M. A. Al-Kamessy

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ON THE OSCILLATION OF SOLUTIONS TO n-th ORDER NONLINEAR DIFFERENTIAL EQUATIONS

M. A. AL-KAMESSY, Bagdad

In paper [2] the oscillation of solutions to the second-order nonlinear differential equation was investigated. Here the results are generalized for a nonlinear differential equation of the n-th order.

A solution $y(t)$ of

$$y^{(n)} + f(t, y(t)) = 0 \tag{1}$$

will be called *oscillatory* if for every $T > 0$ there exists $t_0 > T$ such that $y(t_0) = 0$. Let $\mathscr{A}$ be the class of solutions of (1) which are indefinitely continuable to the right, i. e. $y \in \mathscr{A}$ implies $y(t)$ exists as a solution to (1) on some interval of the form $\langle T_y, +\infty)$. Equation (1) is said to be *oscillatory* if each nontrivial solution from $\mathscr{A}$ is oscillatory. If no solution in $\mathscr{A}$ is oscillatory, equation (1) is said to be *non-oscillatory*. We shall consider only the solutions from the class $\mathscr{A}$, of course, we shall assume that $\mathscr{A} \neq \emptyset$.

In order to prove our main results, we shall use the following (unpublished) lemma proved by V. Šeda.

Lemma 1. *Suppose the sequence $\{y_N(t)\}$, $N = 1, 2, 3, \ldots$ is defined on an interval $\langle T, +\infty)$ and is such that (a) for every $m = 1, 2, 3, \ldots$ there is a natural number n_m with $n_1 \leqq n_2 \leqq \ldots$ having the property that the subsequence $\{y_N(t)\}$, $N = n_m, n_m + 1, \ldots$ is uniformly bounded and equicontinuous in $\langle T, T + m\rangle$ and (b) there exists a constant c such that for any $\varepsilon > 0$ there exists $T(\varepsilon) > T$ with the property that for each $t > T(\varepsilon)$, each $N = 1, 2, \ldots$, $|y_N(t) - c| < \varepsilon$. Then there exists a subsequence $\{y_{N_m}(t)\}$, $m = 1, 2, \ldots$, which is uniformly convergent in $\langle T, +\infty)$ to a continuous function.*

Proof. On the basis of the Ascoli theorem, by mathematical induction, for every $m = 1, 2, 3, \ldots$ there exists a subsequence $\{y_{N_{l,m}}(t)\}$, $l = 1, 2, 3, \ldots$, $N_{1,m} \geqq n_m$, and a continuous function $y_m(t)$ on $\langle T, T + m\rangle$ such that $\{y_{N_{l,m}}(t)\}$ is uniformly convergent to $y_m(t)$ on $\langle T, T + m\rangle$ and $\{y_{N_{l,m+1}}(t)\}$ is a subsequence of $\{y_{N_{l,m}}(t)\}$. Therefore there is a continuous function $y(t)$ on $\langle T, +\infty)$ which is equal to $y_m(t)$ on $\langle T, T + m\rangle$ and for every natural m

there is a function $y_{N_m}(t)$ such that $|y_{N_m}(t) - y(t)| < 1/m$ for $t \in \langle T, T + m \rangle$. Here $N_1 < N_2 < \ldots$ can be supposed. We shall show that $\{y_{N_m}(t)\}$, $m = 1, 2, 3, \ldots$ is uniformly convergent on $\langle T, +\infty)$.

Let $\varepsilon > 0$ be given. There is a $T(\varepsilon/2)$ such that $|y_{N_{m_1}}(t) - y_{N_{m_2}}(t)| \leqq |y_{N_{m_1}}(t) - C| + |C - y_{N_{m_2}}(t)| < \varepsilon$ for every $t > T(\varepsilon/2)$ and for every natural m_1, m_2. When m is so large that $T + m > T(\varepsilon/2)$, and $1/m < \varepsilon/2$, then for every m_1, $m_2 \geqq m$, $t \in \langle T, T + m \rangle$ we have

$$|y_{N_{m_1}}(t) - y_{N_{m_2}}(t)| \leqq |y_{N_{m_1}}(t) - y(t)| + |y(t) - y_{N_{m_2}}(t)| < \varepsilon.$$

From the Cauchy principle the uniform convergence of $\{y_{N_m}(t)\}$ on $\langle T, +\infty)$ follows.

Now, we shall state and prove our main results.

Theorem 1. *Let $n \geqslant 2$ be an integer, $f(t, x)$ be continuous on $S = [0, +\infty) \times (-\infty, +\infty)$, with $a(t)\beta(x) \leqq f(t, x) \leqq b(t)\gamma(x)$ for $(t, x)\varepsilon S$, where (a) $a(t)$ and $b(t)$ are non-negative locally integrable functions, (b) $\beta(x)$ and $\gamma(x)$ are non-decreasing, with $x\beta(x) > 0$ and $x\gamma(x) > 0$ for $x \neq 0$, on $(-\infty, +\infty)$ and for some $\alpha > 0$ $\int\limits_{\alpha}^{+\infty} [\beta(u)^{-1}\,\mathrm{d}u < +\infty$, $\int\limits_{-\alpha}^{-\infty} [\gamma(u)]^{-1}\,\mathrm{d}u < +\infty$. Then*

(i) for n even, equation (1) is oscillatory if and only if,

$$\int\limits_0^{+\infty} t^{n-1}a(t)\,\mathrm{d}t = \int\limits_0^{+\infty} t^{n-1}b(t)\mathrm{d}t = +\infty: \tag{2}$$

(ii) for n odd, any nontrivial solution of the equation (1) is either oscillatory or it tends monotonically to zero as $t \to \infty$ together with all its derivatives of order up to $(n - 1)$ inclusive.

To prove the sufficiency we shall assume that the equation (1) has a non-oscillatory solution. Considering all possible cases which may arise, we shall prove that the existence of such a solution, except for the case when the solution has the property that $\lim\limits_{t\to+\infty} y^{(i)}(t) = 0$, $i = 0, 1, \ldots, n - 1$, and n is odd, contradicts the assumption (2). As to the proof of necessity, assuming that either $\int\limits_0^{+\infty} t^{n-1}a(t)\,\mathrm{d}t < +\infty$ or $\int\limits_0^{+\infty} t^{n-1}b(t)\,\mathrm{d}t < +\infty$, we shall prove the existence of a non-oscillatory solution of a certain integral equation which shall be shown to be a solution of equation (1) and thus contradicting the statement of the theorem.

Proof. Suppose that (2) holds and (1) has a non-oscillatory solution, then for t sufficiently large either $y(t) > 0$ or $y(t) < 0$.

Suppose that $y(t) > 0$ for $t \geqq T > 0$, then $y^{(n)}(t) < 0$ on $[T, +\infty)$. This shows that $y^{(n-1)}(t)$ is non-increasing on $[T, +\infty)$. Denote $\lim\limits_{t \to +\infty} y^{(k)}(t)$ by L_k for $k = 0, 1, 2, \ldots, n-1$. Since $y(t) > 0$, it follows that all $L_k \geqq 0$.

Now, the following cases might arise:

Case 1.1. $L_{n-1} = \ldots = L_1 = 0$.

Let $T \leqq s_1 \leqq \ldots \leqq s_{n-1} \leqq t < +\infty$.

Upon integrating (1) over the interval (s_{n-1}, t) we obtain $y^{(n-1)}(t) - y^{(n-1)}(s_{n-1}) = - \int\limits_{s_{n-1}}^{t} f(u, y(u))du \leqq 0$, and as $t \to +\infty$

$$y^{(n-1)}(s_{n-1}) = \int\limits_{s_{n-1}}^{+\infty} f(u)y(u))du \geqq 0, \quad \text{therefore } y^{(n-1)}(t) \tag{3}$$

is non-decreasing in $[T, +\infty)$. Integrating (3) from s_{n-2} to t we get

$$y^{(n-2)}(t) - y^{(n-2)}(s_{n-2}) = \int\limits_{s_{n-2}}^{t} \int\limits_{s_{n-1}}^{+\infty} f(u, y(u))du\, ds_{n-1} \geqq$$

$$\geqq \int\limits_{s_{n-2}}^{t} (u - s_{n-2}) f(u, y(u))du \geqq 0.$$

Letting $t \to +\infty$,

$$-y^{(n-2)}(s_{n-2}) \geqq \int\limits_{s_{n-2}}^{+\infty} (u - s_{n-2}) f(u, y(u))du \geqq 0.$$

Continuing this process $(n-3)$ times, we get

(1.1a) If n is even then

$$y'(s_1) \geqq \int\limits_{s_1}^{+\infty} \frac{(u - s_1)^{n-2}}{(n-2)!} f(u, y(u))du \geqq 0, \tag{4}$$

which implies that $y(t)$ is non-decreasing in $\langle T, +\infty \rangle$. This shows that the case $L_{n-1} = \ldots = L_1 = 0$, $L_0 = 0$ cannot arise.

Hence we shall assume that $0 < L_0 \leqq +\infty$.

Integration of (4) from T to t yields

$$y(t) \geqq y(t) - y(T) \geqq \int\limits_{T}^{t} \int\limits_{s_1}^{+\infty} \frac{(u - s_1)^{n-2}}{(n-2)!} f(u, y(u))du\, ds_1 \geqq$$

$$\geqq \int\limits_{T}^{t} \frac{(u - T)^{n-1}}{(n-1)!} f(u, y(u))du.$$

Thus

$$(5) \qquad y(t) \geqq \int_T^t \frac{(u-T)^{n-1}}{(n-1)!} a(u)\beta(y(u))\mathrm{d}u > 0.$$

From the monotonicity of β we have

$$(6) \qquad \beta(y(t))\left[\beta\left(\int_T^t \frac{(u-T)^{n-1}}{(n-1)!} a(u)\beta(y(u)\,\mathrm{d}u\right)\right]^{-1} \geqq 1.$$

From (6) it follows that

$$(7) \qquad \int^{+\infty} t^{n-1}a(t)\mathrm{d}t < +\infty, \quad \text{(see [2]: Th. 1)},$$

which is a contradiction to the assumption (2).

(1.1b) If n is odd then

$$(8) \qquad -y'(s_1) \geqq \int_{s_1}^{+\infty} \frac{(u-s_1)^{n-2}}{(n-2)!} f(u, y(u))\mathrm{d}u \geqq 0,$$

which shows that $y(t)$ is non-increasing in $\langle T, +\infty)$.

Integration of (8) from T to t implies that

$$y(T) \geqq y(T) - y(t) \geqq \int_T^t \frac{(u-T)^{n-1}}{(n-1)!} f(u, y(u))\mathrm{d}u.$$

Now, either $L_0 = 0$ or $0 < L_0 < +\infty$.

If $L_0 = 0$, then the solution of (1) together with all its derivatives of the order up to $(n-1)$ included tends monotonically to zero as $t \to +\infty$.

Hence we shall assume that $0 < L_0 < +\infty$. Since $y(t)$ is non-increasing in $[T, +\infty)$, then $\beta(y(t)) \geqq \beta(L_0)$ for $t \geqq T$ and therefore

$$(9) \qquad \frac{y(T)}{\beta(L_0)} \geqq \int_T^t \frac{(u-T)^{n-1}}{(n-1)!} a(u)\mathrm{d}u > 0.$$

From (9) the inequality (7) follows

Case 1.2. Suppose that there exists an integer k, $1 \leqq k \leqq n-2$ such that $L_{n-1} = \ldots = L_{n-k} = 0$ and $0 < L_{n-k-1} \leqq +\infty$, then it follows that $L_{n-k-2} = \ldots = L_1 = L_0 = +\infty$.

Let $T \leqq s_1 \leqq \dots \leqq s_{n-1} \leqq t < +\infty$.

Using the same method of proof as in the case 1.1, we shall obtain

(1.2a) If k is odd, then

$$y^{n-k}(s_1) \geqq \int_{s_1}^{+\infty} \frac{(u - s_1)^{k-1}}{(k-1)!} f(u, y(u))du \geqq 0 \tag{10}$$

(we shall use here s_1 instead of s_{n-k}).

One can easily obtain

$$y^{(n-k)}(s_1) \geqq \int_{s_2}^{+\infty} \frac{(s_2 - s_1)^{k-1}}{(k-1)!} f(u, y(u))du \geqq 0, \tag{11}$$

(12) choose T sufficiently large so that $y^{(i)}(t) > 0$ for all $t \geqq T$ and $i = n - k - 1, \dots, 0$.

Integration of (11), from T to s_2 implies that

$$y^{(n-k-1)}(s_2) \geqq y^{(n-k-1)}(s_2) - y^{(n-k-1)}(T) \geqq \int_{s_2}^{+\infty} \frac{(s_2 - T)^k}{k!} f(u, y(u))du \geqq 0.$$

Again integration of the above inequality from T to s_3 yields

$$y^{(n-k-2)}(s_3) \geqq \int_{s_3}^{+\infty} \frac{(s_3 - T)^{k+1}}{(k+1)!} f(u, y(u))du \geqq 0. \tag{13}$$

Continuing this process $(n - k - 3)$ times, we get

$$y'(s_{n-k}) \geqq \int_{s_{n-k}}^{+\infty} \frac{(s_{n-k} - T)^{n-2}}{(n-2)!} f(u, y(u))du \geqq 0. \tag{14}$$

Once more integration of (14) from T to t shows that

$$y(t) \geqq \int_{T}^{t} \frac{(u - T)^{n-1}}{(n-1)!} f(u, y(u))du > 0$$

and this gives, similarly as in subcase (1.1a), the inequality (7).

(1.2b) If k is even, then

$$(15) \qquad -y^{(n-k)}(s_3) \geqq \int_{s_3}^{+\infty} \frac{(u - s_3)^{k-1}}{(k-1)!} f(u, y(u))du \geqq 0.$$

(Instead of s_{n-k} here s_3 is used.)

From (15) it follows that $y^{(n-k-1)}(t)$ is non-increasing in $(T, +\infty)$, hence $0 < L_{n-k-1} < +\infty$.

Integrating (15) from s_2 to t, we obtain

$$y^{(n-k-1)}(s_2) \geqq \int_{s_2}^{+\infty} \frac{(u - s_2)^k}{k!} f(u, y(u))du \geqq 0, \quad \text{and as} \quad t \to +\infty$$

$$(16) \qquad y^{(n-k-1)}(s_2) \geqq \int_{s_2}^{+\infty} \frac{(u - s_2)^k}{k!} f(u, y(u))du \geqq 0,$$

from which one can obtain the following inequality

$$(17) \qquad y^{(n-k-1)}(s_2) \geqq \int_{s_3}^{+\infty} \frac{(s_3 - s_2)^k}{k!} f(u, y(u))\, du \geqq 0.$$

(18) Choose T large enough so that $y^{(i)}(t) > 0$ for $t > T$ and $i = 0, 1, \ldots, n-k-2$.

Integrating (17) from T to s_3 and using $y^{(n-k-1)}(T) > 0$, we get

$$y^{(n-k-2)}(s_3) \geqq \int_{s_3}^{+\infty} \frac{(s_3 - T)^{k+1}}{(k+1)!} f(u, y(u))\, du \geqq 0.$$

This inequality leads us to the inequality (7), by the method used in proving subcase (1.2a).

Case 1.3. Suppose that $0 < L_{n-1} \leqq +\infty$, then

$$L_{n-2} = \ldots = L_1 = L_0 = +\infty.$$

Let $T \leqq s_1 \leqq \ldots \leqq s_{n-1} \leqq t < +\infty$.

Since $y^{(n-1)}(t)$ is non-increasing in $[T, +\infty)$, we shall consider the case $0 < L_{n-1} < +\infty$.

As before, we shall obtain

$$y^{(n-1)}(s_1) = L_{n-1} + \int_{s_1}^{+\infty} f(u, y(u))\, du \geqq \int_{s_2}^{+\infty} f(u, y(u))\, du \geqq 0.$$

Thus

$$(19) \qquad y^{(n-1)}(s_1) \geqq \int_{s_1}^{+\infty} f(u, y(u))\, du \geqq 0.$$

(20) Let us choose T sufficiently large so that $y^{(i)}(t) > 0$ for $t \geqq T$ and $i = 0, 1, \ldots, n-2$.

Integrating (19) from T to s_2, we shall obtain

$$y^{(n-2)}(s_2) \geqq \int_{s_2}^{+\infty} (s_2 - T) f(u, y(u))\, du \geqq 0.$$

Repeating this process $(n - 3)$ times, we get

$$(21) \qquad y'(s_{n-1}) \geqq \int_{s_{n-1}}^{+\infty} \frac{(s_{n-1} - T)^{k-2}}{(n-2)\,!} f(u, y(u))\, du \geqq 0.$$

Once more integration of (21) from T to t, and using $y'(T) > 0$, implies that

$$y(t) \geqq \int_T^t \frac{(u - T)^{n-1}}{(n-1)\,!} f(u, y(u))\, du > 0,$$

which proves that (7) holds.

In the case $y(t) < 0$ for all $t \geqq T$, we shall have $y^{(n)}(t) > 0$ on $[T, +\infty)$. Hence $y^{(n-1)}(t)$ is non-decreasing in $[T, +\infty)$. Let $L_k^* = \lim\limits_{k \to +\infty} y^{(k)}(t)$, for $k = 0, \ldots, n-1$. Since $y(t)$ is negative in $[T, +\infty)$, it follows that all $L_k^* \leqq 0$.

Now, we shall state without proof all possible cases which arise, since one can obtain them by the same method used in proving the previous cases.

Case 1.4. Suppose $L_{n-1}^* = \ldots = L_1^* = 0$. Let $T \leqq s_1 \leqq \ldots \leqq s_{n-1} \leqq t < +\infty$.

Case 1.5. Suppose that there exists an integer k, $1 \leqq k \leqq n-2$ such that $L_{n-1}^* = \ldots = L_{n-k}^* = 0$ and $-\infty \leqq L_{n-k-1}^* < 0$ then $L_{n-k-2}^* = \ldots = L_0^* = -\infty$. Let $T \leqq s_1 \leqq \ldots \leqq s_{n-1} \leqq t < +\infty$.

Case 1.6. Suppose that $-\infty \leqq L_{n-1} < 0$, then $L_{n-2}^* = \ldots = L_1^* = -\infty$. Let $T \leqq s_1 \leqq \ldots \leqq s_{n-1} \leqq t < +\infty$.

To establish the necessity, we must show that if either $\int_0^{+\infty} t^{n-1} b(t)\, dt < +\infty$ or $\int_0^{+\infty} t^{n-1} a(t)\, dt < +\infty$, then equation (1) has a non-oscillatory solution with the property

$$\lim_{t\to+\infty} y(t) = 1 \quad \text{or} \quad \lim_{t\to+\infty} y(t) = -1 .$$

Case 1.1°. Suppose that $\int\limits_0^{+\infty} t^{n-1}b(t)\,dt < +\infty$.

According to the cases when n is even or odd, respectively, we have two subcases (1.1°a). If n is even, then consider the following integral equation

$$(22) \qquad y(t) = 1 - \int\limits_t^{+\infty} \frac{(s-t)^{n-1}}{(n-1)!} f(s, y(s))\,ds .$$

A non-negative continuous solution of (22) which is bounded on some interval $[T, +\infty)$ is also a solution of (1) with

$$\lim_{t\to+\infty} y(t) = 1 \quad \text{and} \quad \lim_{t\to+\infty} y^{(i)}(t) = 0, \quad i = 1, 2, \ldots, n-1 .$$

Let a positive integer T be chosen such that

$$\gamma(1) \int\limits_T^{+\infty} \frac{s^{n-1}}{(n-1)!}\, b(s)\,ds \leqq \frac{1}{2} .$$

We define for every positive integer $N \geqq T$

$$(23) \qquad y_N(t) = \begin{cases} 1 & \text{for} \quad t \geqq N \\ 1 - \displaystyle\int\limits_{t+(1/N)}^{+\infty} \frac{(s-t-(1/N))^{n-i}}{(n-1)!} (f(s, y_N(s))\,ds & \text{for } T \leqq t \leqq N . \end{cases}$$

This formula defines $y_N(t)$ successively on the intervals

$$\left[N - \frac{k}{N},\; N - \frac{k-1}{N}\right] \quad \text{for} \quad k = 1, \ldots, N(N-T);$$

hence $y_N(t)$ is defined on $[T, +\infty)$.

For $N - \dfrac{1}{N} \leqq t < +\infty$, we have

$$0 \leqq \int\limits_{t+(1/N)}^{+\infty} \frac{(s-t-(1/N))^{n-1}}{(n-1)!} f(s, y_N(s))\,ds \leqq \gamma(1) \int\limits_T^{+\infty} \frac{s^{n-1}}{(n-1)!}\, b(s)\,ds \leqq \frac{1}{2} .$$

Hence $\frac{1}{2} \leqq y_N(t) \leqq 1$ on this interval, and that $\frac{1}{2} \leqq y_N(t) \leqq 1$ on the interval $[T, +\infty)$ can be shown by induction. Consequently for $t \geqq T$, $t \neq N$,

$$|y_N'(t)| \leqq \gamma(1) \int_T^{+\infty} \frac{s^{n-2}}{(n-2)!} f(s, y_N(s))\, ds \leqq \frac{1}{2}.$$

Further, if $\varepsilon > 0$ be given then

$$|y_N(t) - 1| \leqq \int_{t+(1/N)}^{+\infty} \frac{(s - t - (1/N))^{n-1}}{(n-1)!} f(s, y_N(s))\, ds$$

$$\leqq \gamma(1) \int_t^{+\infty} \frac{s^{n-1}}{(n-1)!} b(s)\, ds < \varepsilon \quad \text{for all} \quad t \geqq T(t) > T,$$

$$N = 1, 2, \ldots$$

Using Lemma 1, there exists a uniformly convergent subsequence $\{y_{N(k)}(t)\}_{k=1}^{+\infty}$ on the interval $[T, +\infty)$ of the sequence $\{y_N(t)\}$. Denote its limit as $y(t)$. To find the integral which is satisfied by y, choose any large real number R such that

$$y_{N(k)}(t) = 1 - \int_{t+(1/N(k))}^{R} \frac{(s - t - (1/N(k)))^{n-1}}{(n-1)!} f(s, y_{N(k)}(s))\, ds + \varepsilon(k, R), \tag{24}$$

where

$$\varepsilon(k, R) = - \int_R^{+\infty} \frac{(s - t - (1/N(k)))^{n-1}}{(n-1)!} f(s, y_{N(k)}(s))\, ds,$$

and therefore

$$|\varepsilon(k, R)| \leqq \gamma(1) \int_R^{+\infty} \frac{s^{n-1}}{(n-1)!} b(s)\, ds. \tag{25}$$

Letting $k \to +\infty$, it follows that

$$\liminf_{k\to+\infty} \varepsilon(k, R) \leqq y(t) - 1 + \int_t^R \frac{(s-t)^{n-1}}{(n-1)!} f(s, y(s))\, ds \leqq \limsup_{k\to+\infty} \varepsilon(k, R)$$

and as $R \to +\infty$ in (25).

$\liminf \varepsilon(k, R)$ and $\limsup \varepsilon(k, R)$ approach zero and $y(t)$ satisfies (22).
(1.1°b). If n is odd, then consider the following integral equations.

$$(26) \qquad y(t) = 1 + \int_t^{+\infty} \frac{(s-t)^{n-1}}{(n-1)!} f(s, y(s))\, ds.$$

If (26) has a non-negative continuous and bounded solution $y(t)$ on $[T, +\infty)$, then $y(t)$ is also a solution of equation (1) with $\lim_{t\to+\infty} y(t) = 1$ and $\lim_{t\to+\infty} y^{(i)}(t) = 0$ for $i = 1, 2, \ldots, n-1$.

Following the same method as in the previous subcase, (1.1°a), one can show that (26) has a non-oscillatory solution with the property

$$\lim_{t\to+\infty} y(t) = 1 \quad \text{and} \quad \lim_{t\to+\infty} y^{(i)}(t) = 0 \quad \text{for} \quad i = 1, 2, \ldots, n-1.$$

Case 1.2°. Suppose that $\int_0^{+\infty} t^{n-1} a(t)\, dt < +\infty$. According to the possibilities when n is even or odd, respect., we have two subcases.

(1.2°a). If n is even, then consider the following integral equation

$$(27) \qquad y(t) = -1 - \int_t^{+\infty} \frac{(s-t)^{n-1}}{(n-1)!} f(s, y(s))\, ds.$$

(1.2°b). If n is odd, then consider the following integral equation

$$(28) \qquad y(t) = -1 + \int_t^{+\infty} \frac{(s-t)^{n-1}}{(n-1)!} f(s, y(s))\, ds.$$

By the same method of proof as in the subcase (1.1°a) one can show that both (28) and (27) has a non-oscillatory solution with $\lim_{t\to+\infty} y(t) = -1$ and $\lim_{t\to+\infty} y^{(i)}(t) = 0$, $i = 1, 2, \ldots, n-1$, which is also a solution of (1).

This completes the proof of the theorem.

Theorem 2. *Let $n \geqq 2$ be an integer, $f(t, x)$ be continuous on $S = [0, +\infty) \times \times (-\infty, +\infty)$, with $a(t)\beta(x) \leqq f(t, x) \leqq b(t)\gamma(x)$ for $(t, x)\, \varepsilon S$, where*

(a) $a(t)$ and $b(t)$ are non-negative locally integrable functions,

(b) $\beta(x)$ and $\gamma(x)$ are non-increasing with $x\beta(x) < 0$ and $x\gamma(x) < 0$ for $x \neq 0$, on $(-\infty, +\infty)$ and for some $\alpha > 0$,

$$\int_{\alpha}^{+\infty} \frac{-\mathrm{d}v}{\gamma(v)} < +\infty, \quad \int_{-\alpha}^{-\infty} \frac{-\mathrm{d}v}{\beta(v)} < +\infty.$$

Then

(*i*) *For n even, any non-trivial solution of equation* (1) *is either oscillatory or it tends monotonically to zero or to infinity as* $t \to +\infty$ *together with all its derivatives of the order up to* $(n-1)$ *included, iff,*

$$\int_0^{+\infty} t^{n-1} b(t)\, \mathrm{d}t = \int_0^{+\infty} t^{n-1} a(t)\, \mathrm{d}t = +\infty. \tag{2}$$

(ii) *For n odd, any non-trivial solution of equation* (1) *is either oscillatory or it tends monotonically to infinity as* $t \to +\infty$ *together with all its derivatives of the order up to* $(n-1)$ *included, iff,* (2) *holds.*

The outline of the proof of the sufficiency part in Theorem 2 is the same as that in Theorem 1, except for case 3 and case 6.

One can also prove the necessity part in Theorem 2 in the same way as in Theorem 2.

For this reason we shall prove case 3 and case 6 in the sufficiency part.

Proof. Suppose that (2) holds and (1) has a non-oscillatory solution, say $y(t)$. If $y(t) > 0$, then $y^{(n-1)}(t)$ is nondecreasing in $[T, +\infty)$. If $L_k = \lim_{t\to+\infty} y^{(n)}(t)$, then $L_k \geqq 0$ for $k = 0, 1, \ldots, n-1$.

Now the following cases might arise:

Case 2.1. $L_1 = L_2 = \ldots = L_{n-1} = 0$.
Let $T \leqq s_1 \leqq \ldots \leqq s_{n-1} \leqq t < +\infty$.

Case 2.2. Suppose that there exists a positive integer k, $1 \leqq k \leqq n-2$ such that $L_{n-1} = \ldots = L_{n-k} = 0$ and $0 < L_{n-k-1} < +\infty$ then $L_{n-k-2} = \ldots = L_0 = +\infty$.

Case 2.3. Suppose that $0 < L_{n-1} \leqq +\infty$, then $L_{n-2} = \ldots = L_0 = +\infty$.
Let $T \leqq s_1 \leqq \ldots \leqq s_{n-1} \leqq t < +\infty$.

If $L_{n-1} = +\infty$, then there is nothing to prove. Therefore, we shall assume that $0 < L_{n-1} < +\infty$.

Choose T large enough so that $y^{(i)}(t) > 0$ for $T \leqq t < +\infty$, $i = 0, 1, \ldots, n-1$.

Hence there exists a positive number C_0 such that

$$y^{(n-1)}(s_1) \geqq C_0.$$

Integrating the above inequality from T to s_2 gives

$$y^{(n-2)}(s_2) \geqq C_0(s_2 - T).$$

Continuing this process $(n - 2)$ times, yields

$$y(t) \geqq C_0 \frac{(t - T)^{n-1}}{(n - 1)!}.$$

After multiplying both sides of the above inequality by L_{n-1}, it becomes

$$L_{n-1}y(t) \geqq \frac{C_0(t - T)^{n-1}}{(n - 1)!} L_{n-1} \geqq 0.$$

Integration of (1) from T to t implies that

$$y^{(n-1)}(t) \geqq - \int_T^t f(u, y(u))\, du > 0.$$

Since $L_{n-1} \geqq y^{(n-1)}(t) > 0$ for all $t \geqq T$, then

$$y(t) \geqq \frac{-C_0}{L_{n-1}} \int_T^t \frac{(u - T)^{n-1}}{(n - 1)!} f(u, y(u))\, du > 0.$$

Let $C^* = \dfrac{C_0}{L_{n-1}} > 0$, then the above inequality becomes

$$y(t) \geqq - C^* \int_T^t \frac{(u - T)^{n-1}}{(n - 1)!} f(u, y(u))\, du > 0,$$

which yields

$\int^{+\infty} t^{n-1}b(t)\, dt < +\infty$. This is a contradiction to (2).

In the case when $y(t) < 0$ for $t \geqq T > 0$, then $y^{(n-1)}(t)$ is non-increasing in $[T, +\infty)$. Let $L_k^* = \lim\limits_{t\to+\infty} y^{(i)}(t)$, then all $L_k \leqq 0$ for $k = 0, 1, \ldots, n - 1$.

Now the following cases might arise:

Case 2.4. $L_{n-1}^* = \ldots = L_0^* = 0$.
Let $T \leqq s_1 \leqq \ldots \leqq s_{n-1} \leqq t < +\infty$.

Case 2.5. Suppose that there exists a positive integer k, $1 \leqq k \leqq n - 2$ such that $-\infty \leqq L_{n-k-1}^* < 0$, then $L_{n-k-2}^* = \ldots = L_0^* = -\infty$.

Let $T \leqq s_1 \leqq \ldots \leqq s_{n-1} \leqq t < +\infty$.

Case 2.6. Suppose that $-\infty \leqq L^*_{n-1} < 0$, then $L^*_{n-2} = \ldots = L^*_0 = -\infty$. Let $T \leqq s_1 \leqq \ldots \leqq s_{n-1} \leqq t < +\infty$.

If $L_{n-1} = -\infty$, then there is nothing to prove. Hence we shall assume that

$$-\infty < L_0 < \infty .$$

By a suitable choice of T, we can make $y^{(i)}(t) < 0$, for $T \leqq t < +\infty$ and $i = 0, 1, \ldots, n-1$. Also we can find a real number $C_0 < 0$ such that

$$y^{(n-1)}(s_1) \leqq C_0 .$$

Integrating the above inequality $(n-1)$ times yields

$$y(t) \leqq \frac{C_0(t-T)^{n-1}}{(n-1)\,!} .$$

Following the procedure in case 2.3, one can obtain the following inequality

$$y(t) \leqq -C^* \int\limits_T^t \frac{(u-T)^{n-1}}{(n-1)\,!} f(u, y(u))\, \mathrm{d}u < 0, \text{ where } C^* = \frac{C_0}{L_{n-1}} > 0, \text{ which}$$

yields

$$\int^{+\infty} t^{n-1} a(t)\, \mathrm{d}t < +\infty$$

which contradicts assumption (2).

REFERENCES

[1] APOSTOL, T. M.: Mathematical analysis. Reading, Mass. 1965.

[2] MACKI, J. W.—WONG, S. W.: Oscillation of solutions to second-order nonlinear differential equations. Pacif. J. Math., *24*, 1968.

[3] RUDIN, W.: Principles of mathematical analysis. New York 1964.

[4] WHITTAKER, E. T.—WATSON, G. N.: A course of modern analysis. Cambridge 1952.

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Department of Mathematics
College of Science Bagdad,
Iraq