Stephen K. McKee; Bedford Park

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ORTHOGONAL EXPANSION OF VECTOR-VALUED FUNCTIONS AND MEASURES

STEPHEN K. McKEE, Bedford Park (South Australia)

It is well known that, given a sequence a_n, $n = 0, \pm 1, \pm 2, \ldots$, of complex numbers, the questions (1) does there exist a measure on $[0, 2\pi]$ such that the a_n are the Fourier-Stieltjes coefficients of the measure, and (2) does there exist a function in $L_p([0, 2\pi])$, $1 \leqslant p \leqslant \infty$, such that the a_n are the Fourier coefficients of this function, can both be answered in terms of Cesaro means. That is, the $(C, 1)$ summation process can be used to characterize series involving the trigonometric orthonormal sequence.

This paper firstly considers the above problems for transforms involving other orthonormal sequences of functions on a finite interval, expressing the required conditions in terms of the linear means of certain regular summation methods. Then, in section 3, the a_n are taken to be elements of a vector space and conditions are given under which the a_n are the coefficients of (1) a vector-valued measure, (2) a Pettis integrable vector-valued function, (3) a vector-valued measure with finite total variation, and (4) a Bochner integrable function.

1. Introduction. Let θ_n, $n = 0, 1, 2, \ldots$, be an orthonormal sequence of continuous functions on a finite interval $[a, b]$. Let the real matrix (C_{Nn}), $0 \leqslant n \leqslant N - 1$, $N = 1, 2, \ldots$, represent a regular linear summation process A. For each N and n, put $\lambda_{Nn} = \sum_{k=n}^{N-1} C_{Nk}$. Then

$$\lim_{N\to\infty} \lambda_{Nn} = 1, \quad n = 0, 1, 2, \ldots. \tag{1}$$

Definitions. *For each f in $L_1([a, b])$, the elements*

$$a_n = \int_a^b f(t)\bar{\theta}_n(t)\,\mathrm{d}t, \quad n = 0, 1, 2, \ldots, \tag{2}$$

are called the coefficients of f. For a measure μ on $[a, b]$, the elements

$$(3) \qquad a_n = \int_a^b \bar{\theta}_n(t)\mu\,(\mathrm{d}t), \quad n = 0, 1, 2, \ldots,$$

are called the coefficients of μ.

If f is in L_1 ([a, b]), *the linear means of the expansion* $\sum_{n=0}^{\infty} a_n\theta_n(t)$ of f are

$$\sigma_N(t) = \sum_{n=0}^{N-1} \lambda_{Nn}\, a_n\, \theta_n(t) = \int_a^b K_N(t, s)f(s)\,\mathrm{d}s,$$

where

$$K_N(t, s) = \sum_{n=0}^{N-1} \lambda_{Nn}\, \theta_n\,(t)\bar{\theta}_n(s).$$

Note that for all N and all t and s,

$$(4) \qquad K_N(s, t) = \overline{K_N(t, s)}.$$

In all of our theorems, we assume

Condition U. *The sequence* $\{\theta_n\}$ *and summation method* A *are such that*

$$\lim_{N\to\infty} \int_a^b K_N(t, s)\psi(s)\,\mathrm{d}s = \psi(t)$$

uniformly with respect to t for all continuous functions ψ *on* [a, b].

Lemma 1. (see [5] p. 183). *Condition U implies that the sequence* $\{\theta_n\}$ *is complete and that there exists a constant* $M < \infty$ *such that*

$$\int_a^b |K_N(t, s)|\,\mathrm{d}s \leqslant M, \quad t \in [a, b,], \quad N = 1, 2, \ldots.$$

Example. Condition U holds for the sequence of eigenfunctions of a regular Sturm-Liouville system with (C, 1) summation.

2. Scalar-valued Expansions. Theorem 1. *Let* $\{\theta_n\}$ *be an orthonormal sequence of continuous functions and* A *a regular summation method satisfying Condition U. Given a sequence* a_n, $n = 0, 1, 2, \ldots$, *of complex numbers, there exists*

(i) *a function f in* L_1 *such that the* a_n *are the coefficients of* f;

(ii) *a function f in* L_p, $1 < p \leqslant \infty$, *such that the* a_n *are the coefficients of* f;

(iii) *a complex, regular Borel measure* μ *such that the* a_n *are the coefficients of* μ;

if and only if the linear means σ_N

(i)′ *converge in the* L_1-*norm*;

(ii)′ *are bounded in the* L_p-*norm*;

(iii)′ *are bounded in the* L_1-*norm.*

At least part of this theorem is already known (see [5] p. 215 where the L_p case ($1 < p < \infty$) is given). However, for completeness, we give a proof of all parts using the following lemmas.

Lemma 2. *For each N, define the linear maps $T_N : L_p \to L_p$ (for all p, $1 \leqslant \leqslant p \leqslant \infty$) by*

$$(T_N g)(t) = \int_a^b K_N(t, s) g(s)\, ds, \quad g \in L_p.$$

Then each T_N is continuous for all p, $1 \leqslant p \leqslant \infty$, and there exists a constant $D < \infty$ such that $\|T_N\| \leqslant D$ for all N.

Proof. When $p = 1$, we have, by (4) and Lemma 1, for each g in L_1,

$$\sup_N \|T_N g\| = \sup_N \int_a^b \left| \int_a^b K_N(t, s) g(s)\, ds \right| dt \leqslant \sup_N \int_a^b \left(|g(s)| \int_a^b |K_N(t, s)|\, dt\right) ds$$

$$\leqslant M\|g\|.$$

Therefore $\sup \|T_N\| = D_1 < \infty$. When $p = \infty$, for each N,

$$\|T_N\| = \sup_{\|g\| = 1} \operatorname{ess.sup}_t \left| \int_a^b K_N(t, s) g(s)\, ds \right|$$

$$\leqslant \sup_{\|g\| = 1} \operatorname{ess.sup}_t \int_a^b |K_N(t, s)|\, |g(s)|\, ds = D_2 < \infty.$$

Putting $D = \max(D_1, D_2)$, we have that, for each N, T_N is a linear map of each L_p space into itself ($1 \leqslant p \leqslant \infty$) and T_N is continuous for $p = 1$ and $p = \infty$ with norm at most D. The result now follows from the Riesz convexity theorem ([4] p. 526).

Lemma 3. *If f is in $L_p([a, b])$, $1 \leqslant p < \infty$, then $T_N f$ converges to f in the L_p-norm.*

Proof. Let $\varepsilon > 0$ be given. Then there exists ψ in $C([a, b])$ such that $\|f - \psi\|_p < \varepsilon$. So, by Condition U, for all sufficiently large N,

$$\begin{aligned}\|T_N f - f\| &\leqslant \|T_N(f - \psi)\| + \|T_N\psi - \psi\| + \|\psi - f\| \\ &< \|T_N\|\, \|f - \psi\| + \varepsilon + \varepsilon < D\varepsilon + 2\varepsilon.\end{aligned}$$

Lemma 4. *If f is in $L_\infty([a, b])$, then $T_N f$ converges to f in the weak-star topology of L_∞.*

Proof. For all g in $L_1([a, b])$,

$$\left| \int_a^b ((T_N f)(t) - f(t)) g(t)\, dt \right| = \left| \int_a^b f(s)\left(\overline{(T_N \bar{g})(s) - g(s)}\right) ds \right|$$

$$\leqslant \int_a^b |f(s)|\, |(T_N \bar{g})(s) - \bar{g}(s)|\, ds,$$

which, by Lemma 3 with $p = 1$, tends to zero as $N \to \infty$.

Proof of Theorem 1. Suppose that f is in L_1 and the a_n are the coefficients of f. Then $\sigma_N = T_N f$ and so (i)′ follows from Lemma 3. Also if f is in L_p, (ii)′ follows from Lemma 3 ($1 < p < \infty$) and Lemma 4 ($p = \infty$). If the

a_n are the coefficients of a complex measure μ, then, by (4) and Lemma 1,

$$\|\sigma_N\|_1 = \int_a^b \left| \int_a^b K_N(t, s)\mu\,(\mathrm{d}s)\right| \mathrm{d}t \leqslant \int_a^b \int_a^b |K_N(t, s)|\, \mathrm{d}t|\mu|\,(\mathrm{d}s)$$

$$\leqslant M|\mu|([a, b]).$$

We now show the converse implications. First observe that, if n is fixed and $N > n$, then by (1),

$$(5)\qquad \int_a^b \bar{\theta}_n(t)\sigma_N(t)\,\mathrm{d}t = \int_a^b \bar{\theta}_n(t)\sum_{m=0}^{N-1} \lambda_{Nm}a_m\theta_m(t)\,\mathrm{d}t = \lambda_{Nn}a_n \to a_n, \quad N \to \infty.$$

If the σ_N converge in the L_1-norm, they converge to an integrable function f. So

$$\left|\int_a^b (f(t) - \sigma_N(t))\bar{\theta}_n(t)\,\mathrm{d}t\right| \leqslant \int_a^b |f(t) - \sigma_N(t)|\,|\theta_n(t)|\,\mathrm{d}t$$

$$\leqslant \sup_t |\theta_n(t)|\,\|f - \sigma_N\|.$$

Therefore,

$$\lim_N \left|\int_a^b f(t)\bar{\theta}_n(t)\,\mathrm{d}t - \int_a^b \sigma_N(t)\bar{\theta}_n(t)\,\mathrm{d}t\right| = 0.$$

Hence, by (5),

$$\int_a^b f(t)\bar{\theta}_n(t)\,\mathrm{d}t = a_n.$$

Suppose the σ_N are bounded in L_p-norm, $1 < p \leqslant \infty$. We may as well assume that $\|\sigma_N\| \leqslant 1$, $N = 1, 2, \ldots$. The σ_N then lie in the unit ball of the conjugate space of L_q (where $p^{-1} + q^{-1} = 1$). Since this unit ball is weak-star compact, there is a function f in L_p with $\|f\| \leqslant 1$ such that every weak-star neighbourhood of f contains σ_N for infinitely many values of N. In other words, given any g in L_q, the numbers $\int_a^b \sigma_N(t)g(t)\,\mathrm{d}t$ are near $\int_a^b f(t)g(t)\,\mathrm{d}t$ for infinitely many values of N. But each $\bar{\theta}_n$ is in L_q and so, by (5),

$$a_n = \int_a^b f(t)\bar{\theta}_n(t)\,\mathrm{d}t.$$

Suppose that $\int_a^b |\sigma_N(t)|\,\mathrm{d}t \leqslant \alpha$ for all N. Define, for each N, the scalar-valued map Φ_N on $C([a, b])$ by

$$\Phi_N(\psi) = \int_a^b \psi(t)\sigma_N(t)\,\mathrm{d}t, \qquad \psi \in C([a, b]).$$

Then, for all N, $\|\Phi_N\| \leqslant \alpha$. That is, all the Φ_N are in the (weak-star compact) ball of radius α of the dual space of $C([a, b])$. Hence, there exists a regular Borel measure μ such that, for all $\varepsilon > 0$ and all ψ in $C([a, b])$, there exists N such that

$$(6) \qquad \left| \int_a^b \psi(t)\sigma_N(t)\, \mathrm{d}t - \int_a^b \psi(t)\mu\, (\mathrm{d}t) \right| < \varepsilon .$$

Since each $\bar{\theta}_n$ is in $C([a, b])$, we have, by (5),

$$a_n = \int_a^b \bar{\theta}_n(t)\mu\, (\mathrm{d}t) .$$

3. Vector-valued Expansions. Let X be a quasi-complete, locally convex topological vector space. For each N, let $\Phi_N : C([a, b]) \to X$ be a linear map. The set of maps Φ_N is said to be weakly equi-compact if there is a weakly compact subset W of X such that

$$\{\Phi_N\,(\psi);\ \psi \in C([a, b]),\ \|\psi\| \leqslant 1,\ N = 1, 2, \ldots\} \subset W .$$

Let the sequence $\{\theta_n\}$ and summation matrix A be as above, again satisfying Condition U. Let $\mathscr{B}\,([a, b])$ stand for the σ-algebra of all Borel sets in $[a, b]$.

Theorem 2. *Given a sequence a_n, $n = 0, 1, 2, \ldots$, of elements of X, there exists a regular measure $\mu : \mathscr{B}([a, b]) \to X$ such that the a_n are the coefficients of μ if and only if the set of maps $\Phi_N : C([a, b]) \to X$, $N = 1, 2, \ldots$, defined by*

$$\Phi_N(\psi) = \int_a^b \psi(t)\sigma_N(t)\, \mathrm{d}t, \quad \psi \in C([a, b]),$$

is weakly equi-compact.

Theorem 3. *Given a sequence a_n, $n = 0, 1, 2, \ldots$, of elements of X and a Pettis integrable function f, the a_n are the coefficients of f if and only if*

$$(7) \qquad \lim_N \int_a^b \psi(t)\, (\sigma_N(t) - f(t))\, \mathrm{d}t = 0$$

for all ψ in $C([a, b])$ with $\|\psi\| \leqslant 1$.

Note. It has been proved in [6] that Theorem 2 holds for the Fourier case on any locally compact abelian group. The same paper proves that the convergence of (7) in the Fourier case is uniform. There is evidence to believe that the convergence in Theorem 3 is uniform but we leave this as a conjecture.

Proof of Theorem 2. Suppose that such a measure exists. Then, for each ψ in $C([a, b])$,

$$\Phi_N(\psi) = \int_a^b \psi(t) \int_a^b K_N(t, s)\mu\, (\mathrm{d}s)\, \mathrm{d}t$$
$$= \int_a^b \left(\int_a^b K_N(t, s)\psi(t)\, \mathrm{d}t\right)\mu\, (\mathrm{d}s) .$$

Let $R(\mu) = \{\mu(A);\ A \in \mathscr{B}([a, b])\}$, the range of μ, and let Q be the closed, absolutely convex hull of $R(\mu)$. Then $R(\mu)$ is relatively weakly compact in X (see [7]) and so, by the Krein theorem (eg. [7]), Q is weakly compact. Now, for all ψ in $C([a, b])$ with $\|\psi\| \leqslant 1$, we have, by (4) and Lemma 1,

$$\left| \int_a^b K_N(t, s)\psi(t)\,\mathrm{d}t \right| \leqslant \|\psi\| \int_a^b |K_N(t, s)|\,\mathrm{d}t \leqslant M .$$

But for all measurable φ with $|\varphi(t)| \leqslant 1$ for all t in $[a, b]$,

$$\int_a^b \varphi(t)\mu\,(\mathrm{d}t) \in Q .$$

Therefore, $\Phi_N(\psi)$ is in MQ for all N and all ψ in $C([a, b])$ with $\|\psi\| \leqslant 1$. That is, the set of Φ_N is weakly equi-compact.

Suppose now that the set of Φ_N is weakly equi-compact. Then there exists a weakly compact subset W of X such that $\{\Phi_N(\psi);\ \psi \in C([a, b]),\ \|\psi\| \leqslant 1,\ N = 1, 2, \ldots\} \subset W$. Take x' in X', the dual of X. Then there exists a constant $\alpha_{x'}$ such that

$$|\langle \Phi_N(\psi), x' \rangle| \leqslant \alpha_{x'}$$

for all N and all ψ with $\|\psi\| \leqslant 1$. Therefore, for each N,

$$\sup_{\|\psi\| \leq 1} \left| \int_a^b \psi(t)\langle \sigma_N(t), x' \rangle\,\mathrm{d}t \right| \leqslant \alpha_{x'} ;$$

that is

$$\int_a^b |\langle \sigma_N(t), x' \rangle|\,\mathrm{d}t \leqslant \alpha_{x'} .$$

Therefore, part (iii) of Theorem 1 implies that there exists a scalar-valued measure $\mu_{x'}$ such that

$$\langle a_n , x' \rangle = \int_a^b \bar{\theta}_n(t)\mu_{x'}\,(\mathrm{d}t) \tag{8}$$

and, by (6),

$$\lim_N \langle \Phi_N(\psi), x' \rangle = \int_a^b \psi(t)\mu_{x'}\,(\mathrm{d}t) \tag{9}$$

for all ψ in $C([a, b])$. That is, for each fixed ψ, $\{\langle \Phi_N(\psi), x' \rangle\}$ is convergent for all x' in X'. Thus $\{\Phi_N(\psi)\}$ is weakly Cauchy and therefore weakly convergent since $\{\Phi_N(\psi);\ N = 1, 2, \ldots\}$ is in the weakly compact set $\|\psi\| W$. Denote the weak limit by $\Phi(\psi)$. Then, for all ψ with $\|\psi\| \leqslant 1$, $\Phi(\psi)$ is in W. Since W is weakly compact, Φ is weakly compact (i.e. it takes the unit ball of $C([a, b])$ to a relatively weakly compact set). So, by a theorem of Bartle, Dunford and Schwartz ([6] Proposition 1), there exists a regular measure $\mu : \mathscr{B}([a, b]) \to$

$\to X$ such that

$$\Phi(\psi) = \int_a^b \psi(t)\mu\,(\mathrm{d}t)$$

for all ψ in $C([a, b])$. Taking $\psi = \bar{\theta}_n$ gives, for all x' in X',

$$\langle \Phi(\bar{\theta}_n), x' \rangle = \int_a^b \bar{\theta}_n(t) \langle \mu\,(\mathrm{d}t), x' \rangle .$$

But, by (8) and (9),

$$\langle \Phi(\bar{\theta}_n), x' \rangle = \int_a^b \bar{\theta}_n(t) \mu_{x'}(\mathrm{d}t) = \langle a_n, x' \rangle .$$

Hence

$$a_n = \int_a^b \bar{\theta}_n(t)\, \mu\,(\mathrm{d}t) .$$

Proof of Theorem 3. Suppose that the a_n are the coefficients of f. Let V be an absorbent neighbourhood of 0 in X. For all ψ in $C([a, b])$,

$$\int_a^b \psi(t)(\sigma_N(t) - f(t))\mathrm{d}t = \int_a^b \psi(t)\Big(\int_a^b K_N(t, s) f(s)\, \mathrm{d}s - f(t)\Big)\, \mathrm{d}t$$
$$= \int_a^b f(s)\Big(\int_a^b K_N(t, s)\psi(t)\, \mathrm{d}t - \psi(s)\Big)\, \mathrm{d}s .$$

Also, there exists a constant $\delta > 0$ such that for all γ with $|\gamma| < \delta$,

$$\int_a^b \gamma f(s)\, \mathrm{d}s \in V .$$

But for each ψ in $C([a, b])$ with $\|\psi\| \leqslant 1$, there exists an integer $N_0(\psi)$ such that for all $N > N_0(\psi)$,

$$\Big| \int_a^b K_N(t, s)\psi(t)\, \mathrm{d}t - \psi(s) \Big| < \delta .$$

Hence, if ψ is in $C([a, b])$ with $\|\psi\| \leqslant 1$, $N > N_0(\psi)$ implies

$$\int_a^b \psi(t)(\sigma_N(t) - f(t))\, \mathrm{d}t \in V .$$

Conversely, define

$$\Phi_N(\psi) = \int_a^b \psi(t)\sigma_N(t)\, \mathrm{d}t, \qquad N = 1, 2, \ldots;$$

$$\Phi(\psi) = \int_a^b \psi(t) f(t)\, \mathrm{d}t, \qquad \psi \in C([a, b]),$$

and suppose that $\lim_N \Phi_N(\psi) = \Phi(\psi)$ for all ψ in $C([a, b])$ with $\|\psi\| \leqslant 1$. Then, for all x' in X' and all such ψ,

$$\lim_N \langle \Phi_N(\psi), x' \rangle = \langle \Phi(\psi), x' \rangle .$$

So, for every x' in X',

$$\langle a_n, x'\rangle = \lim_N \lambda_{Nn}\langle a_n, x'\rangle = \lim_N \int_a^b \bar{\theta}_n(t)\langle \sigma_N(t), x'\rangle \,\mathrm{d}t$$
$$= \langle \int_a^b \bar{\theta}_n(t) f(t)\,\mathrm{d}t, x'\rangle$$

and hence

$$a_n = \int_a^b \bar{\theta}_n(t) f(t)\,\mathrm{d}t.$$

Let X be a Banach space. Suppose the sequence $\{\theta_n\}$ and summation method A again satisfy Condition U.

Theorem 4. *Given a sequence a_n, $n = 0, 1, 2, \ldots$, of elements of X, there exists a regular measure $\mu : \mathscr{B}([a, b]) \to X$ of finite total variation such that the a_n are the coefficients of μ if and only if there exists a constant H such that*

$$\int_a^b \|\sigma_N(t)\|\,\mathrm{d}t \leqslant H, \qquad N = 1, 2, \ldots.$$

Theorem 5. *Given a sequence a_n, $n = 0, 1, 2, \ldots$, of elements of X, there exists an X-valued Bochner integrable function f on $[a, b]$ such that the a_n are the coefficients of f if and only if*

$$\lim_{N,J\to\infty} \int_a^b \|\sigma_N(t) - \sigma_J(t)\|\,\mathrm{d}t = 0.$$

These theorems have been proved in [2] for the Fourier case on groups (under certain restrictions).

Proof of Theorem 4. Suppose that such a measure exists. Then, for each N, by (4) and Lemma 1,

$$\int_a^b \|\sigma_N(t)\|\,\mathrm{d}t = \int_a^b \|\int_a^b K_N(t, s)\mu\,(\mathrm{d}s)\|\,\mathrm{d}t$$

$$\leqslant \int_a^b (\int_a^b |K_N(t, s)|\,\mathrm{d}t)\|\mu\|\,(\mathrm{d}s) \leqslant M\|\mu\|([a, b]).$$

Conversely, suppose that $\int_a^b \|\sigma_N(t)\|\,\mathrm{d}t \leqslant H$ for all N. If we define

$$\Phi_N(\psi) = \int_a^b \psi(t)\sigma_N(t)\,\mathrm{d}t, \qquad \psi \in C([a, b]),$$

then $\|\Phi_N\| \leqslant H$ for all N. Since, for each n, $\lim_N \Phi_N(\bar{\theta}_n) = a_n$, we have that $\lim_N \Phi_N(\psi)$ exists for all ψ which are linear combinations of the θ_n and so, as $\|\Phi_N\| \leqslant H$ for all N, we conclude that $\lim_N \Phi_N(\psi)$ exists for all ψ in $C([a, b])$.

Denote this limit by $\Phi(\psi)$. To obtain our required measure, we use Lemma 5 (see [3], § 19, p. 380, 383). For each subset A of $[a, b]$, let $C([a, b], A)$ denote the space of continuous functions on $[a, b]$ vanishing outside A. If $F : C([a, b]) \to \to X$ is a linear map, define for each A,

$$|||F_A||| = \sup \sum \|F(\psi_i)\|$$

where supremum is over all finite families ψ_i in $C([a, b], A)$ with $\sum |\psi_i(t)| \leqslant \leqslant \chi_A(t)$ for all t in $[a, b]$.

Lemma 5. *If $F : C([a, b]) \to X$ is a linear map, then there exists a regular measure $\mu : \mathscr{B}([a, b]) \to X$ with finite variation such that*

$$F(\psi) = \int_a^b \psi(t)\mu\,(\mathrm{d}t), \qquad \psi \in C([a, b]),$$

if and only if

$$|||F_A||| < \infty$$

for all A in $\mathscr{B}([a, b])$.

Let A be in $\mathscr{B}([a, b])$ and let $\{\psi_i;\ i = 1, 2, \ldots n\}$ be a finite family of functions in $C([a, b], A)$ with $\sum_{i=1}^{n} |\psi_i(t)| \leqslant \chi_A(t)$, t in $[a, b]$. Then, for each N,

$$\sum_{i=1}^{n} \|\Phi_N(\psi_i)\| = \sum_{i=1}^{n} \left\| \int_a^b \psi_i(t)\sigma_N(t)\,\mathrm{d}t \right\|$$

$$\leqslant \sum_{i=1}^{n} \int_a^b |\psi_i(t)|\, \|\sigma_N(t)\|\,\mathrm{d}t \leqslant \int_a^b \chi_A(t)\, \|\sigma_N(t)\|\,\mathrm{d}t \leqslant H.$$

Hence $\sum_1^n \|\Phi(\psi_i)\| \leqslant H$ and so $|||\Phi_A||| \leqslant H$. Therefore, there exists a regular measure $\mu : \mathscr{B}([a, b]) \to X$ with finite variation such that

$$\Phi(\psi) = \int_a^b \psi(t)\mu\,(\mathrm{d}t), \qquad \psi \in C([a, b]).$$

But each $\bar{\theta}_n$ is in $C([a, b])$ and so, by (5),

$$a_n = \int_a^b \bar{\theta}_n(t)\mu\,(\mathrm{d}t).$$

Proof of Theorem 5. Suppose that f is Bochner integrable and the a_n are the coefficients of f. Let $\{A_i\}_1^n$ be a finite family in $\mathscr{B}([a, b])$ and $\{\beta_i\}_1^n$ a finite family of vectors in X and define $g : [a, b] \to X$ by

$$g(t) = \sum_{i=1}^{n} \beta_i\, \chi_{A_i}(t).$$

Then,

$$\int_a^b \|\int_a^b K_N(t, s)g(s)\,\mathrm{d}s - g(t)\|\,\mathrm{d}t = \int_a^b \|\sum_{i=1}^n \beta_i(\int_a^b K_N(t, s)\,\chi_{A_i}(s)\,\mathrm{d}s - \chi_{A_i}(t)\|\,\mathrm{d}t$$

$$\leqslant \sum_{i=1}^n (\|\beta_i\| \int_a^b | \int_a^b K_N(t, s)\,\chi_{A_i}(s)\,\mathrm{d}s - \chi_{A_i}(t)|\mathrm{d}t)$$

which, by Lemma 3, tends to 0 as $N \to \infty$. Therefore, since the set of all such g is dense in the space of all Bochner integrable functions and as $\sigma_N(t) = \int_a^b K_N(t, s)f(s)\,\mathrm{d}s$,

$$\lim_N \int_a^b \|\sigma_N(t) - f(t)\|\,\mathrm{d}t = 0.$$

Conversely, suppose that the sequence $\{\sigma_N\}$ is Cauchy in the norm of the space of all Bochner integrable functions. Since this space is Banach, σ_N converges in the Bochner space norm to a Bochner integrable function f. So, for each n,

$$\|\int_a^b (f(t) - \sigma_N(t))\bar{\theta}_n(t)\,\mathrm{d}t\| \leqslant \int_a^b \|f(t) - \sigma_N(t)\|\,|\theta_n(t)|\,\mathrm{d}t$$

$$\leqslant \sup_t |\theta_n(t)|\,\|\sigma_N - f\|_B \to 0, \quad N \to \infty.$$

Thus, by (5),

$$a_n = \int_a^b \bar{\theta}_n(t)f(t)\,\mathrm{d}t.$$

REFERENCES

[1] BARTLE, R. G.—DUNFORD, N.—SCHWARTZ, J. T.: Weak compactness and vector measures. Canad. J. Math., *7*, 1955, 289—305.

[2] BUKOVSKÁ, Z.: Characterization of Fourier transforms of vector valued functions and measures. Mat. časop., *20*, 1970, 109—115.

[3] DINCULEANU, N.: Vector Measures. London 1967.

[4] DUNFORD, N.—SCHWARTZ, J. T.: Linear Operators. Part I, Interscience, New York 1966.

[5] KACZMARZ, S.—STEINHAUS, H.: Theorie der Orthogonalreihen. New York 1951.

[6] KLUVÁNEK, I.: Fourier transforms of vector-valued functions and measures. Studia math., *37*, 1970, 1—12.

[7] TWEDDLE, I.: Weak compactness in locally convex spaces. Glasgow Math. J., *9*, 1968, 123—127.

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The Flinders University of South Australia
Bedford Park, South Australia