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POSITIVITY WITH RESPECT TO THE ROUND CONE

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Dedicated to Professor Štefan SCHWARZ on the occasion of his sixtieth birthday

In this note we shall first find formulae for the minimum of a quadratic non-homogeneous function on a sphere. Then we obtain necessary and sufficient conditions for a quadratic form to be copositive with respect to the round selfdual cone and for a linear operator to be positive with respect to this cone.

We shall choose the coordinate system in an n-dimensional Euclidean space E_n in such a way that the round selfdual cone is given by

$$C_r = \{\mathbf{x} \mid x_1 \geqq (\sum_{k=2}^{n} x_k^2)^{1/2}\}.$$

Vectors will always be real. By the norm $\|\mathbf{z}\|$ of a vector $\mathbf{z}$ we mean the usual Euclidean norm $(\sum z_i^2)^{1/2}$.

We shall also use the Moore-Penrose generalized inverse $\mathbf{A}^+$ of a matrix $\mathbf{A}$ (see e.g. [3]).

Let us prove first a lemma:

Lemma. *Let* $m \geqq 1$, *let* $d_1, d_2, \ldots, d_m, b_1, b_2, \ldots, b_m$ *be real numbers. Then*

$$\min_{\mathbf{x}, \sum_{i=1}^{m} x_i^2 \leqq 1} \left(\sum_{i=1}^{m} d_i x_i^2 + 2 \sum_{i=1}^{m} b_i x_i\right) = f(\lambda_0),$$

where $f(\lambda) = \lambda - \sum_{\substack{i=1 \\ b_i \neq 0}}^{m} \frac{b_i^2}{d_i - \lambda}$ *and* λ_0 *is the minimal real zero of the polynomial*

$$g(\lambda) = \lambda \left(\sum_{\substack{i=1 \\ b_i \neq 0}}^{m} \frac{b_i^2}{(d_i - \lambda)^2} - 1\right) \eta^2(\lambda),$$

$\eta(\lambda)$ being the least common multiple of the polynomials $d_i - \lambda$, $i = 1, \ldots, m$.

Proof. Let us denote $M = \{1, 2, \ldots, m\}$, $M_1 = \{i \in M | b_i \neq 0\}$, $M_2 = M \setminus M_1$,

$$\varphi(\lambda) = \sum_{i \in M_1} \frac{b_i^2}{(d_i - \lambda)^2}.$$

We have

$$g(\lambda) > 0 \quad \text{for} \quad \lambda \to -\infty, \tag{1}$$

and also $g(0) = 0$ so that

$$\lambda_0 \leqq 0. \tag{2}$$

Let us show that

$$\lambda_0 \leqq \min_{i \in M} d_i. \tag{3}$$

Suppose $d_i < \lambda_0$ for some $i \in M$. Then $d_i < 0$ by (2) while

$$g(d_i) = d_i \Big(\sum_{k, d_k = d_i} b_k^2 \Big) (\eta'(d_i))^2 \leqq 0. \tag{4}$$

This is a contradiction to (1) and $g(\lambda_0) = 0$.

By (3), $\varphi(\lambda)$ is continuous in $(-\infty, \lambda_0)$. Since $\varphi(\lambda) < 1$ for $\lambda \to -\infty$, it follows that

$$\varphi(\lambda_0) \leqq 1. \tag{5}$$

Since $\eta'(d_i) \neq 0$, it follows also from (4) that $\lambda_0 = d_i < 0$ only if $b_k = 0$ for all k for which $d_k = d_i$.

Moreover, let us show that $\lambda_0 = d_i = 0$ also implies $b_k = 0$ for all k for which $d_k = d_i$. But this is an easy consequence of (1) and of the fact that then

$$g'(0) = \Big(\sum_{k, d_k = 0} b_k^2 \Big) (\eta'(0))^2.$$

Thus, $\lambda_0 = d_i$ always implies $i \in M_2$.

Define now a vector $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_m)$ by

$$\tilde{x}_i = -b_i/(d_i - \lambda_0) \quad \text{if } i \in M_1,$$

$$\tilde{x}_i = 0 \qquad\qquad \text{if } i \in M_2,$$

with the only exception that if $\lambda_0 = d_s < 0$ for some s, we put $\tilde{x}_s = (1 - \varphi(\lambda_0))^{1/2}$ for exactly one such s.

By (5), $||\tilde{\mathbf{x}}|| \leqq 1$, and $||\tilde{\mathbf{x}}|| < 1$ only if $\lambda_0 = 0$. The equality

$$(d_i - \lambda_0)x_i^2 + 2b_i x_i = (d_i - \lambda_0)(x_i - \tilde{x}_i)^2 - b_i^2/(d_i - \lambda_0),$$

which holds for all x_i's whenever $i \in M_1$ (and thus $d_i - \lambda_0 \neq 0$), yields:

If $\mathbf{x} = (x_i)$ is any vector such that $||\mathbf{x}|| \leqq 1$, then

$$\sum_{i \in M} d_i x_i^2 + 2 \sum_{i \in M} b_i x_i = \lambda_0 \sum_{i \in M} x_i^2 + \sum_{i \in M} (d_i - \lambda_0) x_i^2 + 2 \sum_{i \in M_1} b_i x_i =$$

$$= \lambda_0 \sum_{i \in M} x_i^2 + \sum_{i \in M} (d_i - \lambda_0)(x_i - \tilde{x}_i)^2 - \sum_{i \in M_1} b_i^2/(d_i - \lambda_0) \geqq$$

$$\geqq \lambda_0 - \sum_{i \in M_1} b_i^2/(d_i - \lambda_0) = f(\lambda_0).$$

Moreover, equality is attained for the vector $\tilde{\mathbf{x}}$. The proof is complete.

Theorem 1. *Let* $\mathbf{A}$ *be a symmetric* $m \times m$ *matrix,* $\mathbf{c}$ *an m-dimensional column vector. Then,*

$$\min_{\mathbf{x}, ||\mathbf{x}|| \leqq 1} (\mathbf{x}^T \mathbf{A} \mathbf{x} + 2\mathbf{c}^T \mathbf{x}) = \lambda_0 - \mathbf{c}^T (\mathbf{A} - \lambda_0 \mathbf{I})^+ \mathbf{c},$$

where λ_0 *is the minimal real zero of the polynomial*

$$g(\lambda) = \lambda(\mathbf{c}^T(\mathbf{A} - \lambda \mathbf{I})^{-2} \mathbf{c} - 1)\mu^2(\mathbf{A}; \lambda),$$

where $\mu(\mathbf{A}; \lambda)$ *is the minimal polynomial of* $\mathbf{A}$.

Remark. Instead of $\mathbf{c}^T(\mathbf{A} - \lambda_0 \mathbf{I})^+ \mathbf{c}$ one can take the number $\mathbf{c}^T \mathbf{u}$, where $\mathbf{u}$ is any solution of the system $(\mathbf{A} - \lambda_0 \mathbf{c})\mathbf{u} = \mathbf{c}$.

Proof. If $\mathbf{A}$ is diagonal, $\mathbf{A} = \operatorname{diag}\{d_1, \ldots, d_m\}$, the theorem follows from the Lemma immediately. The remark is also true, since $d_i - \lambda_0 = 0$ implies $c_i - 0$. To prove the general case we use the well known fact that there exists an orthogonal matrix $\mathbf{U}$ and a diagonal matrix $\mathbf{D} = \operatorname{diag}\{d_1, \ldots, d_m\}$ such that

$$\mathbf{A} = \mathbf{U} \mathbf{D} \mathbf{U}^T .$$

Define the vector $\mathbf{b} = \mathbf{U}^T \mathbf{c}$. Then, if we put $\mathbf{U}^T \mathbf{x} = \mathbf{y}$, we have $\min_{\mathbf{x}, ||\mathbf{x}|| \leqq 1} (\mathbf{x}^T \mathbf{A} \mathbf{x} + 2\mathbf{c}^T \mathbf{x}) = \min_{\mathbf{x}, ||\mathbf{x}|| \leqq 1} (\mathbf{x}^T \mathbf{U} \mathbf{D} \mathbf{U}^T \mathbf{x} + 2\mathbf{b}^T \mathbf{U}^T \mathbf{x}) = \min_{\mathbf{y}, ||\mathbf{y}|| \leqq 1} (\mathbf{y}^T \mathbf{D} \mathbf{y} + 2\mathbf{b}^T \mathbf{y})$. On the other hand,

$$g(\lambda) = \lambda(\mathbf{b}^T \mathbf{U}^T \mathbf{U} (\mathbf{D} - \lambda \mathbf{I})^{-2} \mathbf{U}^T \mathbf{U} \mathbf{b} - 1)\mu^2(\mathbf{A}; \lambda) =$$

$$= \lambda(\mathbf{b}^T (\mathbf{D} - \lambda \mathbf{I})^{-2} \mathbf{b} - 1)\mu^2(\mathbf{D}; \lambda),$$

$$\mathbf{c}^T(\mathbf{A} - \lambda_0 \mathbf{I})^+ \mathbf{c} = \mathbf{c}^T \mathbf{U} (\mathbf{D} - \lambda_0 \mathbf{J})^+ \mathbf{U}^T \mathbf{c} = \mathbf{b}^T (\mathbf{D} - \lambda_0 \mathbf{I})^+ \mathbf{b} = \sum_{i \in M_1} \frac{b_i^2}{d_i - \lambda_0}$$

and the general case follows from the Lemma as well.

It is well known (cf. [1], [2]) that a quadratic form $Q(x)$ is called copositive on a selfdual cone $C = C^*$ iff $Q(\mathbf{x}) \geqq 0$ whenever $\mathbf{x} \in C$. Let now $C_r = \{\mathbf{x} \mid x_1 \geqq \geqq (\sum_2^n x_i^2)^{1/2}\}$.

Theorem 2. *Let* $\mathbf{B} = \begin{pmatrix} b_{11}, & \mathbf{b}_1^T \\ \mathbf{b}_1 \ , & \mathbf{B}_{22} \end{pmatrix}$ *be a symmetric matrix. The quadratic form* $(\mathbf{B}\mathbf{x}, \mathbf{x})$ *is copositive on the cone* C_r *iff*

$$b_{11} + \lambda_0 - \mathbf{b}_1^T (\mathbf{B}_{22} - \lambda_0 \mathbf{I}_2)^+ \mathbf{b}_1 \geqq 0$$

where $\mathbf{I}_2$ *is the identity* $n - 1$ *by* $n - 1$ *matrix and* λ_0 *is the minimal real zero of the polynomial*

$$g(\lambda) = \lambda(\mathbf{b}_1^T(\mathbf{B} - \lambda\mathbf{I})^{-2}\mathbf{b}_1 - 1)\mu^2(\mathbf{B}; \lambda),$$

$\mu(\mathbf{B}; \lambda)$ *being the minimal polynomial of* $\mathbf{B}$.

Proof. Clearly $(\mathbf{B}\mathbf{x}, \mathbf{x})$ is copositive on C_r iff it attains nonnegative values for all vectors $\hat{\mathbf{x}} = \begin{pmatrix} 1 \\ \tilde{\mathbf{x}} \end{pmatrix}$, where $\|\tilde{\mathbf{x}}\| \leqq 1$. But $(\mathbf{B}\hat{\mathbf{x}}, \hat{\mathbf{x}}) = b_{11} + 2\mathbf{b}_1^T\tilde{\mathbf{x}} + (\mathbf{B}\tilde{\mathbf{x}}, \tilde{\mathbf{x}})$ and this condition is equivalent to

$$\min_{\tilde{\mathbf{x}}, \|\tilde{\mathbf{x}}\| \leqq 1} ((\mathbf{B}\tilde{\mathbf{x}}, \tilde{\mathbf{x}}) + 2\mathbf{b}_1^T\tilde{\mathbf{x}}) \geqq -b_{11}.$$

From the preceding theorem the assertion follows then immediately.

We can now find a necessary and sufficient condition for a matrix $\mathbf{A}$ to be a positive operator with respect to C_r, i. e. to have the property that $\mathbf{A}\mathbf{x} \in C_r$ for every $\mathbf{x} \in C_r$.

Theorem 3. *A necessary and sufficient condition for a matrix*

$$\mathbf{A} = \begin{pmatrix} a_{11} & \mathbf{a}_1^T \\ \mathbf{a}_2 & \mathbf{A}_n \end{pmatrix}$$

to be a positive operator with respect to the round cone C_r *is that*

$$a_{11} \geqq \|\mathbf{a}_2\|$$

and

$$a_{11}^2 - \|\mathbf{a}_2\|^2 + \lambda_0 - (a_{11}\mathbf{a}_1^T - \mathbf{a}_2^T\mathbf{A}_n)(\mathbf{a}_1\mathbf{a}_1^T - \mathbf{A}_n^T\mathbf{A}_n - \lambda_0\mathbf{I}_2)^+ \, . \, (a_{11}\mathbf{a}_1 - \mathbf{A}_n^T\mathbf{a}_2) \geqq 0,$$

where $\mathbf{I}_2$ *is the* $(n - 1)$*-rowed identity matrix and* λ_0 *is the minimal real zero of the polynomial*

$$\lambda((a_{11}\mathbf{a}_1^T - \mathbf{a}_2^T\mathbf{A}_n)(\mathbf{a}_1\mathbf{a}_1^T - \mathbf{A}_n^T\mathbf{A}_n - \lambda\mathbf{I})^{-2}(a_{11}\mathbf{a}_1 - \mathbf{A}_n^T\mathbf{a}_2) - 1)\mu^2(\lambda),$$

where $\mu(\lambda)$ *is the minimal polynomial of* $\mathbf{a}_1\mathbf{a}_1^T - \mathbf{A}_n^T\mathbf{A}_n$.

Proof. Clearly $\mathbf{A}$ is a positive operator on C_r iff for any $(n-1)$-dimensional vector $\mathbf{x}_2$ and any number x_1 satisfying $x_1 \geqq \|\mathbf{x}_2\|$ we have

$$a_{11}x_1 + \mathbf{a}_1^T\mathbf{x}_2 \geqq \|\mathbf{A}_n\mathbf{x}_2 + \mathbf{a}_2x_1\|.$$

This is equivalent to

$$a_{11}x_1 + \mathbf{a}_1^T\mathbf{x}_2 \geqq 0$$

and

$$(a_{11}x_1 + \mathbf{a}_1^T\mathbf{x}_2)^2 - (\mathbf{x}_2^T\mathbf{A}_n^T + x_1\mathbf{a}_2^T)(\mathbf{A}_n\mathbf{x}_2 + \mathbf{a}_2x_1) \geqq 0.$$

Hence $\mathbf{A}$ is a positive operator iff both conditions

(i) $$a_{11} \geqq \|\mathbf{a}_1\|$$

and

(ii) the quadratic form $(\mathbf{Bx}, \mathbf{x})$ is copositive on C_r,

are fulfilled, where

$$\mathbf{B} = \begin{pmatrix} a_{11}^2 - \|\mathbf{a}_2\|^2, & a_{11}\mathbf{a}_1^T - \mathbf{a}_2^T\mathbf{A}_n \\ a_{11}\mathbf{a}_1 - \mathbf{A}_n^T\mathbf{a}_2, & \mathbf{a}_1\mathbf{a}_1^T - \mathbf{A}_n^T\mathbf{A}_n \end{pmatrix}.$$

From Theorem 2 it follows immediately that this is equivalent to the assertion of the theorem.

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