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CLASSES OF REGULARITY IN SEMIGROUPS

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R. Croisot introduced in [1] the following condition in semigroups: An element a of a semigroup S satisfies the *Condition* (m, n) if there exists an element $x \in S$ such that

$$a = a^m x a^n,$$

where m, n are non-negative integers and a^0 means the void symbol. The set of all elements satisfying the Condition (m, n) is called a *class of regularity* and will be denoted by $\mathscr{R}_S(m, n)$.

First we state some known relations concerning the classes of regularity (see [2]):

(a) $\mathscr{R}_S(0, 0) = S$.

(b) If $m_1 \geqslant m_2$ and $n_1 \geqslant n_2$, then

$$\mathscr{R}_S(m_1, n_1) \subseteq \mathscr{R}_S(m_2, n_2) .$$

(c) If $m_1 \geqslant m_2 \geqslant 2$, then for any n we have:

$$\mathscr{R}_S(m_1, n) = \mathscr{R}_S(m_2, n) .$$

(d) If $n_1 \geqslant n_2 \geqslant 2$, then for any m we have:

$$\mathscr{R}_S(m, n_1) = \mathscr{R}_S(m, n_2) .$$

(e) $\mathscr{R}_S(1, 2) = \mathscr{R}_S(1, 1) \cap \mathscr{R}_S(0, 2)$.

(f) $\mathscr{R}_S(2, 1) = \mathscr{R}_S(1, 1) \cap \mathscr{R}_S(2, 0)$.

These relations imply that there exist at most nine distinct classes of regularity $\mathscr{R}_S(m, n)$, $0 \leqslant m \leqslant 2$, $0 \leqslant n \leqslant 2$, connected by relation (b).

There are at most five distinct classes of regularity in commutative semigroups. In these semigroups classes of regularity for which the sum of numbers m, n is equal, coincide. Moreover, in a commutative semigroup S all non-empty classes of regularity are subsemigroups of S. The situation in non-commutative semigroups is different. In these semigroups non-empty classes of regularity are not necessarily subsemigroups.

The purpose of this paper is to investigate some sufficient conditions for classes of regularity to be subsemigroups of a given semigroup.

A left (right) ideal $L(R)$ of a semigroup S is called *complete* if $SL = L$ $(RS = R)$.

A left ideal L of a semigroup S is called *semiprime* if for every element $a \in S$ and an arbitrary integer n the relation $a^n \in L$ implies $a \in L$.

It may occur in some semigroups that some classes of regularity are empty sets. First we state relatively simple conditions for classes of regularity to be non-empty sets.

Theorem 1. $\mathscr{R}_S(1, 0)$ $(\mathscr{R}_S(0, 1))$ *is non-empty if and only if at least one of the right (left) principal ideals generated by an element of the semigroup S is complete.*

Proof. (a) Let $\mathscr{R}_S(1, 0) \neq \emptyset$. Let $a \in \mathscr{R}_S(1, 0)$. The right principal ideal generated by a we denote by $(a)_R = a \cup aS$. Then we have: $(a \cup aS)\, S = aS \cup aS^2 = aS = a \cup aS$, since $a \in aS$. But it means that $(a)_R$ is a complete ideal.

(b) Let the right principal ideal generated by an element a be complete. Therefore, $(a \cup aS)\, S = aS \cup aS^2 = aS = a \cup aS$. But the last relation implies that $a \in aS$, and it means that $\mathscr{R}_S(1, 0) \neq \emptyset$.

Theorem 2. *If at least one principal right (left) ideal generated by a square of* an element of a semigroup S is semiprime, then $\mathscr{R}_S(2, 0)$ $(\mathscr{R}_S(0, 2))$ *is non-empty.*

Proof. Let a right principal ideal generated by the element a^2 be semiprime. Therefore, $a^2 \in (a^2)_R$ implies that $a \in (a^2)_R$ hence $a \in (a^2 \cup a^2S)$. But the last relation implies that either $a = a^2$, or $a \in a^2S$ and in both cases we obtain that $a \in \mathscr{R}_S(2, 0)$.

Theorem 3. *The class of regularity* $\mathscr{R}_S(1, 1)$ $(\mathscr{R}_S(2, 1)$, $\mathscr{R}_S(1, 2)$, $\mathscr{R}_S(2, 2))$ *is a non-empty set if and only if the semigroup S contains at least one idempotent.*

Proof. (a) If $a = axa$ $(a = a^2xa$, $a = axa^2$, $a = a^2xa^2)$ then $ax(a^2x$, xa^2, $a^2xa)$ is an idempotent of S.

(b) The statement is evident.

Remark 1. It is easy to prove that if S is a semigroup then $\mathscr{R}_S(1, 0)$ is a left ideal of S, or $\mathscr{R}_S(1, 0) = \emptyset$ and $\mathscr{R}_S(0, 1)$ is a right ideal of S or $\mathscr{R}_S(0, 1) = \emptyset$.

It can occur that some of the sets $\mathscr{R}_S(1, 0)$ and $\mathscr{R}_S(0, 1)$ coincides with the semigroup S. We state here one such case.

A semigroup S is called *left (right) simple*, if S contains no left (right) ideal different from S. A semigroup S with zero 0 is called *left (right) 0-simple* if $S^2 \neq 0$ and if 0 is the unique proper left (right) ideal of S.

In [3] it is proved that a semigroup $S(S \neq 0)$ is left simple (left 0-simple) if and only if for every $a \in S(a \neq 0,\ a \in S)$ we have $Sa = S$.

Remark 2. We can prove easily that if a semigroup S is left simple or left 0-simple, then $S = \mathscr{R}_S(0, 1)$.

A simple example can be used in order to show that the preceding condition is only a necessary but not a sufficient one.

Now we show some sufficient conditions in order that other classes of regularity be subsemigroups.

Theorem 4. *Let S be a semigroup, $\mathscr{R}_S(1, 1) \neq \emptyset$ and let one of the following conditions be fulfilled:*

(a) *The product of any two elements of $\mathscr{R}_S(1, 1)$ is an idempotent.*

(b) $\mathscr{R}_S(1, 1) = \mathscr{R}_S(1, 0) \cap \mathscr{R}_S(0, 1)$.

(c) *The set of all idempotents of S is a subsemigroup of S.*

(d) *Any two idempotents of S commute. Then $\mathscr{R}_S(1, 1)$ is a subsemigroup of S and in the case of* (d) *$\mathscr{R}_S(1, 1)$ is an inverse subsemigroup or S.*

Proof. (a) The statement is evident.

(b) The statement follows from Remark 1.

(c) Let $a, b \in \mathscr{R}_S(1, 1)$. Therefore $a = axa$, $b = byb$ for some $x, y \in S$. It is easy to prove that ax, xa, by, yb are idempotents of S. Then: $ab = (axa)(byb) = a(xa)(by)b$. According to the assumption the product of two idempotents is an idempotent too therefore $(xa)(by)$ is an idempotent. Hence we have:

$$ab = a(xa)(by)\,b = a(xaby)\,b = a(xaby)^2\,b =$$

$$= a(xaby)(xaby)\,b = (axa)(by)(xa)(byb) =$$

$$= ab(yx)\,ab = ab \,.\, z \,.\, ab, \text{ where } z = yx \in S.$$

(d) Let e_1, e_2 be idempotents of S such that $e_1 . e_2 = e_2 . e_1$. Then $(e_1 . e_2)(e_1 . e_2) = e_1(e_2 . e_1)\,e_2 = e_1(e_1 . e_2)\,e_2 = (e_1 . e_1)\,(e_2 . e_2) = e_1 . e_2$. It follows that the condition (c) is fulfilled and therefore $\mathscr{R}_S(1, 1)$ is a subsemigroup of S. From [3] it is known that a semigroup S is inverse if all elements of S are regular and if any two idempotents of S commute. But $\mathscr{R}_S(1, 1)$ consists only of regular elements of S, and accordintg to the assumption any two idempotents of S commute, hence (c) implies that $\mathscr{R}_S(1, 1)$ is a subsemigroup of S.

Corollary. *If a semigroup S contains only one idempotent, then $\mathscr{R}_S(1, 1)$ is an inverse subsemigroup of S.*

The following examples of semigroups show that the conditions (b), (d) are not necessary ones

Example 1 Let $S = \{a, b, c, d\}$ be a semigroup with the multiplication table:

	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	b	c
d	a	b	c	d

$\mathscr{R}_S(1, 0) = \mathscr{R}_S(0, 1) = \{a, b, c, d\}$, $\mathscr{R}_S(1, 1) = \{a, b, d\}$, but $\mathscr{R}_S(1, 1)$ is a subsemigroup.

Example 2. Let $S = \{a, b, c, d\}$ be a semigroup with the multiplication table:

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	c	d
d	d	d	d	d

$\mathscr{R}_S(1, 1) = \{a, c, d\}$ is a subsemigroup, $a^2 = a, d^2 = d$, but $ad = a$, $da = d$.

Remark 3. Elements of $\mathscr{R}_S(1, 1)$ have one-sided identities of the form: ax, xa. Elements of $\mathscr{R}_S(2, 0)$ have right identities of the form ax. But we cannot assert that all one-sided identities of elements of $\mathscr{R}_S(1, 1)$, $\mathscr{R}_S(2, 0)$ and $\mathscr{R}_S(0, 2)$ have such a form.

Example 3. Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table:

	a	b	c	d
a	a	a	c	d
b	a	a	c	d
c	c	c	d	a
d	d	d	a	c

$\mathscr{R}_S(1, 1) = \{a, c, d\}$. $c = cxc$ for the unique element $x = d$, $dc = cd = a$. The element dc is a right (and also a left) identity of the element c, but for the element b we have moreover: $cb = c$.

Left (right) identities of elements of $\mathscr{R}_S(1, 1)$ are called left (right) regular identities. But for one-sided identities of elements of $\mathscr{R}_S(2, 0)$ and $\mathscr{R}_S(0, 2)$ no special name is used. Therefore, for our need we introduce:

Definition 1. *Left identities of an element $a \in \mathscr{R}_S(0, 2)$ of the form xa and right identities of an element $a \in \mathscr{R}_S(2, 0)$ of the form ax are called local left identities, and local right identities respectively, or shortly local one-sided identities*

Theorem 5. *Let S be a semigroup, $\mathscr{R}_S(2, 0) \neq \emptyset$ and let any of the following conditions be fulfilled:*

(a) *The product of any two elements of* $\mathscr{R}_S(2, 0)$ *is an idempotent*

(b) *The product of local right identities of the elements* $a, b \in \mathscr{R}_S(2, 0)$ *is a right identity of the element* ab

(c) *Every local right identity of any element of* $\mathscr{R}_S(2, 0)$ *belongs to the centre* Z *of the semigroup* S. *Then* $\mathscr{R}_S(2, 0)$ *is a subsemigroup of* S.

Proof. (a) The statement is evident.

(b) Let $a, b \in \mathscr{R}_S(2, 0)$, therefore $a = a^2x$, $b = b^2y$, and $x, y \in S$. Then $a = a(ax)$, $b = b(by)$. According to the assumption we have $ab = ab(ax)(by)$, $ba = ba(by)(ax)$. Then $ab = (ab)(ax)(by) = a(ba)(xby) = a[ba(by)(ax)](xby) = (ab)(ab)(yax)(xby) = (ab)^2(yax)(xby) = (ab)^2z$, where $z = (yax)(xby) \in S$.

(c) We shall show that (c) implies (b) Let $a, b \in \mathscr{R}_S(2, 0)$. Then $ab = a(ax) b(by) = a(axb)(by) = a(bax)(by) = (ab)(axby)$. Hence the proof follows from (b).

Analogously we can prove

Theorem 5'. *Let* S *be a semigroup,* $\mathscr{R}_S(0, 2) \neq \emptyset$ *and any of the following conditions be fulfilled:*

(a) *The product of any two elements of* $\mathscr{R}_S(0, 2)$ *is an idempotent.*

(b) *The product of local left identities of the elements* $a, b \in \mathscr{R}_S(0, 2)$ *is a left identity of the element* ab.

(c) *Every local right identity of any element of* $\mathscr{R}_S(0, 2)$ *belongs to the centre* Z *of the semigroup* S. *Then* $\mathscr{R}_S(0, 2)$ *is a subsemigroup of* S.

Lemma 1. $\mathscr{R}_S(2, 2) = \mathscr{R}_S(2, 1) \cap \mathscr{R}_S(1, 2)$.

Proof. (a) From p. 299, (b) we have $\mathscr{R}_S(2, 2) \subseteq \mathscr{R}_S(2, 1)$, $\mathscr{R}_S(2, 2) \subseteq \mathscr{R}_S(1, 2)$, therefore $\mathscr{R}_S(2, 2) \subseteq \mathscr{R}_S(2, 1) \cap \mathscr{R}_S(1, 2)$.

(b) Let $a \in \mathscr{R}_S(2, 1) \cap \mathscr{R}_S(1, 2)$, hence $a = a^2xa$, $a = aya^2$. Then $a = a^2xa = a^2xaya^2 = a^2(xay)a^2 = a^2za^2$, where $z = xay \in S$ and it follows that $a \in \mathscr{R}_S(2, 2)$.

Theorem 6. *Let* $E \subseteq Z$, *where* E *is the set of all idempotents and* Z *is the centre of a semigroup* S. *Then each of classes of regularity* $\mathscr{R}_S(1, 1)$, $\mathscr{R}_S(2, 1)$, $\mathscr{R}_S(1, 2)$, *and* $\mathscr{R}_S(2, 2)$ *is a subsemigroup of* S, *or an empty set.*

Proof. The statement that $\mathscr{R}_S(1, 1)$ is a subsemigroup of S under our assumption follows from Theorem 4, (d).

Let now $a, b \in \mathscr{R}_S(2, 1)$, therefore $a = a^2xa$, $b = b^2yb$, for some $x, y \in S$. It is easy to prove that the elements a^2x, b^2y are idempotents of S. Then

$$\begin{aligned} ab &= (a^2xa)\,(b^2yb) = (a^2x)\,a(b^2y)\,b = (a^2x)\,(b^2y)\,(ab) = a(ax)\,b(by)\,(ab) = \\ &= (a^2xa)\,(ax)\,(b^2yb)\,(by)\,(ab) = a(a^2x)\,(axb)\,(b^2y)\,(by)\,(ab) = \\ &= a(a^2x)\,(ax)\,(b^2y)\,(b^2y)\,(ab) = a(b^2y)\,(a^2x)\,(ax)\,(b^2y)\,(ab) = \\ &= (ab)\,(by)\,(a^2x)\,(ax)\,(b^2y)\,(ab) = (ab)\,(a^2x)\,(by)\,(ax)\,(b^2y)\,(ab) = \\ &= (ab)\,(a^2x)\,(b^2y)\,(by)\,(ax)\,(b^2y)\,(ab) = (ab)\,a(ax)\,(b^2y)\,(by)\,(ax)\,(ab) = \end{aligned}$$

$$= (ab)\, a(b^2y)\,(ax)\,(by)\,(ax)\,(ab) = (ab)\,(ab)\,(by)\,(ax)\,(by)\,(ax)\,(ab) =$$
$$= (ab)^2 z(ab), \text{ where } z = (by)\,(ax)\,(by)\,(ax) \in S\,.$$

Analogously we can prove the statement that $\mathscr{R}_S(1, 2)$ is a subsemigroup and the statement, concerning $\mathscr{R}_S(2, 2)$ follows from Lemma 1.

Remark 4. From [2] (pp. 139, 424) it is known that an element $a \in S$ is totally regular if and only if a belongs to some subgroup of the semigroup S. Moreover, S is totally regular if and only if $S = \mathscr{R}_S(2, 2)$. From the above we have:

Corollary. *Let* $\emptyset \neq E \subseteq Z$. Then the union of all subgroups of the semigro of S is a subsemigroup of S.

Remark 5. Other conditions for the classes of regularity $\mathscr{R}_S(2, 1)$, $\mathscr{R}_S(1, 2)$ and $\mathscr{R}_S(2, 2)$ to be subsemigroups of S can be obtained by means of statements (e) (f) quoted in the introduction and Lemma 1, by combining the conditions of Theorem 4 with the conditions of Theorem 5 and Theorem 5′, respectively.

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