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INTERSECTION GRAPHS OF SEMILATTICES

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This paper is a continuation of [3]. We shall use similar concepts as in [3], but some results for semilattices will be substantially different from the results for lattices.

The intersection graph of an algebra A is by definition the graph whose vertices are proper subalgebras of A and in which two vertices are joined by an edge if and only if the corresponding subalgebras have a non-empty intersection.

The study of these graphs was begun by J. Bosák [2] for semigroups. In [3] we have studied intersection graphs of lattices. Here we shall investigate these graphs for semilattices.

Similarly as a lattice, a semilattice can be defined either algebraically (a commutative semigroup all of whose elements are idempotents), or set-theoretically (a partially ordered set in which to any two elements there exists their supremum). Thus a subsemilattice of a given semilattice can be defined in two ways; we shall speak about algebraic and set-theoretical subsemilattices. The operation in a semilattice will be called multiplication and denoted by $\circ$; its result will be called product.

An algebraic subsemilattice of a semilattice S is by definition a non-empty subset of S which is closed with respect to the multiplication (i.e. with any two elements it contains also their product).

A set-theoretical subsemilattice of S is by definition a non-empty subset of S which is a semilattice with respect to the ordering induced by the ordering of S.

Every algebraic subsemilattice of a semilattice S is simultaneously its set-theoretical subsemilattice, but the converse assertion is not true. This can be shown analogously as in [3] for lattices.

Thus we shall distinguish algebraic intersection graphs of semilattices and set-theoretical ones. We shall introduce as in [3] even the third type of intersection graphs of semilattices, namely the interval intersection graphs. If $a \leqq b$ in a semilattice S, then the interval $\langle a, b\rangle$ is the set of all elements $x \in S$ for which $a \leqq x \leqq b$ holds. The interval $\langle a, b\rangle$ is evidently an algebraic subsemilattice of S; the converse assertion is not true, as shown in [3].

The algebraic intersection graph of the semilattice S will be denoted by $GA(S)$, the set-theoretical intersection graph by $GS(S)$, the interval intersection graph by $GI(S)$.

From the above given definitions it follows that each one-element subset of a semilattice S is its algebraic subsemilattice, set-theoretical subsemilattice and interval.

Now we shall express some theorems whose proofs are the same as the proofs of analogous theorems in [3]. For distinguishing them from the theorems which will be proved here we shall number them by Roman numerals, other theorems will be numbered by Arabic numerals.

Theorem I. *The system of all one-element subsets of a finite semilattice S with more than one element is a maximal internally stable* [1] *set in any of the graphs $GA(S)$, $GS(S)$, $GI(S)$, while any other internally stable set in any of these graphs has a less number of vertices.*

Corollary. *The internal stability numbers of the graphs $GA(S)$, $GS(S)$, $GI(S)$ for a finite semilattice S with more than one element are pairwise equal and are equal to the cardinality of S.*

Theorem II. *Let the set-theoretical intersection graph $GS(S)$ of a finite semilattice S with more than one element be given. Then the set of elements of S and the relation of comparability on it can be reconstructed.*

Theorem III. *Let the algebraic intersection graph $GA(S)$ of a finite semilattice S with more than one element be given. Then the set of elements of S and the relation of comparability on it can be reconstructed.*

Now we shall prove a theorem.

Theorem 1. *Let the algebraic intersection graph $GA(S)$ of a finite semilattice S with more than one element be given. Then the set of elements of S can be reconstructed and for any two incomparable elements a, b of S their product $a \bigcirc b$ can be found.*

Proof. The reconstructibility of the set of elements of S follows from Theorem III. If two elements a, b are incomparable, then there exists an algebraic subsemilattice of S consisting of the elements a, b, $a \bigcirc b$ and no other algebraic subsemilattice with three or less elements containing a and b exists. Thus in the graph $GA(S)$ we find a vertex which is joined with the vertices $\{a\}$ and $\{b\}$ and moreover with exactly one further vertex corresponding to a one-element subsemilattice. This further vertex corresponds to the one-element subsemilattice $\{a \bigcirc b\}$.

Theorem IV. *Let the interval intersection graph $GI(S)$ of a finite semilattice S with more than two elements be given. Then the (undirected) Hasse diagram of S can be reconstructed.*

The proof is again the same as the proof of an analogous theorem in [3]. Here we speak about the Hasse diagram as an undirected graph. If this diagram has to determine uniquely the semilattice S, it must be drawn in a certain position, which cannot be performed with the help of this theorem.

Theorem 2. *Let S be a finite semilattice with at least two elements with the property that each element of S either is minimal, or covers at least two elements. Let the algebraic intersection graph $GA(S)$ of S be given. Then we can reconstruct the semilattice S uniquely.*

Proof. Let x and y be two elements of S such that x covers y. There exists at least one element $z \neq y$ which is covered by x (the element x is not minimal). We can recognize that $x > y$, because there exists a subsemilattice $\{x, y, z\}$ and y and z are incomparable, therefore x must be the supremum of y and z in $\{x, y, z\}$. By this method for each pair of elements such that one covers the other we can recognize which inequality holds for them. (We need not know beforehand which pairs of elements have the property that one covers the other; we may investigate all pairs of comparable elements and for pairs with this property we always succeed.) Then by transitivity we recognize this also for other pairs of comparable elements.

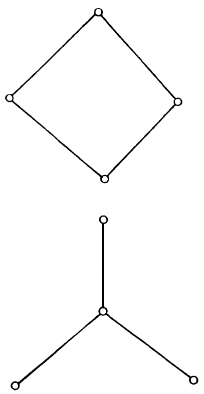

Fig. 1

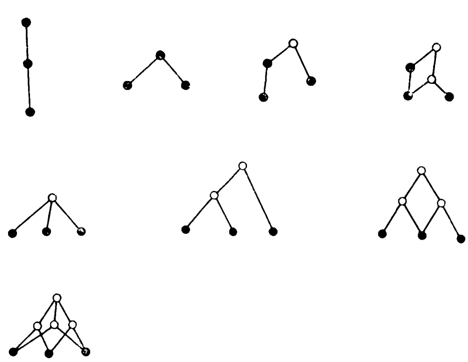

Fig. 2

Fig. 1. shows that in Theorem 2 the assumption that each element of S either is minimal or covers at least two vertices cannot be omitted. In this figure we see the Hasse diagrams of two non-isomorphic finite semilattices which do not satisfy this assumption and whose algebraic intersection graphs are isomorphic.

For set-theoretical intersection graphs the situation is substantially different.

Theorem 3. *Let S be a finite semilattice with at least two elements. Let the set-theoretical intersection graph $GS(S)$ of S be given. Then we can reconstruct the semilattice S uniquely up to isomorphism.*

Proof. According to Theorem II we can reconstruct the set of elements of S and the relation of comparability on it. Now let a, b be two distinct comparable elements of S. If there exists some element c incomparable with b and comparable with a and $\{a, b, c\}$ is a subsemilattice of S, we see that $b < a$ and also $c < a$. If there exists some element d incomparable with a and comparable with b and $\{a, b, d\}$ is a subsemilattice of S, we see that $a < b$. Now if the set of elements of S comparable with a is equal to the set of elements comparable with b, we look for two elements x, y which are mutually incomparable and both comparable with a and b and which have the property that either $\{a, x, y\}$ is a subsemilattice of S and $\{b, x, y\}$ is not, or $\{b, x, y\}$ is a subsemilattice of S and $\{a, x, y\}$ is not. We consider always set-theoretical subsemilattices. In the first case $x < a$, $x > b$, therefore $b < a$. In the second case $a < b$. If there does not exist such a pair x, y, the interval bounded by a, b (i.e. $\langle a, b\rangle$ if $a < b$ or $\langle b, a\rangle$ if $b < a$) is a chain. In this case we say that the pair a, b has the property P. We transfer the pairs with the property P to the end of our procedure; first we determine the inequalities for all pairs without the property P according to the above description. Then the set of elements belonging to the pairs with the property P can be partitioned into classes so that the two elements p, q belong to the same class if and only if $p < r$ is equivalent to $q < r$ for each element $r \in S$ which is comparable with both p, q and such that the pairs p, r and q, r have not the property P (this means that we know whether $p < r$ or $r < p$ and the same for q). Evidently each pair of elements with the property P is a subset of some of these classes and the elements of each class form a chain. Thus in each class we choose a complete ordering arbitrarily and we obtain always isomorphic semilattices.

Now a natural question arises: If some graph is given, how can we recognize, whether it is isomorphic to an intersection graph of a semilattice?

This question can be answered simply. We proceed according to the proof of Theorem 2 or Theorem 3. If we reconstruct some semilattice S, we see that the given graph is isomorphic to an intersection graph of a semilattice. If we must stop the procedure at some step (for example, if to some two vertices of the considered internally stable set of the maximal cardinality no vertex exists with the property that it is joined with both of them and with no other vertex of this set), or if we obtain a partially ordered set which is not a semilattice, we see that this graph is not isomorphic to an intersection graph of a semilattice.

For algebraic intersection graphs we can use also some other procedure for answering this question.

We introduce some notation. Let G be an undirected graph with the vertex set V. Suppose that G contains exactly one internally stable set M of the maximal cardinality and this set M has the property that to any pair of the elements x, y of M exactly one vertex of $V - M$ exists which is joined with both x, y and at most with one vertex of $M - \{x, y\}$; if such a vertex exists, we denote it by $u(x, y)$. Let M_0 be a subset of M of the cardinality 3. Let M_1 be the union of M_0 and the set of all vertices $u(x, y)$ for $x \in M_0$, $y \in M_0$ which exist. Further let M_2 be the union of M_1 and the set of all vertices $u(x, y)$ for $x \in M_1$, $y \in M_1$ which exist. Let $G(M_0)$ be the induced subgraph of G with the vertex set consisting of the vertices of M_2 and all vertices of $V - M$ which are joined with some, but not all, vertices of M_2 and with no vertices of $M - M_2$.

Lemma 1. *If G is an algebraic intersection graph of some semilattice S, then $G(M_0)$ is the algebraic intersection graph of the subsemilattice S_0 of S generated by the elements corresponding to the vertices of M_0.*

Proof. For the sake of simplicity we consider the elements of M directly as elements of S. If $u(x, y)$ exists for some $x \in M_0$, $y \in M_0$, then evidently $u(x, y) = x \bigcirc y$; if it does not exist, then $x \bigcirc y = x$ or $x \bigcirc y = y$. Therefore M_1 consists of M_0 and all products of two elements of M_0 and M_2 consists of M_0 and all products of at most three elements of M_0. As $|M_0| = 3$, the set M_2 is the algebraic subsemilattice S_0 of S generated by M_0. Each vertex of $V - M$ which is joined with elements of M_2, but not with elements of $M - M_2$, corresponds to a subsemilattice of S which contains only elements of M_2, i.e. to a subsemilattice of S_0. If it is joined with all the vertices of M_2, then it corresponds to S_0 itself and cannot belong to the intersection graph of S_0 (it is not a proper subsemilattice of S_0).

Lemma 2. *Let S be a groupoid with at least three elements. Then S is a semilattice, if and only if each subgroupoid of S generated by three elements is a semilattice.*

Proof. If S is a semilattice, then each of its subgroupoids is a semilattice. Suppose that each subgroupoid of S generated by three elements is a semilattice. Let a, b, c be arbitrary three elements of S, let $S(a, b, c)$ be the subgroupoid of S generated by a, b, c. Then for the elements a, b, c the idempotence law, the commutativity law and the associativity law in S (a, b, c) hold and they must hold also in S. As a, b, c were chosen arbitrarily and each of the mentioned laws contains at most three predicate variables, this holds for arbitrary three elements of S and S is a semilattice.

Theorem 4. *Let G be an undirected graph with the vertex set V. The necessary and sufficient conditions for G to be isomorphic to an algebraic intersection graph of a semilattice with at least three elements are the following:*

(i) *G has exactly one internally stable set M of the maximal cardinality and this cardinality is at least three;*

(ii) *to any two vertices x, y of M there exists exactly one vertex of* $V - M$ *which is joined with x and y and at most with one vertex of* $M - \{x, y\}$;

(iii) *if* $M_0 \subseteq M$, $|M_0| = 3$, *then the above described graph* $G(M_0)$ *is isomorphic to the algebraic intersection graph of some semilattice generated by three elements.*

The proof follows from Lemma 1 and Lemma 2.

In Fig. 2 there are Hasse diagrams of all possible semilattices which are generated by three elements. The generators are denoted by black circles, other elements by white circles. The reader can construct the corresponding intersection graphs himself.

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