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NOTE ON THE NILPOTENCY IN COMPACT H-SEMIGROUPS

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K. Numakura introduced (see [1] and [2]) the notion of the nilpotent element of a topological semigroup S with respect to an ideal of S. By extending this notion we obtain three kinds of nilpotency (as in paper [3] in the case of semigroups without topology). The present paper deals with relations among these kinds of nilpotency and with some properties of sets of nilpotent elements of compact H-semigroups. Some of the results we obtained in the case of compact H-semigroups are conformable with the corresponding results of paper [3].

By a general topological space we mean a set S with a family $\mathfrak{S}$ of subsets of S (called open sets), which satisfies the following conditions:

1) S and $\emptyset$ (the empty set) belong to $\mathfrak{S}$.
2) The union of an arbitrary subfamily of $\mathfrak{S}$ belongs to $\mathfrak{S}$.
3) The intersection of every finite subfamily of $\mathfrak{S}$ belongs to $\mathfrak{S}$.

The topological space S is called compact if from any covering of this space by open sets a finite family of open sets can be chosen which covers the space S (finite covering).

By a topological semigroup (H-semigroup) [compact H-semigroup] we mean a general topological (Hausdorff) [compact Hausdorff] space together with a continuous associative multiplication.

In a topological semigroup S we can introduce the following notions.

Definition 1. *Let S be a topological semigroup and M a subset of S*

a) *Let $x \in S$ and for every neighbourhood U of M let there exist a positive integer N such that for all integers $n \geqslant N$ (for almost all n) $x^n \in U$ holds Then the element x will be called strongly nilpotent with respect to M.*

b) *Let $x \in S$ and let every neighbourhood U of M contain x^n for infinitely many positive integers n. Then the element x will be called weakly nilpotent with respect to M.*

c) *Let $x \in S$ and let every neighbourhood U of M contain at least one power x^n. Then the element x will be called almost nilpotent with respect to M.*

The set of all strongly nilpotent elements with respect to M will be denoted by $N_1(M)$, the set of all weakly nilpotent elements with respect to M will be denoted by $N_2(M)$ and the set of all almost nilpotent elements with respect to M will be denoted by $N_3(M)$.

Remark 1. From definition 1 it is clear that each strongly nilpotent element with respect to M is weakly nilpotent with respect to M and each weakly nilpotent element with respect to M is almost nilpotent with respect to M. Thus $N_1(M) \subseteq N_2(M) \subseteq N_3(M)$.

The following examples show that these notions differ even if S is a compact H-semigroup.

Example 1. Let $S = \langle 0, 1 \rangle$ with the ordinary multiplication as operation and with the ordinary topology. Then S is a compact H-semigroup and the element $x = \frac{1}{2}$ is almost nilpotent with respect to $M = \{\frac{1}{4}\}$, but it is not weakly nilpotent with respect to M.

In the same semigroup let us put $M = \left\{ \frac{1}{2^{2k}} \middle| k = 1, 2, \ldots \right\}$. M is a subsemigroup, $x = \frac{1}{2}$ is weakly nilpotent with respect to M but it is not strongly nilpotent with respect to M.

Thus a weakly nilpotent element with respect to M need not be strongly nilpotent with respect to M even if M is a subsemigroup of a compact H-semigroup. Further we shall show that a weakly nilpotent element with respect to M need not be strongly nilpotent with respect to M even if M is a closed subsemigroup of a compact H-semigroup S.

Example 2. $S_1 = \{0, 1\}$ with the addition mod 2 as operation is a compact H-semigroup and $S_2 = \{0\} \cup \left\{ \frac{1}{2^k} \middle| k = 1, 2, \ldots \right\}$ with the ordinary multiplication as operation is also a compact H-semigroup (S_1 with the discrete topology and S_2 with the usual relative topology). Thus their direct product $S' = S_1 \times S_2$ is a compact H-semigroup too. Let $S = \{(0, 0), (1, 0)\} \cup \left\{ \left(0, \frac{1}{2^{2k}}\right) \middle| k = 1, 2, \ldots \right\} \cup \left\{ \left(1, \frac{1}{2^{2k-1}}\right) \middle| k = 1, 2, \ldots \right\}$. S is a closed subsemigroup of S' and therefore it is also a compact H-semigroup. $(0, 0)$ is an idempotent of S, thus $M = \{(0, 0)\}$ is a closed subsemigroup of S, $x = (1, \frac{1}{2})$ is a weakly nilpotent element with respect to M but it is not strongly nilpotent with respect to M.

Remark 2. In the case of M being an open subset of the topological semigroup S, the element x of S is strongly (weakly) [almost] nilpotent with respect to M if and only if it is strongly (weakly) [almost] nilpotent with respect to M in the sense of paper [3].

Lemma 1. *Let S be a topological semigroup and let S be a T_1-space. Let M be a subsemigroup of S. Then every almost nilpotent element with respect to M is weakly nilpotent with respect to M,* i. e. *if M is a subsemigroup then $N_3(M) = N_2(M)$.*

Proof. If an almost nilpotent element x were not weakly nilpotent then a neighbourhood U of M would exist which would contain only a finite number of powers x^n. If some of them, for example x^n were contained in M then M would contain infinitely many further powers of the element x (namely all powers x^{kn}, $k = 1, 2, \ldots$) and the element x would be weakly nilpotent. If no power x^n were contained in M, then an open set V could be formed, which would contain M but it would contain no power of the element x. This is impossible.

Lemma 2. *Let S be a compact H-semigroup and M a left (right) [two sided] ideal of S. Then every weakly nilpotent element with respect to M is strongly nilpotent with respect to M.*

Proof. Let us denote $X = \{x^k \mid k = 1, 2, \ldots\}$. Then $\bar{X}$ (the closure of the set X) is a closed subset of S. Further $SM \subseteq M$, therefore $\bar{X}M \subseteq M$ too. Let W be any neighborhood of the ideal M. Then W is also a neighbourhood of each element $m \in M$. From $y \in \bar{X}$ and $m \in M$ if follows that $ym \in M$ and W is a neighbourhood of the element ym. Thus there exist a neighbourhood $U_m(y)$ of the element y and a neighbourhood $V_y(m)$ of the element m such that $U_m(y)\ V_y(m) \subseteq W$. Let us fix an element $m \in M$ and denote $\bigcup_{y \in \bar{X}} U_m(y) = U_m$. The open sets $U_m(y)$, $y \in \bar{X}$ cover the closed set $\bar{X}$. In consequence of the compactness of S a finite number of such open sets is sufficient for it. Let these sets be $U_m(y_1)$, $U_m(y_2)$, $\ldots$, $U_m(y_k)$. Hence $U_m = \bigcup_{i=1}^{k} U_m(y_i)$. Let. us further denote $V(m) = \bigcap_{i=1}^{k} V_{y_i}(m)$ and $V = \bigcup_{m \in M} V(m)$ Under these conditions V is a neighbourhood of the ideal M and U_m is a neighbourhood of the set $\bar{X}$ (and of the set X too). Clearly $U_m\ V(m) \subseteq W$, therefore $XV(m) \subseteq W$ and hence $XV \subseteq W$. But V contains at least one power x^n, thus W contains almost all powers of the element x (as it contains the elements x^{n+1}, x^{n+2}, x^{n+3}, $\ldots$). Hence x is strongly nilpotent with respect to M, q. e. d.

We now introduce another characterization of the weak nilpotency of an element, which will be used in proving lemma 8.

Lemma 3. *Let S be a topological semigroup and let S be a T_1-space. Let M be a subset of S. Then an element $x \in S$ is weakly nilpotent with respect to M if and only if at least one of the following conditions holds:*

a) *for infinitely many positive integers n, $x^n \in M$ holds*

b) *at least one accumulation point of the sequence* $\{x^k\}_{k=1}^{\infty}$ *is contained in* M.

Proof. If condition a) holds then x is clearly weakly nilpotent with respect to M. If condition b) holds then in every neighbourhood of M (which is a neighbourhood of each accumulation point contained in M of the sequence $\{x^k\}_{k=1}^{\infty}$) there are contained infinitely many powers x^n. Thus x is a weakly nilpotent element with respect to M.

Let neither condition *a*) nor condition b) hold. Than at most finitely many powers $x^{n_1}, x^{n_2}, \ldots, x^{n_k}$ are contained in M. These elements have such neighbourhoods that apart from these elements, do not contain further elements of $\{x^k\}_{k=1}^{\infty}$ and the other elements $m \in M$ have neighbourhoods which contain no elements of $\{x^k\}_{k=1}^{\infty}$. Thus there exists such an open set V that is a neighbourhood of M and which contains at most a finite number of powers x^n. Hence x is not weakly nilpotent with respect to M.

In the case of M being a subsemigroup of S we can formulate the weak nilpotency with respect to M in quite another way.

Lemma 4. *Let* S *be a topological semigroup and let* S *be a* T_1*-space. Let* M *be a subsemigroup of* S*. Then an element* $x \in S$ *is weakly nilpotent with respect to* M *if and only if at least one of the following conditions holds:*

a) *at least one power* x^n *is contained in* M,

b) *at least one accumulation point of the sequence* $\{x^k\}_{k=1}^{\infty}$ *is contained in* M.

The proof follows from the fact that in the case of M being a semigroup, $x^n \in M$ holds for a positive integer n if and only if infinitely many powers of the element x are contained in M.

Lemma 5. *Let* S *be a topological semigroup. Let* $M_\varkappa$, $\varkappa \in K$ *be subsets of* S*. Then* $\bigcup_{\varkappa \in K} N_3(M_\varkappa) = N_3(\bigcup_{\varkappa \in K} M_\varkappa)$.

Proof. a) For every $\varkappa \in K$ we have $M_\varkappa \subseteq \bigcup_{\varkappa \in K} M_\varkappa$. Therefore for every $\varkappa \in K$, $N_3(M_\varkappa) \subseteq N_3(\bigcup_{\varkappa \in K} M_\varkappa)$ holds. Hence $\bigcup_{\varkappa \in K} N_3(M_\varkappa) \subseteq N_3(\bigcup_{\varkappa \in K} M_\varkappa)$.

b) Let $x \in N_3(\bigcup_{\varkappa \in K} M_\varkappa)$. Then every neighbourhood W of the set $\bigcup_{\varkappa \in K} M_\varkappa$ contains some powers of the element x. Thus there exists a $\varkappa_0 \in K$ such that every neighborhood U of the set $M_{\varkappa_0}$ contains a power of x. If it were not so then for each $\varkappa \in K$ there would exist such a neighbourhood $U_\varkappa$ of the set $M_\varkappa$ that would contain no power of x. Then the neighbourhood $W = \bigcup_{\varkappa \in K} U_\varkappa$ of the set $\bigcup_{\varkappa \in K} M_\varkappa$ would contain no power of x. This is a contradiction. Hence $x \in N_3(M_{\varkappa_0})$ and therefore $N_3(\bigcup_{\varkappa \in K} M) \subseteq \bigcup_{\varkappa \in K} N_3(M_\varkappa)$. This together with a) gives $N_3(\bigcup_{\varkappa \in K} M_\varkappa) = \bigcup_{\varkappa \in K} N_3(M_\varkappa)$.

Lemma 6. *Let S be a topological semigroup. Let M_1 and M_2 be subsets of S. Then $N_2(M_1 \cup M_2) = M_2(N_1) \cup N_2(M_2)$.*

Proof. a) $N_2(M_1) \cup N_2(M_2) \subseteq N_2(M_1 \cup M_2)$.

b) Let $x \in N_2(M_1 \cup M_2)$. Then every neighbourhood W of the set $M_1 \cup M_2$ contains infinitely many powers x^n of the element x. If x were contained neither in $N_2(M_1)$ nor in $N_2(M_2)$ then a neighbourhood U of the set M_1 and a neighbourhood V of the set M_2 would exist which would contain only a finite number of powers of x. Then the neighborhood $U \cup V$ of the set $M_1 \cup M_2$ would contain only a finite number of powers of x. But this is impossible.

The following example shows that $N_1(M_1 \cup M_2) = N_1(M_1) \cup N_1(M_2)$ need not hold.

Example 3. Let $S = \{0\} \cup \left\{\frac{1}{2^k} \middle| k = 1, 2, \ldots\right\}$ with the ordinary multiplication as operation and with the usual topology. S is a compact H-semigroup. Let $M_1 = \left\{\frac{1}{2^{2k}} \middle| k = 1, 2, \ldots\right\}$ and $M_2 = \left\{\frac{1}{2^{2k+1}} \middle| k = 1, 2, \ldots\right\}$. Then $\frac{1}{2} \in N_1(M_1 \cup M_2)$ but $\frac{1}{2} \notin N_1(M_1) \cup N_1(M_2)$.

Lemma 7. *Let S be a compact H-semigroup and let M_1 and M_2 be closed subsets of S. Then $N_1(M_1 \cap M_2) = N_1(M_1) \cap N_1(M_2)$.*

Proof. a) If $A \subseteq B$ then $N_1(A) \subseteq N_1(B)$. Thus $N_1(M_1 \cap M_2) \subseteq N_1(M_1) \cap N_1(M_2)$.

b) Let $x \in N_1(M_1) \cap N_1(M_2)$ i. e. $x \in N_1(M_1)$ and $x \in N_1(M_2)$. Let us choose any neighbourhood U of the set $M_1 \cap M_2$(1). Then $M_1 \setminus U$ and $M_2 \setminus U$ are closed, disjoint sets. Therefore there exist such open sets $U_1 \supseteq M_1 \setminus U$ and $U_2 \supseteq M_2 \setminus U$ which are disjoint too ($U_1 \cap U_2 = \emptyset$). The sets $V_1 = U_1 \cup U$ and $V_2 = U_2 \cup U$ are also open sets and V_1 is a neighbourhood of the set M_1 and V_2 is a neighbourhood of the set M_2. Hence beginning with a positive integer all powers of the element x are contained in V_1 and in V_2, i. e. beginning with a positive integer all powers of the element x are contained in $V_1 \cap V_2$. But $V_1 \cap V_2 = (U_1 \cup U) \cap (U_2 \cup U) = (U_1 \cap U_2) \cup (U_1 \cap U) \cup (U \cap U_2) \cup (U \cap U) = U$ (because $U_1 \cap U_2 = \emptyset$, $U_1 \cap U \subseteq U$, $U \cap U_2 \subseteq U$, $U \cap U = U$). Thus, beginning with a positive integer all powers of the element x are contained in U. This means that $x \in N_1(M_1 \cap M_2)$ and therefore $N_1(M_1) \cap N_1(M_2) \subseteq N_1(M_1 \cap M_2)$.

(1) If U is a neighbourhood of at least one of the sets M_1 and M_2 then clearly almost all powers of the element x are contained in U. In the following we consider only the case where U is neither a neighbourhood of M_1 nor a neighbourhood of M_2.

The following example shows the fact that M_1 and M_2 are closed to be essential.

Example 4. Let $S = \{r \mid r \in \langle 0,1 \rangle\} \cup \left\{1 + \frac{1}{2^k} \middle| \ k = 1, 2, \ldots\right\}$ with the following operation in S:

Let $r_1 r_2$ be the ordinary product of the real numbers r_1 and r_2 if $r_1, r_2 \in \langle 0, 1 \rangle$,

$$\left(1 + \frac{1}{2^k}\right)\left(1 + \frac{1}{2^l}\right) = \left(1 + \frac{1}{2^{k+l}}\right) \text{ for } k, l = 1, 2, \ldots,$$

$$r\left(1 + \frac{1}{2^k}\right) = \left(1 + \frac{1}{2^k}\right) \cdot r = r \text{ for } r \in \langle 0, 1 \rangle \text{ and } k = 1, 2, \ldots.$$

S is a semigroup which is a compact H-semigroup if we take for the topology in S the usual relative topology.

Let $M_1 = S \setminus \{1\}$ and $M_2 = \{r \mid r \in \langle 0, 1 \rangle\}$. M_1 is a subsemigroup and M_2 is even an ideal of S but M_1 is not a closed set. $M_1 \cap M_2 = \{r \mid r \in \langle 0, 1 \rangle\}$. $x = (1 + \frac{1}{2})$ is a strongly nilpotent element with respect to M_1 and M_2, thus $x \in N_1(M_1) \cap N_1(M_2)$. But on the other hand x is not even almost nilpotent with respect to $M_1 \cap M_2$. Hence $x \notin N_1(M_1 \cap M_2)$.

Remark 3. This example shows simultaneously that even if S is a compact H-semigroup $N_2(M_1) \cap N_2(M_2) = N_2(M_1 \cap M_2)$ need not hold even if M_1 is a subsemigroup and M_2 an ideal of S.

If M_1 and M_2 are two-sided ideals, lemma 7 can be improved (it holds without the condition that M_1 and M_2 are closed).

Lemma 8. *Let S be a compact H-semigroup and let M_1 and M_2 be two-sided ideals of S. Then $N_1(M_1 \cap M_2) = N_1(M_1) \cap N_1(M_2)$.*

Proof. a) $N_1(M_1 \cap M_2) \subseteq N_1(M_1) \cap N_1(M_2)$.

b) Let $x \in N_1(M_1) \cap N_1(M_2)$. Then $x \in N_1(M_1)$ and $x \in N_1(M_2)$ and we have the following possibilities (lemmas 4 and 2):

1) For some positive integers n_1 and n_2, $x^{n_1} \in M_1$ and $x^{n_2} \in M_2$ holds.

2) M_1 contains an accumulation point m_1 of the sequence $\{x^k\}_{k=1}^{\infty}$ and M_2 contains an accumulation point m_2 of the sequence $\{x^k\}_{k=1}^{\infty}$.

3) There exists a positive integer n such that $x^n \in M_1$ and M_2 contains an accumulation point m_2 of the sequence $\{x^k\}_{k=1}^{\infty}$

4) There exists a positive integer n such that $x^n \in M_2$ and M_1 contains an accumulation point m_1 of the sequence $\{x^k\}_{k=1}^{\infty}$.

In the first case $x^{n_1+n_2} \in M_1 \cap M_2$ and $x \in N_1(M_1 \cap M_2)$.

In the second case $m_1 m_2 \in M_1 \cap M_2$ and $m_1 m_2$ is an accumulation point of the sequence $\{x^k\}_{k=1}^{\infty}$. In fact for every neighbourhood W of $m_1 m_2$ there exists a neighbourhood U of m_1 and a neighbourhood V of m_2 such that

$UV \subseteq W$. But U and V contain infinitely many members of the sequence $\{x^k\}_{k=1}^{\infty}$ and therefore W contains infinitely many members of this sequence too. Hence $x \in N_1(M_1 \cap M_2)$.

In the third case we have $x^n m_2 \in M_1 \cap M_2$ and it is easy to see (as in the second case) that $x^n m_2$ is an accumulation point of the sequence $\{x^k\}_{k=1}^{\infty}$ i. e. $x \in N_1(M_1 \cap M_2)$.

In the fourth case there is $x^n m_1 \in M_1 \cap M_2$ and $x^n m_1$ is an accumulation point of the sequence $\{x^k\}_{k=1}^{\infty}$, i. e. we have again $x \in N_1(M_1 \cap M_2)$.

Thus in all four possible cases we obtain $x \in N_1(M_1 \cap M_2)$. In this way we proved that $N_1(M_1) \cap N_1(M_2) \subseteq N_1(M_1 \cap M_2)$ and this together with a) gives $N_1(M_1 \cap M_2) = N_1(M_1) \cap N_1(M_2)$.

From the following example we can see that even if S is a compact H-semigroup, $N_3(M_1) \cap N_3(M_2) = N_3(M_1 \cap M_2)$ need not hold.

Example 5. Let $S = \{0\} \cup \left\{ \frac{1}{2^k} \middle| k = 0, 1, 2, \ldots \right\}$ with the ordinary multiplication as operation and with the usual relative topology. S is a compact H-semigroup. Let $M_1 = \{1, \frac{1}{2}\}$ and $M_2 = \{1, \frac{1}{4}\}$ (these are open sets). Then $N_3(M_1) = \{1, \frac{1}{2}\}$, $N_3(M_2) = \{1, \frac{1}{2}, \frac{1}{4}\}$ and $N_3(M_1) \cap N_3(M_2) = \{1, \frac{1}{2}\} \neq \{1\} = $
$= M_1 \cap M_2 = N_3(M_1 \cap M_2)$. (The same results hold by the discrete topology, only S is not a compact space.)

Remark 4. Lemmas 6, 7 and 8 can be extended by induction from two subsets M_1 and M_2 to any finite number of subsets $M_\varkappa$, $\varkappa \in K$.

From the foregoing lemmas follow

Theorem 1. *Let S be a topological semigroup. Then the mappings $M \to N_2(M)$ and $M \to N_3(M)$ are endomorphisms of the $\cup$-semilattice of all subsets of S.*

Theorem 2. *Let S be a compact H-semigroup. Then*

a) *the mapping $M \to N_1(M)$ is a homomorphism of the $\cap$-semilattice of all closed subsets of S into the $\cap$-semilattice of all subsets of S,*

b) *the mapping $M \to N_1(M)$ is a homomorphism of the lattice of all two-sided deals of S into the lattice of all subsets of S.*

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