_j+1,x_j+1)} d\mu_{j+1} \dots \int\limits_{\Sigma(k_m-\delta_m,x_m)} f(\Xi, u_{k\ 1}, \mathbf{D}^1_\varrho u_{k-1}, \dots, \mathbf{D}^{n-\nu}_\varrho u_{k\ 1})\, d\mu_m \ \le$$

$$\le \| D_{\delta_1\dots\delta_m} G_0(X) - D_{\delta_1\dots\delta_m} G_0(Y) \| +$$

$$+ K \left\{ \sum_{j\ 1}^{m} |x_j - y_j| \frac{a_j^{k_j\ \delta_j-1}}{(k_j - \delta_j - 1)!} \prod_{\substack{l-1\\ l\neq j}}^{m} \frac{a_1^{k_l\ \delta_l}}{(k_l - \delta_l)!} \right\}$$

and

$$\| D_{\delta_1\dots\delta_m} T_\varrho u_k(X) \| \le \max_{X\in R} \| D_{\delta_1\dots\delta_m} G_0(X) \| + K \prod_{j\ 1}^{m} \frac{a_j^{k_j-\delta_j}}{(k_j - \delta_j)!}$$

for X, $Y \in R$ and for every $(\delta_1 \dots \delta_m) \in \Delta_\varrho(\delta)$, $k = 0, 1, \dots$. The above inequalities guarantee the equicontinuity and uniform boundedness of the sequence iterates. As a consequence of the generalized Ascoli theorem we are able to choose successively a subsequence $\{T^{k_\omega}_\varrho u_0(X)\}_{\omega=1}^\infty$, which converges to $\psi(X)$ in the metric (13) together with the corresponding sequence of derivatives $\{D_{\delta_1\dots\delta_m} T^{k_\omega}_\varrho u_0(X)\}_{\omega=1}^\infty$ such that $\lim_{\omega\to\infty} \bar{d}[D_{\delta_1\dots\delta_m} T^{k_\omega}_\varrho u_0(X),\ D_{\delta_1\dots\delta_m}\psi(X)]\quad 0$ for $(\delta_1 \dots \delta_m) \in \Delta_\varrho(\delta)$, whence it follows that the function $\psi(X)$ is from $M^*_\varrho(R)$.

We must show that the sequence $\{T^{k_\omega}_\varrho u_0(X)\}_{\omega\ 1}^\infty$ tends to $\psi(X)$ also in the distance given by (34).

Notice that for $\varepsilon > 0$ there exists (on account of the continuity $B_{uv}(X)$ in R) $\delta > 0$ such that for all ω we have

$$\sup_{O} \frac{\sum_{i=0}^{n-\nu} [\tilde{\mathbf{P}}_\varrho^i X), \| \mathbf{D}_\varrho^i T_\varrho^{k_\omega} u_0(X) - \mathbf{D}_\varrho^i \psi(X) \|]}{x_1^{k_1-1} \dots x_m^{k_m-1}} < \varepsilon,$$

where $O = \bigcup_{j=1}^{m} O_j$ and $O_j = \{X : X \in R^\circ \text{ and } 0 < x_j \leq \delta\}$. Since the sequence $\{T_\varrho^{k_\omega} u_0(X)\}_{\omega=1}^{\infty}$ converges to $\psi(X)$ in the metric (22), we may choose a positive integer $N(\delta)$ such that for $\omega \geq N(\delta)$

$$\sup_{R-O} \{\sum_{i=0}^{n-\nu} [\tilde{\mathbf{P}}_\varrho^i(X), \| \mathbf{D}_\varrho^i T_\varrho^{k_\omega} u_0(X) - \mathbf{D}_\varrho^i \psi(X) \|]\} < \varepsilon \delta^{k_1+\dots+k_m-m}.$$

Hence we obtain

$$\tilde{d}_\varrho^*(T^{k_\omega} u_0, \psi) \leq \max \left\{ \sup_{O} \frac{\sum_{i=0}^{n-\nu} [\tilde{\mathbf{P}}_\varrho^i(X), \| \mathbf{D}_\varrho^i T_\varrho^{k_\omega} u_0(X) - \mathbf{D}_\varrho^i \psi(X) \|]}{x_1^{k_1-1} \dots x_m^{k_m-1}} ; \right.$$

$$\left. \sup_{R-O} \frac{\sum_{i=0}^{n-\nu} [\tilde{\mathbf{P}}_\varrho^i(X), \| \mathbf{D}_\varrho^i T_\varrho^{k_\omega} u_0(X) - \mathbf{D}_\varrho^i \psi(X) \|]}{x_1^{k_1-1} \dots x_m^{k_m-1}} \right\} < \varepsilon$$

for all $\omega \geq N(\delta)$, i. e. $\lim_{\omega \to \infty} \tilde{d}_\varrho^*(T_\varrho^{k_\omega} u_0, \psi) = 0$, which ends the proof of Theorem 6.

Remark 5. Theorem 6 was proved specially for the problem (1_ϱ), (2) if $\varrho = 1, 2$ and besides we have supposed that $a_1 \dots a_n \sum_{\Delta_2(\gamma)} p_{\gamma_1 \dots \gamma_m} \leq 1$. However, in case we confine ourselves to the proof of existence and uniqueness only, we may omit the above-mentioned restriction for the domain R and formulate the following theorem for the problem (1_1), (2).

Theorem 7. *If the continuous mapping* $f(X, \mathbf{U}_1^0, \mathbf{U}_1^1, \dots, \mathbf{U}_1^{n-m})$, *defined on* E_1 *into B satisfied the condition* (32) *from Theorem* 6 *for* $\varrho = 1$, *the condition*

$$\| f(X, \mathbf{U}_1^0, \mathbf{U}_1^1, \dots, \mathbf{U}_1^{n-m}) - f(X, \mathbf{V}_1^0, \mathbf{V}_1^1, \dots, \mathbf{V}_1^{n-m}) \| \leq$$

$$< x_1^{k_1} \dots x_m^{k_m} \sum_{i=0}^{n-m} [\tilde{\mathbf{P}}_1^i(X), \| \mathbf{U}_1^i - \mathbf{V}_1^i \|]$$

in E_1^0, *where* $\tilde{\mathbf{P}}_1^i$ *is the vector given in* Theorem 6 *and moreover* $\sum_{\Delta_1(\gamma)} p_{\gamma_1 \dots \gamma_m} < 1$. *Then there exists one and only one solution of the problem* (1_1), (2) *from the class* $M_1(R)$.

The proof of the existence is similar to that of Theorem 2 from paper [5], where the Shauder principle of the fixed point is used. The uniqueness can be proved by the classical method applied for ordinary differential equations. (See [1].)

Theorem 8. *If the continuous mapping* $f(X, \mathbf{U}_0^0, \mathbf{U}_0^1, \ldots, \mathbf{U}_0^{n-1})$ *defined on* E_0 *into* B, *satisfies the condition*

$$|f(X, \mathbf{U}_0^0, \mathbf{U}_0^1, \ldots, \mathbf{U}_0^{n-1}) - f(X, \mathbf{V}_0^0, \mathbf{V}_9^1, \ldots, \mathbf{V}_0^{n-1}) \leq$$

$$\leq x_1^{-k_1} \ldots x_m^{k_m} \sum_{i=0}^{n\ 1} [\tilde{\mathbf{P}}_0^i(X), \|\mathbf{U}_0^i - \mathbf{V}_0^i\|]$$

in E_0^0 *and moreover* $\sum_{\Delta_0(\gamma)} \tilde{p}_{\gamma_1 \ldots \gamma_m} \leq 1$. *Then the problem* (1_0), (2) *has at most one solution from the class* $M_0(R)$.

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Katedra matematickej analýzy
Prírodovedeckej fakulty
Univerzity Komenského,
Bratislava