Matematický časopis

Ivan Kolář

On the Prolongations of Differentiable Distributions

Matematický časopis, Vol. 23 (1973), No. 4, 317--325

Persistent URL: http://dml.cz/dmlcz/126578

Matematický časopis 23 (1973), No. 4

ON THE PROLONGATIONS OF DIFFERENTIABLE DISTRIBUTIONS

IVAN KOLÁŘ, Brno

Using the theory of jets, we clarify some problems in the foundations of the higher order geometry of differentiable distributions. We first show that one can naturally introduce two kinds of prolongations of such a distribution. Then we establish an invariant algorithm for each kind of these prolongations. Meeting the semi-holonomic contact elements in the course of the prolongations of the second kind, we explain from the conceptual point of view the well-known analogy between some geometric properties of manifolds with connection and differentiable distributions on homogeneous spaces. In conclusion, we outline how to treat the distributions on spaces with Cartan connection. Our considerations are in the category C^∞. The standard terminology and notations of the theory of jets are used throughout the paper.

An m-dimensional distribution Δ on a differentiable manifold M is usually introduced as an assignment to each point $x \in M$ an m-dimensional subspace of $T_x(M)$, $m < n = \dim M$, [2]. Since we aim at higher orders, we shall equivalently define Δ as a cross-section of the fibered manifold $K^1_m(M)$ of all regular contact m^1-elements on M. As $K^1_m(M)$ is an associated fibre bundle of the symbol $(M, K^1_{n,m}, L^1_n, H^1(M))$, where $K^1_{m,n} = K^1_{m,0}(\mathbf{R}^n)$, the indirect form of Δ, [6], is a mapping $H^1(M) \to K^1_{n,m}$. Let $\hat{K}^1_{n,m} \subset K^1_{n,m}$ be the subspace of all elements transversal with respect to the canonical projection $\mathbf{R}^n \to \mathbf{R}^m$ and let

$$\hat{H}^1(M) = \{u \in H^1(M);\ u^{-1}(\Delta(x)) \in \hat{K}^1_{n,m},\ x = \beta u\}\ .$$

On $\hat{K}^1_{n,m}$, there are natural coordinates y^J_p, [7]. Hence we have the coordinate functions a^J_p: $\hat{H}^1(M) \to \mathbf{R}$ of Δ; in other words, $\mathrm{d}x^J = a^J_p(u)\,\mathrm{d}x^p$ are the equations of the subspace $u^{-1}(\Delta(x)) \subset T_0(\mathbf{R}^n)$. Let φ be the canonical form of $H^1(M)$, [2], [4]. By a certain analogy to [7], Proposition 1, we shall say that

$$(1)\qquad \varphi^J = a^J_p\varphi^p, \qquad \begin{aligned} &p, q, \ldots = 1, \ldots, m,\\ &J, K, \ldots = m+1, \ldots, n, \end{aligned}$$

are the formal equations of Δ.

In accordance with the general theory of prolongations of geometric object fields, the cross-section $\Delta: M \to K^1_m(M)$ is prolonged to a cross-section $j^1\Delta$: $M \to J^1K^1_m(M)$, where $J^1K^1_m(M)$ means the first prolongation of fibered manifold $K^1_m(M) \to M$. This cross section will be called the (first) prolongation of the first kind of Δ. By [6], $J^1K^1_m(M)$ is a fibre bundle associated with $W^1(H^1(M)) = \tilde{H}^2(M)$. But there is a natural reduction $H^2(M)$ of $\tilde{H}^2(M)$ and it is more appropriate to consider $J^1K^1_m(M)$ as a fibre bundle associated with $H^2(M)$. Analogously, the cross-section $j^r\Delta: M \to J^rK^1_m(M)$ will be said to be the r-th prolongation of the first kind of Δ, the space $J^rK^1_m(M)$ being considered as a fibre bundle associated with $H^{r+1}(M)$. On the other hand, using some specific properties of distributions, one can introduce another kind of prolongations of Δ. We first recall some notions.

By a regular holonomic or semi-holonomic or non-holonomic contact m^r-element on M is meant a set XL^r_m or $Y\bar{L}^r_m$ or $Z\tilde{L}^r_m$, where X or Y or Z is a regular holonomic or semi-holonomic or non-holonomic m^r-velocity on M; we shall denote the fibered manifold of all such elements by $K^r_m(M)$ or $\bar{K}^r_m(M)$ or $\tilde{K}^r_m(M)$, respectively. A semi-holonomic contact m^r-element $Y\bar{L}^r_m$ is called holonomic if it contains a holonomic m^r-velocity, i. e. if there holds $Y\bar{L}^r_m = Y'\bar{L}^r_m$, $Y' \in T^r_m(M)$. Similarly, a non-holonomic contact m^r-element is said to be semi-holonomic, if it contains a semi-holonomic m^r-velocity. Let ϱ^s_r: $\tilde{K}^r_m(M) \to \tilde{K}^s_m(M)$, $s \leqslant r$, be the jet projection. In general, a cross section Δ_1: $M \to \tilde{K}^r_m(M)$ will be called a non-holonomic m^r-distribution on M. Naturally, the cross-section $j^s\Delta_1: M \to J^s\tilde{K}^r_m(M)$ will be said to be the s-th prolongation of the first kind of Δ_1. Moreover, let $\Delta: M \to K^1_m(M)$ be a distribution of the first order on M. Let $\Delta(x) = XL^1_m$, $X = j^1_0\psi$, where ψ is a mapping of $\mathbf{R}^n$ into M. Then $\Delta_1\psi$ is a mapping of $\mathbf{R}^m$ into $\tilde{K}^r_m(M)$ and, by [9], $j^1_0(\Delta_1\psi)$ is identified with an element of $\tilde{K}^{r+1}_m(M)$, which will be denoted by $j^1\Delta_1(\Delta)(x)$. One sees easily that this definition is correct. The cross section $j^1\Delta_1(\Delta): M \to \tilde{K}^{r+1}_m(M)$ will be called the prolongation of Δ_1 with respect to Δ. In particular, if $\Delta = \varrho^1_r(\Delta_1)$, then $\Delta'_1 = j^1\Delta_1(\varrho^1_r(\Delta_1))$ will be said to be the (first) prolongation of the second kind of Δ_1 or the weak prolongation of Δ_1. By the iteration $\Delta^{(s)}_1 = (\Delta^{(s-1)}_1)'$ we introduce the s-th prolongation of the second kind or the s-th weak prolongation of Δ_1.

Lemma 1. *Let Δ be an m-dimensional distribution of the first order on M. Then all weak prolongations of Δ are semi-holonomic, i.e. $\Delta^{(s)}: M \to \bar{K}^{s+1}_m(M)$ for every s.*

Proof. According to [9], choose an auxiliary fibering $\pi: U \to U_1$, $U_1 \subset \mathbf{R}^m$, on a neighbourhood U of a point $x \in M$ in such a way that $\Delta(y)$ is transversal with respect to π for every $y \in U$. Then the restriction of Δ to U can be identified with a distribution $\delta: U \to J^1(U, \pi, U_1)$ on fibered manifold (U, π, U_1)

in the sense of [3]. By definition, the s-th weak prolongation $\Delta^{(s)}$ of Δ corresponds to the s-th prolongation $\delta^{(s)}$ of δ, [3]. By Proposition 3 of [3], $\delta^{(s)}$ is semi-holonomic. Hence $\Delta^{(s)}$ is also semi-holonomic, QED.

Let (E, p, B) be a fibered manifold, $\dim B = m$. A distribution $\delta : E \to J^1E$ will be said to be involutive, if it is involutive as an m-dimensional distribution on the differentiable manifold E in the classical sense, [2].

Lemma 2. *A distribution $\delta : E \to J^1E$ is involutive if and only if its first prolongation δ' is holonomic, i.e. $\delta' : E \to J^2E$.*

The proof consists in a simple evaluation in some local coordinates.

Taking into account the above identification, we obtain

Proposition 1. *A distribution $\Delta : M \to K^1_m(M)$ is involutive if and only if its first weak prolongation is holonomic, i.e. $\Delta' : M \to K^2_m(M)$.*

By definition, we now deduce immediately

Proposition 2. *If Δ is involutive, then $\Delta^{(s)}$ is holonomic for every s. Moreover, if ζ is the germ of the integral manifold of Δ at $x \in M$, then $\Delta^{(s)}(x) = k^{s+1}_x\zeta$ (= the contact element of order $s + 1$ determined by ζ).*

Remark 1. Starting from involutive distributions, we can give an instructive description of the difference between the prolongations of the first and of the second kinds of a distribution $\Delta : M \to K^1_m(M)$. If Δ is involutive, it is locally represented by an $(n - m)$-parameter family of m-dimensional submanifolds of M. Investigating the prolongations of the second kind, we consider each of these submanifolds separately and we construct its higher order contact elements. Constructing the prolongations of the first kind, we use essentially even the integral submanifolds in a neighbourhood of the corresponding point. Further, if Δ is not involutive, the difference between both kinds of prolongations can be shortly expressed by an analogy to the previous situation. Investigating the prolongations of the second kind, we consider Δ as a "non-holonomic" generalization of an m-dimensional submanifold of M, while in the case of the prolongations of the first kind Δ is considered as a „non-holonomic" generalization of an $(n - m)$-parameter family of m-dimensional submanifolds of M.

Since the prolongations of the first kind of Δ are, in fact, the prolongations of a geometric object field in pure form, an algorithm for such prolongations is described in [6]. The only specific feature is that we apply the reduction of $W^1(H^r(M))$ to $H^{r+1}(M)$ at every step. Consequently, we have to restrict the canonical form of $W^1(H^r(M))$ to $H^{r+1}(M)$, but this restriction is just the canonical form of $H^{r+1}(M)$. Thus, we have to find an invariant algorithm for the weak prolongations only. To get a "lemma", we shall first study the prolongations of arbitrary geometric object fields with respect to a distribution.

Consider a principal fibre bundle $P(M, G)$ and an associated fibre bundle $E = E(M, F, G, P)$. Denote by $\hat{K}^1_m(E)$ the space of all regular contact m^1-elements on E transversal with respect to the bundle projection $p: E \to M$. The space $\hat{K}^1_m(E)$ has a natural structure of an associated fibre bundle of the symbol $(M, A, G^1_n, W^1(P))$, where we set $A = \hat{K}^1_{m,0}(F \times \mathbf{R}^n)$, cf. [6]. Introduce $q: F \times \mathbf{R}^n \to \mathbf{R}^m$, $(z, (x^1, \ldots, x^n)) \mapsto (x^1, \ldots, x^m)$ and denote by $\hat{A} \subset A$ the subspace of all elements transversal with respect to q. Then every element of $\hat{A}$ can be identified with a 1-jet of $\mathbf{R}^m$ into $F \times \mathbf{R}^{n-m}$ with source $0 \in \mathbf{R}^m$ and with target in $F \times \{0\}$, $0 \in \mathbf{R}^{n-m}$. In this sense we shall write $\hat{A} = T^1_m(F) \times L^1_{n\ m,m}$. If z^A are some local coordinates on F, then the corresponding local coordinates on $\hat{A}$ are

$$z^A,\ z^A_p,\ y^J_p\,. \tag{2}$$

Let $\Delta: M \to K^1_m(M)$ be a distribution and let $\sigma: M \to E$ be a geometric object field. Then we define the prolongation $j^1\sigma(\Delta)$ of σ with respect to Δ as follows. The jet $j^1_x\sigma$ being identified with an n-dimensional subspace V of $T_{\sigma(x)}(E)$, we denote by $j^1\sigma(\Delta)(x)$ the m-dimensional subspace of V over $\Delta(x)$. By the natural identification, $j^1\sigma(\Delta)$ is a cross section of $\hat{K}^1_m(E)$. By [6], if $\bar{b}^A$ are the coordinate functions of σ, then the coordinate functions $b^A = \beta^*\bar{b}^A$ and b^A_i of $j^1\sigma$ satisfy

$$\mathrm{d}b^A + \xi^A_\varkappa(b^B)\Theta^\varkappa = b^A_i\Theta^i, \qquad i, j, \ldots = 1, \ldots, n, \quad \varkappa = n+1, \ldots, n + \dim G, \tag{3}$$

where $\mathrm{d}z^A + \xi^A_\varkappa(z^B)\omega^\varkappa = 0$ are the equations of the fundamental distribution on $G \times F$ and $(\Theta^i, \Theta^\varkappa)$ is the canonical form of $W^1(P)$. Let $\chi: W^1(P) \to H^1(M)$ be the canonical projection, [5]. Considering the formal equations (1) of Δ, introduce $\tilde{a}^J_p = \chi^* a^J_p$. (Since $\Theta^i = \chi^*\varphi^i$ holds, we may say that $\Theta^J = \tilde{a}^J_p\Theta^p$ are the formal equations of Δ on $W^1(P)$.) Then the coordinate functions of $j^1\sigma(\Delta)$ corresponding to z^A, y^J_p of (2) are b^A, $\tilde{a}^J_p$. Let us denote by c^A_p the coordinate functions of $j^1\sigma(\Delta)$ corresponding to z^A_p.

Proposition 3. *There holds*

$$c^A_p = b^A_J\tilde{a}^J_p + b^A_p. \tag{4}$$

Proof. This follows directly from the above identification of $\hat{A}$ and $T^1_m(F) \times L^1_{n\ m,m}$.

Remark 2. From the formal point of view, (4) looks like if one substitutes the formal equations of Δ into the right-hand side of (3).

In particular, if we take a non-holonomic m^r-distribution $\Delta_1: M \to \tilde{K}^r_m(M)$ as the above cross section σ, then $j^1\Delta_1(\Delta)$ seems to have two meanings. On the one hand, by our first definition, it is a cross section of $\tilde{K}^{r+1}_m(M)$, while, on the other hand, it is a cross section of $\hat{K}^1_m(\tilde{K}^r_m(M))$ by the second definition. But both these cross sections are mutually identified in the sense of [9]. so that our notation is correct. Thus, we may apply Proposition 3 for investigating the weak prolongations of a distribution $\Delta: M \to K^1_m(M)$. The coordinate functions of the successive weak prolongations of Δ can be treated by a recurrent algorithm starting from the formal equations (1) of Δ. Let

$$\mathrm{d}y^J_p + \Psi^J_p(y^K_q, \omega^i_j) = 0$$

$$\vdots$$

(5) $$\mathrm{d}y^J_{p_1\dots p_r} + \Psi^J_{p_1\dots p_r}(y^K_q, \dots, y^K_{q_1\dots q_r}, \omega^i_j, \dots, \omega^i_{j_1\dots j_r}) = 0$$

be the equations of the fundamental distribution on $L^r_n \times \bar{K}^r_{n,m}$ deduced by the recurrent algorithm established in [8]. Set

$$\hat{H}^r(M) = \{u \in H^r(M);\ j^1_r(u) \in \hat{H}^1(M)\}\,.$$

(In general, j^s_r means the canonical projection of r-jets into s-jets, $s \leqslant r$.)

Proposition 4. *Assume by induction that we have deduced the coordinate functions* $a^J_p, \dots, a^J_{p_1\dots p_r}: \hat{H}^r(M) \to \mathbf{R}$ *of the* $(r-1)$*-st weak prolongation* $\Delta^{(r-1)}$ *of* Δ. *Let* $a^J_p = j^{r*}_{r+1}\bar{a}^J_p, \dots, a^J_{p_1\dots p_r} = j^{r*}_{r+1}\bar{a}^J_{p_1\dots p_r}, a^J_{p_1\dots p_r p_{r+1}}: \hat{H}^{r+1}(M) \to \mathbf{R}$ *be the coordinate functions of the r-th weak prolongation* $\Delta^{(r)}$ *of* Δ, *let* $\varphi = (\varphi^i, \varphi^i_j, \dots, \varphi^i_{j_1\dots j_r})$ *be the canonical form of* $H^{r+1}(M)$ *and let* $b^J_{p_1\dots p_r i}$ *be the functions determined by*

(6) $$\mathrm{d}a^J_{p_1\dots p_r} + \Psi^J_{p_1\dots p_r}(a^K_q, \dots, a^K_{q_1\dots q_r}, \varphi^i_j, \dots, \varphi^i_{j_1\dots j_r}) = b^J_{p_1\dots p_r i}\varphi^i\,.$$

Then it holds

(7) $$a^J_{p_1\dots p_r p_{r+1}} = b^J_{p_1\dots p_r K}a^K_{p_{r+1}} + b^J_{p_1\dots p_r p_{r+1}}\,.$$

Proof. This follows easily from Proposition 1 of [6], Proposition 3 and the relation $\Delta^{(r)} = j^1\Delta^{(r-1)}(\Delta)$.

For the next step of the algorithm, we extend (5) to the equations of the fundamental distribution on $L^{r+1}_n \times \bar{K}^{r+1}_{n,m}$ in the way described in [8], Proposition 3.

Assume now that M is a homogeneous space with a fundamental group G. Fix a point $c \in M$ and denote by H its stability group. Then G has a natural structure of a principal fibre bundle over M with structure group H and $\bar{K}^r_m(M)$ can be considered as an associated fibre bundle of the symbol $(M, \bar{K}^r_{m,c}(M), H, G)$. For the sake of simplicity, we shall restrict ourselves to those homogeneous spaces for which one can find such a basis ω^α of $\mathfrak{g}^*$ that there holds

(8) $$\mathrm{d}\omega^i = c^i_{j\lambda}\omega^j \wedge \omega^\lambda, \qquad \lambda, \mu, \dots = n+1, \dots, \dim G,$$

$$\mathrm{d}\omega^\lambda = c^\lambda_{j\mu}\omega^j \wedge \omega^\mu + \frac{1}{2}\, c^\lambda_{\mu\nu}\omega^\mu \wedge \omega^\nu\,,$$

provided $\omega^i = 0$ are the differential equations of H. (As remarked in [7], a great number of homogeneous spaces investigated in the classical differential geometry are of type (8).) We denote by K the subgroup $\omega^\lambda = 0$ of G and we shall consider a local coordinate system $\varkappa$ on F corresponding to the canonical coordinates on K determined by e_i, where e_α means the basis of $\mathfrak{g}$ dual to ω^α. By means of $\varkappa$, $\bar{K}^r_{m,c}(M)$ is locally identified with $\bar{K}^r_{n,m}$. Set

$$\hat{G} = \{g \in G, g^{-1}(\Delta(\pi(g))) \in \hat{K}^1_{n,m}\},$$

where π is the bundle projection of $G(M, H)$. Thus, we have the coordinate functions $a^J_p: \hat{G} \to \mathbf{R}$ of Δ. For analogous reasons to (1), we shall say that

$$\omega^J = a^J_p \omega^p \tag{9}$$

are the formal equations of Δ. In this situation, the coordinate functions of the successive weak prolongations of Δ can be treated by the following recurrent algorithm. Let π^λ be the restriction of ω^λ to H and let

$$\text{(10)}\quad \mathrm{d}y^J_p + \Psi^J_{p\lambda}(y^K_q)\pi^\lambda = 0,$$

$$\vdots$$

$$\mathrm{d}y^J_{p_1\ldots p_r} + \Psi^J_{p_1\ldots p_r\lambda}(y^K_q, \ldots, y^K_{q_1\ldots q_r})\pi^\lambda = 0$$

be the equations of the fundamental distribution on $H \times \bar{K}^r_{n,m}$ deduced according to [8].

Proposition 5. *Assume by induction that we have found the coordinate functions $a^J_p, \ldots, a^J_{p_1\ldots p_r}: \hat{G} \to \mathbf{R}$ of the $(r-1)$-st weak prolongation $\Delta^{(r-1)}$ of Δ. Let $b^J_{p_1\ldots p_r i}$ be the functions determined by*

$$\text{(11)}\quad \mathrm{d}a^J_{p_1\ldots p_r} + \Psi^J_{p_1\ldots p_r\lambda}(a^K_q, \ldots, a^K_{q_1\ldots q_r})\omega^\lambda = b^J_{p_1\ldots p_r i}\omega^i.$$

Then the remaining coordinate functions $a^J_{p_1\ldots p_r p_{r+1}}: \hat{G} \to \mathbf{R}$ of the r-th weak prolongation of Δ satisfy

$$\text{(12)}\quad a^J_{p_1\ldots p_r p_{r+1}} = b^J_{p_1\ldots p_r K} a^K_{p_{r+1}} + b^J_{p_1\ldots p_r p_{r+1}}.$$

Proof. This is a direct consequence of (7) and of [7].

For the next step of the algorithm, we extend (10) to the equations of the fundamental distribution on $H \times \bar{K}^{r+1}_{n,m}$ in the way described in [8], Proposition 4.

Remark 3. To clarify fundamental ideas, we have used the "frame field of order zero" in our previous investigations. In practice, it is useful to apply a convenient specialization of frames. (A modern explanation of Cartan's method of specialization of frames is given in [13].) In particular if H acts transitively on $K^1_{m,c}(M)$, then one can use the "frame field of the first order" of Δ characterized by $a^J_p = 0$. Under this specialization of frames, the prolongation procedure is essentially simplified. Some concrete evaluations of this sort for 2-dimensional distributions on a 3-dimensional projective space can be found, e. g., in [11].

Remark 4. Since the $(r-1)$-st prolongation of the first kind $j^{r-1}\Delta$ of Δ is a cross section of $J^{r-1}K^1_m(M)$ and $J^{r-1}K^1_m(M)$ is an associated fibre bundle of the symbol $(M, J^{r-1}_c K^1_m(M), H, G)$, it is natural to introduce a geometric object of the order r of the first kind for m-dimensional distributions on M as an equivariant mapping μ of H-space $J^{r-1}_c K^1_m(M)$ into another H-space S. Let μ_2 be the induced mapping of $J^{r-1}K^1_m(M)$ into $E = E(M, S, H, G)$, cf. [7]. Then the composition $\mu_2 j^{r-1}\Delta : M \to E$ is the value of μ on Δ. On the other hand, dealing with the weak prolongations of Δ, we obtain a more interesting situation. Since $\Delta^{(r-1)}$ is a cross section of $\bar{K}^r_m(M) = \bar{K}^r_m(M)(M, \bar{K}^r_{m,c}(M), H, G)$, we are led to equivariant mappings of H-space $\bar{K}^r_{m,c}(M)$. In [8], where we have treated the manifolds with connection, such an equivariant mapping $\mu : \bar{K}^r_{m,c}(M) \to S$ was said to be a semi-holonomic geometric m^r-object on M. This mapping is extended to a mapping $\mu_2 : \bar{K}^r_m(M) \to E = E(M, S, H, G)$ and the composition $\mu_2 \Delta^{(r-1)} : M \to E$ is the value of μ on Δ.

Remark 5. We have shown that one meets the semi-holonomic geometric m^r-objects both in the theory of m-dimensional manifolds with connection and in the case of the weak prolongations of m-dimensional distributions. This explains numerous analogies between geometric properties of manifolds with connection and of differentiable distributions. Our results also give a conceptual explanation of the following fact, which was pointed out in the coordinate form by Kovancov, [10]. By Propositions 1 and 2, if a distribution Δ is holonomic in the second order, then every weak prolongation of Δ is holonomic. On the other hand, there are manifolds with connection holonomic in the second order which are not holonomic in the higher orders, see [9].

Remark 6. Let $M_1 \subset M$ be a k-dimensional submanifold and let Δ be an m-dimensional distribution on M_1, i. e. $\Delta : M_1 \to K^1_m(M_1)$. In this case we shall say that Δ is an immersed distribution. (The first systematic explanation of the use of Cartan's methods for the investigation of immersed distributions was presented by Ščerbakov, [12].) Then $\Delta^{(r-1)}(x)$ is a semi-holonomic contact m^r-element on M which is in a natural incidence relation with the contact element $k^r_x M_1$ determined by M_1, $x \in M_1$. Combining in a convenient way the algorithm for the fundamental fields of submanifolds, [7], with our previous results, one can obtain an invariant method of investigation for immersed distributions. But a detailed explanation of this subject is beyond the scope of this paper.

As another natural generalization, one can treat a distribution on the base of a space with Cartan connection. Let $P(B, G)$ be a principal fibre bundle, let M be a homogeneous space with fundamental group G, let C be a connection of the first order on the groupoid PP^{-1} associated with P and let σ be a cross section of the associated fibre bundle $E = E(B, M, G, P)$. According to [4], a space with Cartan connection of type M can be defined as a quadruple $(P(B, G), M, C, \sigma)$ satisfying the following two conditions: a) $\dim B = \dim M$, b) $C^{-1}(x)(\sigma)$ is regular for every $x \in B$. Further, let $\Delta : B \to K^1_m(B)$ be an m-dimensional distribution on B. To define the weak developments of Δ, we first explain how to develop contact elements by means of elements of connection. It will be sufficient to treat the semi-holonomic case only. According to Ehresmann, [1] (for terminology and notations see [3]), if X is a semi-holonomic r-element of connection on PP^{-1} over $x \in B$ and if Z is a semi-holonomic r-jet of a manifold N into E, $\beta Z \in E_x$, then the development $X^{-1}(Z)$ of Z by means of X is defined by

$$X^{-1}(Z) = (X^{-1}pZ) \,.\, Z \in \bar{J}^r(N, E_x), \tag{13}$$

where p: $E \to B$ in the bundle projection. This operation is immediately extended to contact elements on E as follows. If η is a semi-holonomic contact k^r-element on E at a point of E_x, $\eta = Y\bar{L}^r_k$, then the semi-holonomic contact k^r-element $(X^{-1}(Y))\bar{L}^r_k$ on E_x is well determined by η, since $(X^{-1}(Y))A = ((X^{-1}pY) \,.\, Y)A = (X^{-1}pYA) \,.\, YA$, $A \in \bar{L}^r_k$. This contact element will be said to be the development of η by means of X and will be denoted by $X^{-1}(\eta)$. We now introduce the r-th weak development $\lambda^r(\Delta)$ of Δ by

$$\lambda^r(\Delta)(x) = [C^{(r-1)}]^{-1}(x)(\sigma\Delta^{(r-1)}(x)) \in \bar{K}^r_m(E_x)\,, \tag{14}$$

where $C^{(r-1)}$ is the $(r-1)$-st prolongation of C according to Ehresmann, [1], and $\sigma\Delta^{(r-1)}(x) \in \bar{K}^r_m(E)$ is the image of $\Delta^{(r-1)}(x) \in \bar{K}^r_m(B)$ by σ. The following consideration shows that $\lambda^r(\Delta)$ can be constructed in a natural way by means of C only. We shall proceed by induction. The first weak development of Δ is the cross section $x \mapsto C^{-1}(x)(\sigma\Delta(x))$ of $\bigcup_{x\in B} K^1_m(E_x)$. Consider the $(r-1)$-st weak development $\lambda^{r-1}(\Delta) : B \to \bigcup_{x\in B} \bar{K}^{r-1}_m(E_x) = W$ and denote by V the tangent space to $\lambda^{r-1}(\Delta)$ at $\lambda^{r-1}(\Delta)(x)$. Then the m-dimensional subspace of V over $\Delta(x)$ is identified with a contact m^1-element ζ on W. Since W is a fibre bundle associated with P, the development $C^{-1}(x)(\zeta)$ of ζ by $C(x)$ is a contact m^1-element on $\bar{K}^{r-1}_m(E_x)$. By [9], $C^{-1}(x)(\zeta)$ is identified with a contact m^r-element on E_x. Analogously to [3] and [8], we deduce that this contact element coincides with $\lambda^r(\Delta)(x)$. Furthermore, combining our previous procedure with the method established in [8], one can obtain an invariant method of investigation for distributions on the base of a space with Cartan connection. Finally, we remark that one can similarly treat a distribution immersed in a space with Cartan connection, or, in other words, a distribution on the base of a manifold with connection in the sense of [8].

Remark 7. After this paper has been finished, there appeared the book [14], a great part of which is devoted to differential geometry of differentiable distributions. In this connection, we underline that our paper explains the foundations of the higher order geometry of differentiable distributions in the intrinsic form and that we outline some further generalizations of this subject (in particular, Proposition 4 implies immediately an invariant method of investigation for differentiable distributions on spaces with a fundamental Lie pseudogroup, cf. [7]).

REFERENCES

[1] EHRESMANN, C.: Sur les connexions d'ordre supérieur, In: Atti del V° Congresso dell'Unione Matematica Italiana, 1955, Roma Cremonese, 344—346.

[2] KOBAYASHI, S.—NOMIZU, K.: Foundations of Differential Geometry I. New York London—Sydney 1963.

[3] KOLÁŘ, I.: On the torsion of spaces with connection. Czechoslovak Math. J., *21* (96), 1971, 124—136.

[4] KOLÁŘ, I.: Higher order torsions of spaces with Cartan connection. Cahiers Topologie Géom. Différentielle, *12*, 1971, 137—146.

[5] KOLÁŘ, I.: Canonical forms on the prolongations of principle fibre bundles. Rev. math. pures et appl., *16*, 1971, 1091—1106.

[6] KOLÁŘ, I.: On the prolongations of geometric object fields. An. Sti. Univ. „Al. I. Cuza" Iasi, *17*, 1971, 437—446.

[7] KOLÁŘ, I.: On the invariant method in differential geometry of submanifolds, to appear.

[8] KOLÁŘ, I.: On manifolds with connection, Czechoslovak Math. J., *23* (98), 1973, 34—44.

[9] KOLÁŘ, I.: Higher order torsions of manifolds with connection, Arch. Math. (Brno), *8*, 1972, 149 —156.

[10] KOVANCOV, M. I.: The geometry of manifolds which are immersed in a space with a fundamental group connection (Ukrainian), Visnik Kiiv. Univ., *9*, 1967, 72—85.

[11] MIHAILESCU, T.: Geometrie diferentiala proiectiva. Bucuresti 1958.

[12] ŠČERBAKOV, R. N.: On the method of the framing of submanifolds (Russian). Tr. Tomsk. Univ., *168*, 1963, 5—11.

[13] ŠVEC, A.: Cartan's method of specialization of frames. Czechoslovak Math. J., *16* (91), 1966, 552—599.

[14] Trudy geometričeskogo seminara, III, Moskva 1971.

Received May 23, 1972.

Matematický ústav ČSAV

pobočka v Brně