Matematicko-fyzikálny časopis

Juraj Virsik

The Representation of Inertial Particles in the Lie Algebra of the Lorentz Group

Matematicko-fyzikálny časopis, Vol. 16 (1966), No. 1, 82--90

Persistent URL: `http://dml.cz/dmlcz/126719`

MATEMATICKO-FYZIKÁLNY ČASOPIS SAV, 16, 1, 1966

THE REPRESENTATION OF INERTIAL PARTICLES IN THE LIE ALGEBRA OF THE LORENTZ GROUP

JURAJ VIRSIK, Bratislava

Some Lie properties of the general Lorentz group are investigated and an application of them to the space-time structure of the special relativity theory is given.

All the matrices dealt with are supposed to be real. The Lie group of regular $(n \times n)$-matrices $X = [x^{ik}]$ is denoted by $GL(n, R)$. Let $G = [g_{ab}]$ be a fixed regular diagonal $(n \times n)$-matrix. The matrices X with elements x^{ik} satisfying the $\frac{n(n+1)}{2}$ equations

$$F^{kl} \equiv g_{ab}x^{ak}x^{bl} - g^{kl} = 0; \quad k \leq l \tag{1}$$ [1]

form a subgroup $\mathfrak{G}(G)$ of $GL(n, R)$. This can be easily established observing that (1) is equivalent to $X^*GX = G$, where X^* denotes the transpose of X. In other words, $\mathfrak{G}(G)$ is the general Lorentz group of matrices X which leave invariant the quadratic form

$$\xi \to \xi^* G \xi \tag{2}$$

on R^n. Note that $\mathfrak{G}(E)$ (E the unity matrix) is the orthogonal group $\mathfrak{O}(n)$ and $\mathfrak{G}(L)$, with L, the diagonal matrix of the form

$$\xi_1^2 + \xi_2^2 + \xi_3^2 - c^2\xi_4^2, \quad (c > 0), \tag{3}$$

is the usual full Lorentz group.

Next we shall explicitly show that $\mathfrak{G}(G)$ is a Lie subgroup of $GL(n, R)$ and point out a concrete local chart of $\mathfrak{G}(G)$ containing the unity element E

Lemma. *The Jacobian of* (1), *i. e.*

---

[1] Summation over repeated indices. No „geometrical" difference is made between upper and lower indices.

$$\det\left(\frac{\partial F^{kl}}{\partial x^{ij}}\right) \quad \textit{with } k \leq l,\ i \leq j \tag{4}$$

taken at the point E, is non-zero.

Proof. Direct differentiation of (1) gives

$$\frac{\partial F^{kl}}{\partial x^{ij}}(E) = g_{il}\delta_{jk} + g_{ki}\delta_{jl} \tag{5}$$

(δ is the usual Kronecker symbol). This is a matrix of order $\frac{n(n+1)}{2}$ for $i \leq j$, $k \leq l$. Let us suppose there exists a non-trivial system of numbers y^{kl} ($k \leq l$) satisfying the $\frac{n(n+1)}{2}$ linear equations

$$\sum_{k \leq l} (g_{il}\delta_{jk} + g_{ki}\delta_{jl})\, y^{kl} = 0; \qquad i \leq j. \tag{6}$$

If we define $y^{kl} = 0$ for $k > l$ and denote $Y = [y^{kl}]$ then (6) asserts that the matrix $G(Y^* + Y)$ has zero elements on its principal diagonal and above it. Furthermore it is also symmetric and hence $G(Y^* + Y) = 0$ i. e. $Y = 0$. Thus the determinant (4) is necessarily non-zero.

Applying the implicit function theorem one can find a neighbourhood $\mathscr{U}_0$ of the origin $\mathcal{O}$ in R^N $\left(N = \frac{n(n-1)}{2}\right)$, a neighbourhood $\mathscr{U}(E)$ of the matrix E in $\mathfrak{G}(G)$ ($\mathfrak{G}(G)$ provided with the topology induced by the natural topology in R^{n^2}) and a homeomorphism φ_0: $\mathscr{U}(E) \to \mathscr{U}_0$. This φ_0 has the properties:

$$i > j \Rightarrow [\varphi_0^{-1}(x^{ij})]^{kl} = \begin{cases} x^{kl} \text{ for } k > l \\ h^{kl}(x^{ij}) \text{ for } k \leq l \end{cases} \tag{7a}$$

where h^{kl} ($k \leq l$) are the (analytic) functions obtained by „solving the equations (1) with respect to x^{kl} ($k \leq l$)", and

$$\varphi_0(E) = \mathcal{O}. \tag{7b}$$

Thus the pair $(\mathscr{U}(E), \varphi_0)$ defines a local chart on $\mathfrak{G}(G)$. It can be easily shown that the family of charts $(A \cdot \mathscr{U}(E), \varphi_A)$ for all $A \in \mathfrak{G}(G)$, where $\varphi_A(X) = \varphi_0(A^{-1}X)$, provides $\mathfrak{G}(G)$ with the structure of an analytic submanifold of $GL(n, R)$. Moreover $\mathfrak{G}(G)$ is a topological group with the topology induced by the topology in $GL(n, R)$. Hence it is an N-dimensional Lie subgroup of $GL(n, R)$.

Lemma. *The functions in* (7a) *satisfy the equations*

$$\frac{\partial h^{ab}}{\partial x^{ij}}(\mathscr{C}) = -g_{ib}\tilde{g}_{ja}$$

for $a \leq b$, $i > j$, *with* $\tilde{g}_{ab} = 0$ *for* $a \neq b$, $\tilde{g}_{aa} = \frac{1}{g_{aa}}$ *for* $a = 1, 2, \ldots, n$.

Proof. Differentiation of (1) provides

$$\frac{\partial F^{kl}}{\partial x^{ij}}(E) + \sum_{a \leq b} \frac{\partial F^{kl}}{\partial x^{ab}}(E) \frac{\partial h^{ab}}{\partial x^{ij}}(\mathscr{C}) = 0;\ k \leq l,\ i > j.$$

i. e. using (5)

$$g_{il}\delta_{jk} + g_{ki}\delta_{jl} + \sum_{a \leq b} (g_{al}\delta_{bk} + g_{ka}\delta_{bl}) \frac{\partial h^{ab}}{\partial x^{ij}}(\mathscr{C}) = 0. \tag{8}$$

Note here that $g_{ki}\delta_{jl} = 0$ for all $k \leq l$, $i > j$. Given a fixed pair $(i > j)$, (8) is a system of $\frac{n(n+1)}{2}$ equations possessing a unique solution (cf. the lemma above) Hence it suffices to show

$$\sum_{a \leq b} (g_{al}\delta_{bk}g_{ib}\tilde{g}_{ja} + g_{ka}\delta_{bl}g_{ib}\tilde{g}_{ja}) = g_{il}\delta_{jk}. \tag{9}$$

This, however, is evident: The first summand in the bracket is zero for each $k \leq l$ and $a \leq b$. The second one is non-zero only if $l = i$, $k = j$ with both $a = l$, $b = k$ and its value is g_{ii}. The same, of course, is true about the right hand side. Thus the lemma is proved.

The Lie algebra $gL(n, R)$ of $GL(n, R)$ consists of all the $(n \times n)$-matrices and the product is given by $(A, B) \to AB - BA$ (multiplication of matrices). Each $A \in gL(n, R)$ can be written in the vector form

$$A = \sum_{k,l} a^{kl} \frac{\partial}{\partial x^{kl}}(E).$$

Let $\mathfrak{g}(G)$ be the Lie algebra of $\mathfrak{G}(G)$. It is a subalgebra of $gL(n, R)$ and the homeomorphism φ_0 defines a canonical basis

$$U_{ij} = \sum_{k,l} u_{ij}^{kl} \frac{\partial}{\partial x^{kl}}(E), \quad i > j \tag{10}$$

where U_{ij} are the vectors in $\mathfrak{g}(G)$ associated with the coordinates given by the mapping φ_0, i. e.

$$U_{ij}(f)(E) = \frac{\partial(f \cdot \varphi_0^{-1})}{\partial x^{ij}}(\mathscr{C}), \quad i > j$$

for each function f differentiable in a neighbourhood of E in $GL(n, R)$.

Applying (7a) and the preceding lemma one finds

$$\Gamma_{ij}(f)(E) = \frac{\partial f}{\partial x^{ij}}(E) + \sum_{k \leq l} \frac{\partial f}{\partial x^{kl}}(E) \frac{\partial h^{kl}}{\partial x^{ij}}(\mathcal{O}) =$$

$$= \left(\frac{\partial}{\partial x^{ij}} - \sum_{k \leq l} g_{il}\tilde{g}_{jk} \frac{\partial}{\partial x^{kl}} \right)(f)(E), \quad i > j.$$

Comparison with (10) gives

$$u^{kl}_{ij} = \delta^k_i \delta^l_j \qquad \text{for } k > l,$$

$$u^{kl}_{ij} = -g_{il}\tilde{g}_{jk} \qquad \text{for } k \leq l.$$

The elements $T = [t^{ij}] \in \mathfrak{g}(G)$ can be expressed in the form

(11) $$t^{kl} = \sum_{i>j} \vartheta^{ij} u^{kl}_{ij} = \begin{cases} \vartheta^{kl} & \text{for } k > l \\ -\sum_{i>j} g_{il}\tilde{g}_{jk}\vartheta^{ij} & \text{for } k \leq l. \end{cases}$$

Note that the last expression is zero if $k = l$.

Proposition. *The matrix $T \in gL\,(n, R)$ is an element of the Lie algebra $\mathfrak{g}(G)$ if and only if*

(12) $$T^* + GTG^{-1} = O.$$

Proof. Let $T \in \mathfrak{g}(G)$. Then the (a, b) – element of the matrix on the left hand side of (12) is

(13) $$t^{ba} + g_{ai}t^{ij}\tilde{g}_{jb} = t^{ba} + g_{aa}t^{ab}\tilde{g}_{bb}. \quad (^2)$$

If $b > a$ this is equal to

$$\vartheta^{ba} - g_{aa}g_{bb}\tilde{g}_{aa}\vartheta^{ba}\tilde{g}_{bb} = 0.$$

If $b < a$, (13) gives

$$-g_{aa}\tilde{g}_{bb}\vartheta^{ab} + g_{aa}\vartheta^{ab}\tilde{g}_{bb} = 0.$$

The case $a = b$ is evident.

Conversely let t^{ab} satisfy (13). A similar consideration yields (11) q. e. d.

Each element T of the Lie algebra $gL(n, R)$ generates a one-parameter subgroup $\Gamma_T = \{\Gamma_T(\theta)\}_{\theta \in R}$ of $GL(n, R)$ with $\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\Gamma(\theta)\right]_{\theta=0} = T$, i. e. $\Gamma_T(\theta) = e^{T\theta}$. Particularly $T \in \mathfrak{g}(G)$ induces $e^{T\theta} \in \mathfrak{G}(G)$ for all $\theta \in R$. The basis $U_{ij}(i > j)$ of $\mathfrak{g}(G)$ generates $\frac{n(n-1)}{2}$ one-parameter subgroups $\Gamma_{ij} = \{e^{U_{ij}\theta}\}_{\theta \in R}$.

(2) No summation applied in the rest of the proof.

The exponential mapping exp: $\mathfrak{g}(G) \to \mathfrak{G}(G)$ given by $\exp T = \Gamma_T(1) = e^T$ provides a homeomomorphism of a neighbourhood of the origin in $\mathfrak{g}(G)$ onto a neighbourhood of E in $\mathfrak{G}(G)$ (cf. [1]).

For the sake of simplicity and physical interpretation we shall restrict our following considerations to the case $G = L$. $\mathfrak{G}^+$ will denote the proper Lorentz group, i. e. the component in $\mathfrak{G}(L)$ containing E. It consists of space and time orientation preserving Lorentz transformations. The matrices $\Gamma_{ij}(\theta)$ can be given now an explicit form. We have $\Gamma_{ij}(\theta) = e^{\Gamma_{ij}\theta}$ or, after having solved the corresponding differential equations,

$$\Gamma_{21}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{or}$$

$$\Gamma_{41}(\theta) = \begin{pmatrix} \cosh\theta/c & 0 & 0 & c\sinh\theta/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/c\,.\sinh\theta/c & 0 & 0 & \cosh\theta/c \end{pmatrix}$$

respectively, with similar expresions for $\Gamma_{31}(\theta)$, $\Gamma_{32}(\theta)$, or for $\Gamma_{42}(\theta)$, $\Gamma_{43}(\theta)$ respectively. Hence the one-parameter group Γ_{ij} $(4 > i > j)$ represents all the space rotations in the (i, j)-coordinate plane while Γ_{4j} $(j = 1, 2, 3)$ correspond to parallel frames moving along the j-th axis. (³)

The subgroup of $\mathfrak{G}^+$ consisting of matrices of the type

$$\begin{pmatrix} P_3 & 0 \\ 0 & 1 \end{pmatrix},$$

where P_3 is a (3×3)-orthogonal matrix with $\det P_3 > 0$, is denoted by $\mathfrak{D}$. It is clearly a Lie subgroup of $\mathfrak{G}^+$ its Lie algebra being the vector subspace $\mathfrak{r}$ of $\mathfrak{g}(L)$ generated by the vectors Γ_{21}, Γ_{31}, Γ_{32}. The vectors Γ_{41}, Γ_{42}, Γ_{43} generate a vector subspace $\mathfrak{m} \subset \mathfrak{g}(L)$ so that $\mathfrak{g}(L) = \mathfrak{r} \oplus \mathfrak{m}$. Clearly $\mathfrak{r}$ is a subalgebra of $\mathfrak{g}(L)$ but this is not true about $\mathfrak{m}$. Nevertheless there is a (local) homeomorphism of a neighbourhood of the origin in $\mathfrak{m}$ onto a neighbourhood of the unity class in the space $\mathfrak{G}^+/\mathfrak{D}$ of right cosets $\mathfrak{D}.X$. This homeomorphism is a restriction of the mapping

$$\pi \exp : \mathfrak{m} \to \mathfrak{G}^+/\mathfrak{D}, \tag{14}$$

where π is the projection in $\mathfrak{G}^+/\mathfrak{D}$ and $\mathfrak{G}^+/\mathfrak{D}$ is provided with the induced coset topology (cf. [1] Ch. II. Lemma 4.1).

(³) „Parallel" means here always including orientation.

Our next task is to show that in this special case the mapping (14) is a homeomorphism on the whole of $\mathfrak{m}$ onto $\mathfrak{G}^+/\mathfrak{D}$. We shall first prove that (14) is a one-to-one mapping of $\mathfrak{m}$ onto $\mathfrak{G}^+/\mathfrak{D}$.

One can give a physical interpretation to the space $\mathfrak{G}^+/\mathfrak{D}$. The matrices of $\mathfrak{G}^+$ represent inertial observers of the special relativity theory, one observer being pointed out as corresponding to the unity matrix E. We shall call him the original observer. Each coset of $\mathfrak{G}^+/\mathfrak{D}$ represents a class of observers moving with a common 3-velocity vector but their frames (of orthogonal space coordinates) arbitrarily turned. Thus a coset of $\mathfrak{G}^+/\mathfrak{D}$ can be characterized by inertial observers without frames: we shall identify them with inertial material particles and call them simply particles. A right coset of $\mathfrak{G}^+/\mathfrak{D}$ will be called an IP-coset.

An inertial particle can be equipped with a canonical frame — a frame with its axes parallel to those of the original observer. This canonical frame of the particle defines a Lorentz matrix of special kind. Let us call it an IP-matrix. From the intuitive point of view it is quite natural that the correspondence between IP-cosets and IP-matrices is a one-to-one. Nevertheless we shall give a mathematically strict proof of this statement (cf. the proposition bellow).

It is known that each $X \in \mathfrak{G}^+$ can be written as $X = P \,.\, S$, where $P \in \mathfrak{D}$ and S is an IP-matrix. Moreover each IP-matrix ($\neq E$) has the form (cf [2])

$$\begin{pmatrix} E_3 + \dfrac{q-1}{v^2} W_3 & -q\,\mathbf{v}^* \\ -q/c^2\,\mathbf{v} & q \end{pmatrix}$$

with $v = \sqrt{v_1^2 + v_2^2 + v_3^2} < c$; $\quad q = \left(1 - \dfrac{v^2}{c^2}\right)^{-\frac{1}{2}}$; $\quad W_3 = [v_i v_j]$,

where v_1, v_2, v_3 are the components of the velocity vector $\mathbf{v}$ of the particle with respect to the coordinate system of the original observer.

Proposition. *Each IP-coset contains one and only one IP-matrix.*

Proof. As stated above, each coset of $\mathfrak{G}^+/\mathfrak{D}$ contains an IP-matrix. Suppose a coset contains two IP-matrices, i. e. $S_2 = PS_1$ for some IP-matrices S_1, S_2; $P \in \mathfrak{D}$. Then direct calculation gives

$$(15) \qquad PS_1 = \begin{pmatrix} P_3 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} E_3 + \dfrac{q-1}{v^2\,\mathbf{v}} W_3 & -q\mathbf{v}^* \\ -q/c^2 & q \end{pmatrix} =$$

$$= \begin{pmatrix} P_3 + \dfrac{q-1}{v^2} P_3 W_3 & -qP_3\,\mathbf{v}^* \\ -q/c^2\mathbf{v} & q \end{pmatrix}$$

and comparing the lower rows of PS_1 and of S_2 one gets $S_1 = S_2$.

Hence there is a one-to-one correspondence between the points of the open ball $v_1^2 + v_2^2 + v_3^2 < c^2$ and the IP-cosets. (15) gives an explicit expression of this correspondence: If $X = [x^{ij}] \in \mathfrak{G}^+$, the triple (v_1, v_2, v_3) corresponding to the class of X is given by

$$v_j = -\frac{c^2 x^{4j}}{x^{44}}. \tag{16}$$

Now one can find the explicit form of the mapping (14) simply by computing the elements of the fourth row of the matrix $\exp T$, $T \in \mathfrak{m}$. For this purpose let $T = t_1 C_{41} + t_2 C_{42} + t_3 C_{43}$. The matrix

$$\exp T\theta = \Gamma_T(\theta) \quad (\theta \in R)$$

is the solution of the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \Gamma_T(\theta) = \Gamma_T(\theta) \,.\, T$$

with $\Gamma_T(0) = E$. Denoting $\Gamma_T(\theta) = [\gamma_{ik}(\theta)]$ and $t = \sqrt{t_1^2 + t_2^2 + t_3^2}$ one gets

$$\gamma_{4k}(1) = \frac{t_k}{ct} \sinh c\,t; \quad k \neq 4, \quad t \neq 0$$

$$\gamma_{44}(1) = \cosh c\,t,$$

or, with respect to (16) and $e^O = E$,

$$\begin{aligned} v_k &= -\frac{c\,t_k}{t} \operatorname{tgh} c\,t && \text{for } t \neq 0 \\ v_k &= 0 && \text{for } t = 0. \end{aligned} \tag{17}$$

This formulae can be inverted in a unique way

$$\begin{aligned} t_k &= -\frac{v_k}{cv} \operatorname{arccosh} q && \text{for } v \neq 0 \\ t_k &= 0 && \text{for } v = 0 \end{aligned} \tag{18}$$

Thus the mapping (14) is a one-to-one. It is also a homeomorphism as one can easily see from (17) and (18) realizing that the topology in $\mathfrak{G}/\mathfrak{O}$ is such that π is continuous and open. We may sum this up in the

Theorem. *The mapping π . exp is a homeomorphism of the linear subspace $\mathfrak{m} \subset \mathfrak{g}(L)$ onto the space $\mathfrak{G}^+/\mathfrak{D}$ of IP-cosets. This homeomorphism is given by* (17) *resp.* (18) *and maps IP-cosets corresponding to particles moving along the k-th axis onto vectors in $\mathfrak{m}$ colinear with U_{1k}. Moreover it represents the family of particles moving in a given direction as a subspace of colinear vectors in $\mathfrak{m}$.*

Note that in our considerations the inertial particle is completely characterized by its 3-velocity vector and no attention is payed to its position say in the zero moment of the original observer. So we can always suppose the particle passing the origin of the original observer (and also of the others) at this moment.

Up to this time we have used the one-to-one correspondence between particles and IP-cosets provided all the measurings have been made with respect to the original observer. If $\mathbf{p}$ denotes the particle in view and $h(\mathbf{p}, E)$ the corresponding coset of $\mathfrak{G}^+/\mathfrak{D}$ then $h(\mathbf{p}, E)$ is given by the triple (v_1, v_2, v_3) describing the 3-velocity vector components of the particle from the point of view of the original observer. Calculating the velocity vector with respect to another observer, say given by the matrix $X_0 \in \mathfrak{G}^+$, one obtains in general an other triple (v_1', v_2', v_3') defining an another IP-coset. In order to get explicitly this new triple it suffices to calculate the lower row in the matrix YX_0^{-1}, where Y is an arbitrary matrix of the IP-coset given by the triple (v_1, v_2, v_3). Formally it can be shown that the homeomorphism (14) defines a unique analytic structure on $\mathfrak{G}^+/\mathfrak{D}$ with the property that $\mathfrak{G}^+$ is a Lie transformation group of $\mathfrak{G}^+/\mathfrak{D}$ (cf. [1] Th. 4.2).

We may connect with each particle $\mathbf{p}$ and each observer given by $X_0 \in \mathfrak{G}$ an IP-coset $h(\mathbf{p}, X_0)$ defined by

$$h(\mathbf{p}, X_0) = h(\mathbf{p}, E) \, . \, X_0^{-1}.$$

In accordance with the considerations above the triple (v_1, v_2, v_3) corresponding to the IP-coset $h(\mathbf{p}, X_0)$ is nothing else but the 3-velocity components of the particle with respect to the observer represented by the matrix X_0.

On the other hand the linear subspace $\mathfrak{m} \subset \mathfrak{g}(L)$ may be considered as a linear space of right invariant vector fields on $\mathfrak{G}^+$. Hence there is a canonical one-to-one correspondence $T \to \check{X}_0(T)$ between the vectors of $\mathfrak{m}$ and the vectors of a linear subspace $\mathfrak{m}(X_0)$ of the tangent space to $\mathfrak{G}^+$ at X_0. Let $\log\colon \mathfrak{G}^+/\mathfrak{D} \to \mathfrak{m}$ denote the inverse of the homeomorphism (14). Given a fixed particle $\mathbf{p}$ one can define a continuous vector field on $\mathfrak{G}^+$ by

$$X_0 \to F_{\mathbf{p}}(X_0) = (\check{X}_0 \log)\, h(\mathbf{p}, X_0).$$

It is not difficult to see that this is even an analytic vector field on $\mathfrak{G}^+$. The field $X_0 \to F_{\mathbf{p}}(X_0)$ is uniquely defined by $F_{\mathbf{p}}(E) = \log h(\mathbf{p}, E)$ and for a fixed $X_0 \in \mathfrak{G}^+$ the correspondence $\mathbf{p} \to F_{\mathbf{p}}(X_0)$ is a one-to-one. The physical meaning of this field can be found in the following: Given $F_{\mathbf{p}}(X_0)$ one calculates its components t_k with respect to the basis $\hat{X}_0$ (U_{4k}) $(k = 1, 2, 3)$, uses (17) and gets the components of the 3-velocity of the particle measured by the observer connected with the matrix X_0. In particular $F_{\mathbf{p}}(X_0) = 0$ means that the particle $\mathbf{p}$ is in rest with respect to X_0.

REFERENCES

[1] Helgason S., *Differential geometry and symmetric spaces*, New York 1962.

[2] Фок В. А., *Теория пространства, времени и тяготения*, Москва 1961.

Received January 27, 1965.

ČSAV, *Matematický ústav Slovenskej akadémie vied, Bratislava*