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A NOTE ON THE GENERALIZATION OF JEGOROFF'S THEOREM

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In various papers concerning the problem of extension of the wellknown Jegoroff theorem on measurable functions we can see two main kinds of approach. One takes into account the extension of this theorem in cases when the range space of the considered transformations is some topological space, the other when the normal convergence of a sequence of transformations is replaced by the Moore-Smith convergence of a net of transformations. Thus e. g. Kvačko [3] proved Jegoroff's theorem for a sequence of measurable transformations with values in a separable metric space but his proof contains an error, see [4]. His result is covered by that of Neubrunn [5]. The latter replaced the class of separable metric spaces by a certain class of topological spaces containing all separable metric spaces. Tolstov [10] and Weston [7] showed by a counterexample that the ordinary Jegoroff theorem cannot be extended in the case of a family of real measurable functions $f(t, x)$ with t a real continuous parameter. Frumkin [8] showed that it is possible if the ordinary notion of uniform convergence is replaced by „*ess. unif. weaker*" convergence. His result was generalized by Zakon [6], who also simplified Frumkin's proof. Further Zakon [4] introduced the notions „*ess. unif.*" and „*ess. unif. weaker*" convergence of a net of transformations if the range space is a uniform space, and the notion of „*ess.*" properties of uniform spaces, and proved Jegoroff's theorem for a sequence (quasicountable net) of measurable transformations with values in ess. metrizable and ess. separable uniform spaces in the first (second) sense. It can easily be seen that if the sets „E_n^t" are measurable then the first statement is based on the standard proof of the ordinary Jegoroff theorem as given e. g. in [1, p. 90], the second statement is in the same way based on the results of [6]. In the first part of this paper we give an alternative proof for the measurability of „E_n^t". Further this part contains some remarks concerning the „*ess.*" notions introduced by Zakon in [4]. In the second part of this paper we shall study the possibility of the extension of Jegoroff's theorem in the case of a net of measurable transformations with values in some uniform space. It will be shown that such extension in general fails both for the ordinary uniform convergence of a sequence of measurable transformations as well as for „*ess. unif.*“ convergencies of a net of transformations. We shall construct a sequence of measurable transformations mapping a finite measure space into a uniform space converging everywhere pointwise to a measurable transformation which

(a) fails to converge almost uniformly in the ordinary sense;

(b) fails to converge almost ess. uniformly in the weaker sense;

(c) fails to converge almost ess. uniformly.

1. The notion of the measure space $(X, \mathbf{S}, m)$ is used as in [1], i. e. $\mathbf{S}$ is a σ-algebra of subsets of the space X, m is a measure on $\mathbf{S}$. The uniform space will be denoted by a pair $(Y, \mathscr{U})$, where $\mathscr{U}$ is some uniformity for Y, see [2, p. 176], or by a triplet $(Y, \mathbf{N}, \mathbf{I})$ where $\mathbf{N}$ is some uniform neighbourhood system for Y and $\mathbf{I}$ is its grader [4]. Members of $\mathbf{N}$ are denoted by N_y^i where $y \in Y$ and $i \in \mathbf{I}$. It is well known that each uniformity $\mathscr{U}$ on Y determines some uniform neighbourhood system on Y. Therefore these notions are used equivalently. Further we consider each uniform space as a topological space with the uniform topology generated by its uniformity. Some mapping f from the measure space $(X, \mathbf{S}, m)$ to the uniform space $(Y, \mathscr{U})$ is said to be measurable if for each open set $O \subset Y$ the set $f^{-1}(O)$ is measurable. A net of transformations $\{f_t, t \in T\}$ from a measure space $(X, \mathbf{S}, m)$ into a uniform space $(Y, \mathscr{U})$ is said to converge uniformly to the transformation f iff for each $U \in \mathscr{U}$, there is $t_U \in T$ such that for each $t \geqslant t_U$ and for each $x \in X$ we have $f_t(x) \in U[f(x)]$. It is obvious that this definition is equivalent to the following given by Zakon in [4] : A net of transformations $\{f_t, t \in T\}$ is said to converge uniformly to f iff for each $i \in \mathbf{I}$ there exists $t_i \in T$ such that for each $t \geqslant t_i$ and for each $x \in X$ we have $f_t(x) \in N^i_{f(x)}$. The almost uniform convergence is derived from the uniform convergence as usual. Now we can recall the definitions of ess. unif. convergencies as given in [4]. A net of transformations $\{f_t, t \in T\}$ is said to converge *m-essentially uniformly* (*m-ess. unif* or *ess. unif.*, if ambiguity is excluded) to the transformation f iff for each $i \in \mathbf{I}$ there is $t_i \in T$ and $Z_i \subset X$, $m(Z_i) = 0$ such that for each $t \geqslant t_i$ and an arbitrary $x \in (X\text{-}Z_i)$ we have $f_t(x) \in N^i_{f(x)}$. A net of transformations $\{f_t, t \in T\}$ is said to converge *m-essentially uniformly in the weaker sense* (abbr. as *m-ess. unif. weaker* or *m-ess. unif. (w)* or only *ess. unif. (w)* if ambiguity is excluded) to the transformation f iff for each $i \in \mathbf{I}$ there is $t_i \in T$ such that for each $t \geqslant t_i$ there exists a set $Z_t^i \subset X$, $m(Z_t^i) = 0$ such that for an arbitrary $x \in (X - Z_t^i)$, $f_t(x) \in N^i_{f(x)}$. The almost ess. unif. convergencies are derived as usual.

Note 1. It can easily be seen that for each net of transformations mapping $(X, \mathbf{S}, m)$ into $(Y, \mathscr{U})$ the following holds

(1) the almost uniform convergence of a considered net always implies the almost ess. unif. convergence;

(2) the almost ess. unif. convergence implies the almost ess. unif. (w) convergence.

Note 2. For each sequence $\{f_n\}_{n=1}^{\infty}$ of transformations from $(X, \mathbf{S}, m)$ to $(Y, \mathcal{U})$ the almost ess unif. convergence and the almost ess. unif (w) convergence are equivalent.

Note 3. If for the range space of transformations f_n, $n = 1, 2, \ldots$ a separable pseudometrizable uniform space is considered then all three kinds of almost uniform convergence are equivalent.

Lemma 1. *Suppose that f, g are measurable transformations from the measure space $(X, \mathbf{S}, m)$ into a separable pseudometrizable uniform space $(Y, \mathcal{U})$. Then the set $E = \{x : x \in X, (f(x), g(x)) \in U, U$ is an open set in the product space $(Y \times Y)$, $U \in \mathcal{U}\}$ is measurable.*

Proof. Since a separable pseudometrizable uniform space possesses a countable base of uniformity it satisfies the second axiom of countability. We consider the countable base of open sets formed by the sets $\{\{U_k[y_i]\}_{k=1}^{\infty}\}_{i=1}^{\infty}$ where $\{y_i\}_{i=1}^{\infty}$ is a countable dense set in $(Y, \mathcal{U})$ and $\{U_k\}_{k=1}^{\infty}$ is a countable base of the uniformity formed by open members of $\mathcal{U}$. Now let U be an arbitrary open member of the uniformity $\mathcal{U}$. Then

$$U = \bigcup_{l=1}^{\infty} (U_{k_l}[y_{i_l}] \times U_{k_l}[y_{j_l}]),$$

thus

$$E = \{x : x \in X, (f(x), g(x)) \in U\} = \bigcup_{l=1}^{\infty} \{x : x \in X, (f(x), g(x)) \in$$

$$\in (U_{k_l}[y_{i_l}] \times U_{k_l}[y_{j_l}])\} = \bigcup_{l=1}^{\infty} (x : x \in X, f(x) \in U_{k_l}[y_{i_l}]) \cap (x : x \in X, g(x) \in$$

$$\in U_{k_l}[y_{j_l}])$$

It follows immediately that the set E is measurable. Q. e. d.

Corollary. *Suppose that $\{f_t, t \in T\}$ is a quasicountable net or a sequence of measurable transformations mapping a finite measure space $(X, \mathbf{S}, m)$ into a separable uniform space $(Y, \mathcal{U})$ with a countable base of the uniformity $\mathcal{U}$. Then the almost everywhere convergence of the considered net to the measurable transformation f implies the almost ess. unif. (w) convergence to f.*

Proof. Let us define the following sets:

$$E_n^t = \{x : x \in X, (f(x), f_t(x)) \in U_n\},$$

$t \in T$, $n = 1, 2, \ldots$ $\{U_n\}_{n=1}^{\infty}$ is the countable base of the uniformity $\mathcal{U}$. Then these sets are measurable by the foregoing lemma and with mere notational changes it is possible to apply the same proof as in [6]. Q. e. d.

Note 4. If f is a measurable transformation and $\{f_t, t \in T\}$ is a seqeunce as in the corollary, converging everywhere pointwise to f then, by Note 3 the almost uniform convergence of $\{f_t, t \in T\}$ to f follows.

2. The aim of this section is to show that no type of almost uniform convergence follows in general from the pointwise convergence almost everywhere. For this end a corresponding counterexample is constructed.

We shall consider the following finite measure space $(X, \mathbf{S}, m)$. $X = \langle 0,1 \rangle$, $\mathbf{S}$ is a σ-algebra of the Boreal sets of X, m is the Lebesque measure defined on $\mathbf{S}$. As a uniform space $(Y, \mathscr{U})$ the following pair is considered: Y is the set of all mappings $f(t, x)$, mapping the set $\langle 0,1 \rangle \times \langle 0, \infty)$ into the set $\{0,1\}$ and taking the following form:

For an arbitrary fixed $t_0 \in \langle 0,1 \rangle$

$$f(t_0, x) = 0 \quad \text{if} \quad x \in \langle 0, x_0), \qquad f(t_0, x) = 1 \quad \text{if} \quad x \geqslant x_0,$$

where x_0 is a positive real number or ∞.

$\mathscr{U}$ is the uniformity generated in the sense of [2, p. 187] by the system $\mathbf{P}$ of pseudometrics for Y.

$$\mathbf{P} = \{d_{(\tau, \xi)}(f(t, x), g(t, x)), (\xi, \xi) \in (\langle 0, 1 \rangle \times \langle 0, \infty))\}$$

where $d_{(\tau, \xi)}(f(t, x), g(t, x)) = |f(\tau, \xi) - g(\tau, \xi)|$.

Now let us define the following sequence of transformations $\{F_n\}_{n=1}^{\infty}$, each of them mapping $(X, \mathbf{S}, m)$ into $(Y, \mathscr{U})$.

For an arbitrary $z \in \langle 0,1 \rangle$, $F_n(z) = f_n^z(t, x) \in (Y, \mathscr{U})$, where

(1) $\quad f_n^z(t, x) = 0 \quad \text{if} \quad t < 1 - z \quad \text{and} \quad x \in \left\langle 0, \left(1 - \dfrac{z}{1-t}\right) \cdot n \right)$;

(2) $\quad f_n^z(t, x) = 1 \quad \text{if} \quad t < 1 - z \quad \text{and} \quad x \geqslant \left(1 - \dfrac{z}{1-t}\right) \cdot n$;

(3) $f_n^z(t, x) = 0 \quad \text{if} \quad t \geqslant 1 - z \quad \text{and} \quad x \in \langle 0, \infty)$.

Now it will be proved that

(1) the sequence $\{F_n\}_{n=1}^{\infty}$ converges everywhere on X to the measurable transformation F_0 defined as follows:

$$F_0(z) = f_0^z(t, x) \in (Y, \mathscr{U}) \text{ where}$$
$$f_0^z(t, x) = 0 \text{ on all } \langle 0,1 \rangle \times \langle 0, \infty);$$

(2) each of the transformations F_n, $n = 1, 2, \ldots$ is measurable;

(3) the sequence $\{F_n\}_{n=1}^{\infty}$ does not converge, either almost uniformly or ess. uniformly in the weaker sense, to F_0.

(1) To prove the pointwise convergence of the sequence $\{F_n\}_{n=1}^{\infty}$ to F_0 it is

sufficient to show that for an arbitrary $z \in \langle 0,1\rangle$, $\varepsilon > 0$ and $(\tau, \xi) \in \langle 0,1\rangle \times \times \langle 0, \infty)$ there is an integer n_0 such that $n > n_0$ implies

$$d_{(\tau,\xi)}(F_n(z), F_0(z)) < \varepsilon$$

i. e.

$$d_{(\tau,\xi)}(F_n(z), F_0(z)) = |f_n^z(\tau, \xi) - f_0^z(\tau, \xi)| = f_n^z(\tau, \xi) < \varepsilon.$$

It is evident that if $\tau < 1 - z$ and $n > \left[\dfrac{\tau - 1}{z - 1 + \tau} \cdot \xi\right]$ then

$$d_{(\tau,\xi)}(F_n(z), F_0(z)) = 0;$$

if $\tau \geqslant 1 - z$, $n = 1, 2, \ldots$ then

$$d_{(\tau,\xi)}(F_n(z), F_0(z)) = 0. \qquad \text{Q. e. d.}$$

(2) Now it must be proved that each of the transformations F_n, $n = 1, 2, \ldots$ is measurable. It is obvious when $n = 0$. In case $n > 0$ we consider the sets

$$F_n^{-1}({}_{(\tau,\xi)}O_{y_0}^{\varepsilon})$$

where

$${}_{(\tau,\xi)}O_{y_0}^{\varepsilon} = \{y | y \in Y,\ d_{(\tau,\xi)}(y, y_0) < \varepsilon,\ y_0 \in (Y, \mathscr{U}),\ (\tau, \xi) \in (\langle 0,1\rangle \times \langle 0, \infty))\}.$$

Let $y_0 = f(t, x)$ such that if $t = \tau$ then

$$f(\tau, x) = 0 \text{ for } x \in \langle 0, x')$$
$$f(\tau, x) = 1 \text{ for } x \geqslant x'.$$

When $\varepsilon > 1$ it is evident that $F_n^{-1}({}_{(\tau,\xi)}O_{y_0}^{\varepsilon}) = \langle 0,1\rangle \in \mathbf{S}$.
When $\varepsilon \leqslant 1$ there are several possibilities shown below:

		$F_n^{-1}({}_{(\tau,\xi)}O_{\varepsilon}^{y_0})$	
(a) $n < x'$	$\xi < n$	$\langle 0, a) \cup \langle 1 - \tau, 1\rangle$	$a \in \langle 0,1\rangle$
	$\xi = n$	$\langle 1 - \tau, 1\rangle$	
	$n < \xi < x'$	$\langle 1 - \tau, 1\rangle$	
	$\xi \geqslant x'$	$\langle 0,1 - \tau)$	
(b) $n = x'$	$\xi < n$	$\langle 0, a) \cup \langle 1 - \tau, 1\rangle$	$a \in \langle 0,1\rangle$
	$\xi \geqslant n$	$\langle 0,1 - \tau)$	
(c) $n > x'$	$\xi < x'$	$\langle 0, a) \cup \langle 1 - \tau, 1\rangle$	$a \in \langle 0,1\rangle$
	$\xi = x'$	$\langle b, 1 - \tau)$	$b \in \langle 0,1\rangle$
	$x' < \xi < n$	$\langle b', 1 - \tau)$	$b' \in \langle 0,1\rangle$
	$\xi \geqslant n$	$\langle 0,1 - \tau)$	

By the definition of the uniformity $\mathscr{U}$ it is obvious that the system of all finite intersections of sets ${}_{(\tau,\xi)}O_y^{\varepsilon}$, where $(\tau, \xi) \in \langle 0,1\rangle \times \langle 0, \infty)$ and $\varepsilon > 0$, $y \in \in Y$, form a base of open sets in $(Y, \mathscr{U})$. Thus for each open set $O \subset Y$ the following holds

$$F_n^{-1}(O) = \bigcup_{\gamma \in \Gamma} I_\gamma$$

where each I_γ takes one of the following forms: $\langle 0, a)$, $\langle b, c)$, $\langle d, 1\rangle$, a, b, c, $d \in \langle 0,1\rangle$. Therefore the system $\{I_\gamma\}_{\gamma \in \Gamma}$ contains at most countably many disjoint sets. Let us choose them and denote them by $I_1, I_2, \ldots$ Now if $\{I_{\gamma_i}\}_{\gamma_i \in \Gamma_i}$ denotes the system of all I_γ satisfying $I_\gamma \cap I_i \neq \emptyset$ it follows immediately that

$$\bigcup_{\gamma \in \Gamma} I_\gamma = \bigcup_{i=1}^{\infty} \bigcup_{\gamma_i \in \Gamma_i} (I_{\gamma_i} \cup I_i).$$

Further the set $\bigcup_{\gamma_i \in \Gamma_i} (I_{\gamma_i} \cup I_i)$ takes one of the following forms: $\langle b, c\rangle$, $\langle b, c)$, $(b, c\rangle$, (b, c), $b, c \in \langle 0,1\rangle$ and thus the set $F_n^{-1}(O)$ is measurable. Q. e. d.

(3) Finally it will be shown that the constructed sequence fails to be convergent almost uniformly in the ordinary sense as well as almost ess. uniformly and thus according to Note 2 it fails to be convergent almost ess. uniformly in the weaker sense.

It can easily be seen that $\{F_n\}_{n=1}^{\infty}$ does not converge uniformly on any set which would have both a point $z_0 \in (0,1\rangle$ and at the same time a sequence of numbers $\{z_n\}_{n=1}^{\infty}$, $\lim_n z_n = z_0$, $z_n < z_0$, $n = 1, 2, \ldots$ In fact, let z_0 be an arbitrary fixed point from $(0,1\rangle$. Let us suppose arbitrary $1 > \varepsilon > 0$ and $\xi \in (0, \infty)$. Then for a sufficiently large integer n and for each $z \in \left(z_0 - \xi \cdot \frac{z_0}{n}, z_0\right)$ the following holds

$$d_{(1-z_0,\xi)}\,(F_n(z), F_0(z)) = 1.$$

It follows that if on a set $E \subset X$ the considered sequence tends uniformly to F_0 then E the relation

$$E \subset \langle 0,1\rangle - \bigcup_{z \in (0,1\rangle} (z - \varepsilon_z, z),$$

for some $\varepsilon_z < 0$ satisfies. Then E is at most countable thus measurable and $m(E) = 0$.

Similarly we can see that the considered sequence does not converge ess. uniformly on any set which together with some point $z_0 \in (0,1\rangle$ for each $\eta > 0$ contains some set $A(\eta, z_0) \subset (z_0 - \eta, z_0)$, such that $m(A(\eta, z_0)) > 0$. Thus each set F, on which the considered sequence can ess. uniformly conver to F_0, is of the following form:

$$F \subset \langle 0,1\rangle - \bigcup_{z \in (0,1\rangle} A_z$$

where for each $z \in (0,1\rangle$, $A_z \subset (z - \eta_z, z)$, for some $\eta_z > 0$ and $m(A_z) = m(z - \eta_z, z) > 0$.

But $$\langle 0,1\rangle - \bigcup_{z\in(0,1\rangle} A_z = \Big(\bigcup_{z\in(0,1\rangle} (z-\eta_z, z) \cup Z\Big) - \bigcup_{z\in(0,1\rangle} A_z$$

where $m(Z) = 0$. Further there are countably many $z_n \in (0,1\rangle$ so that

$$\bigcup_{z\in(0,1\rangle} (z-\eta_z, z) - \bigcup_{z\in(0,1\rangle} A_z = \bigcup_{n=1}^{\infty} (z_n - \eta_{z_n}, z_n) - \bigcup_{z\in(0,1\rangle} A_z \subset \bigcup_{n=1}^{\infty} (z_n - \eta_{z_n}, z_n) - \bigcup_{n=1}^{\infty} A_{z_n}$$

where $m(\bigcup_{n=1}^{\infty} (z_n - \eta_{z_n}, z_n)) = m(\bigcup_{n=1}^{\infty} A_{z_n})$. Thus the only sets on which the considered sequence can converge ess. uniformly to F_0 are the sets of the measure zero. Q. e. d.

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