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MODULAR AND DISTRIBUTIVE EQUALITIES IN LATTICES

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In this paper we deal with "local" equalities characterizing the modularity and the distributivity of lattices as indentities. The main results are as follows.

§ 1. The modularity (1) (see below) of a pair of elements b, c of the lattice S and the modular equality (2) of a triple of elements $a, b, c \in S$ is characterized by means of the set $B(a, b, c)$ (defined before Theorem 1) — Theorems 1 and 2, respectively.

In § 2 the following problem is pursued. J. v. Neumann [4] p. 108 proved that an arbitrary equality from the six distributive ones for a triple $a, b, c \in S$, which we can obtain from (3) and (3*) (see below) by all permutations of the elements a, b, c together with the modularity of the sublattice $\langle a, b, c\rangle$ generated in S by a, b, c, implies (and, of course, is then equivalent to) the distributivity of the lattice $\langle a, b, c\rangle$. M. Kolibiar [5] has verified that if we omit the requirement of modularity of the lattice $\langle a, b, c\rangle$, only the validity of all six distributive equalities together with (4) suffices (and is necessary) for the distributivity of the lattice $\langle a, b, c\rangle$. This condition is exact, i. e. none from these seven distributive equalities for a, b, c can be omitted. Theorem 3 of our paper shows that the (selfdual) distributive condition (4) together with the upper or lower covering condition or with the semimodularity, respectively, guarantees (and is necessary for) the distributivity of the lattice $\langle a, b, c\rangle$. The same assertion follows from the condition $B(a, b, c) \neq \emptyset$ together with all "non-self-dual" distributive equalities for a, b, c. By Theorem 3, J. v. Neumann's Theorem [4] cited above (also see [7] Th. 35) is recovered, too. In Lemma 8, the lattice $\mathfrak{L}$ with three generators and with (4) as the defining relation is described ((4) is the median equality). This lattice has 24 elements.

§ 1.

Let a, b, c be arbitrary elements of the lattice S. It is well-known that the modularity of the lattice S is characterized by any of the following identities:

(1) $a \vee (b \wedge c) = (a \vee b) \wedge c, \quad a \leqslant c$
(1*) $a \wedge (b \vee c) = (a \wedge b) \vee c, \quad a \geqslant c$
(2) $a \vee [b \wedge (c \vee a)] = (a \vee b) \wedge (a \vee c)$
(2*) $a \wedge [b \vee (c \wedge a)] = (a \wedge b) \vee (a \wedge c)$.
(See, e. g. [7] § 32.)

In a similar way the distributivity can be characterized, e. g. by means of the following equivalent identities:
(3) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(3*) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(4) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ (median identity) ([7] §§ 30 and 33).

A quite new situation arises if we investigate the equalities (1) and (1*) for a fixed pair $b, c \in S$ and others for a fixed triple $a, b, c \in S$ (i. e. if we consider these relations "locally"). The equivalences mentioned above lose the validity, in general, these equalities do not admit either to permute a, b, c. In the paper [5] examples of lattices are given, not fulfilling one and only one of the seven equalities which result from (3), (3*) and (4) by permuting a, b, c (see also [1] V § 3, Ex. 5).

In this context we define (see [6] Def. 1.1):
The pair b, c of elements of S is said to be *modular* or *dual modular*, respectively, if (1) or (1*) is true. We will use the following concept to characterize modular pairs (see [2]).

By the symbol abc we mean that the following equalities are satisfied

$$(a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (b \vee c).$$

Denote $B(a, b, c) = \{x \in S : axb, bxc, cxa\}$.

Theorem 1. *Let b, c be elements of the lattice S. The pair b, c is modular in $S \Leftrightarrow B(a, b, c) \neq \emptyset$ for all $a \in S$, $a \leqslant c$.*

The proof is analogous to that of Theorem 2 [2]. 1. Assume that the pair b, c is not modular in S. There exists $a_0 \leqslant c$ such that

$$a_0 \vee (b \wedge c) < (a_0 \vee b) \wedge c.$$

Let $x \in B(a_0, b, c)$. Then

$$a_0 = a_0 \wedge c \leqslant (a_0 \vee x) \wedge (x \vee c) = x = (a_0 \wedge x) \vee (x \wedge c) \leqslant a_0 \vee c = c,$$

thus $a_0 \leqslant x \leqslant c$. It follows
$(b \vee a_0) \wedge c \leqslant (b \vee x) \wedge (x \vee c) = x = (b \wedge x) \vee (x \wedge a_0) \leqslant (b \wedge c) \vee a_0 <$ $< (a_0 \vee b) \wedge c$, a contradiction.

2. Let the pair b, c be modular in S, $s = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$, $a \leqslant c$. Then by (1)

$$s = a \vee (b \wedge c) = (a \vee b) \wedge c, \quad \text{thus} \quad a \leqslant s \leqslant c,\ b \wedge c \leqslant s \leqslant a \vee b.$$

From this one deduces easily $a \vee s = s$, $b \vee s = b \vee a \vee (b \wedge c) = b \vee a$, $c \vee s = c$, $a \wedge s = a$, $b \wedge s = b \wedge c \wedge (a \vee b) = b \wedge c$, $c \wedge s = s$. Then asb holds for $(a \vee s) \wedge (s \vee b) = s \wedge (b \vee a) = s$, $(a \wedge s) \vee (s \wedge b) = a \vee (b \wedge c) = s$, further bsc since $(b \vee s) \wedge (s \vee c) = (b \vee a) \wedge c = s$, $(b \wedge s) \vee (s \wedge c) = (b \wedge c) \vee s = s$ and also csa because of $(c \vee s) \wedge (s \vee a) = c \wedge s = s$, $(c \wedge s) \vee (s \wedge a) = s \vee a = s$. Finally, $s \in B(a, b, c)$ for all $a \in S$, $a \leqslant c$. The assertion has been proved.

With respect to $B(a, b, c) = B(x, y, z)$ being fulfilled for an arbitrary permutation x, y, z of the elements a, b, c, by Theorem 1, we easily deduce the condition of modularity of any pair of the elements a, b, c.

From Theorem 1 the condition of the dual modularity follows as well. For reason of definiteness let us write $B_S(a, b, c)$. If T denotes the lattice dual to S, clearly, $B_S(a, b, c) = B_T(a, b, c)$ will hold for arbitrary $a, b, c \in S$. Hence from Theorem 1 it follows immediately

b, c is the dual modular pair in $S \Leftrightarrow b, c$ is the modular pair in $T \Leftrightarrow B_T(a,b,c) \neq \emptyset$ for all $a \in T$, $a \leq_T c \Leftrightarrow B_S(a, b, c) \neq \emptyset$ for all $a \in S$, $a \;{}_S{\geq}\; c$.

Thus we have obtained the following

Corollary 1. *The pair b, c is dual modular in $S \Leftrightarrow B(a, b, c) \neq \emptyset$ for all $a \in S$, $a \geqslant c$.*

Let a, b, c be given elements of S and let us denote

$$t = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a), \quad t^* = (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$$

$t \leqslant t^*$ holds and the relation (4) can be replaced by $t = t^*$.

Lemma 1. *$t \leqslant s \leqslant t^*$ holds for all $s \in B(a, b, c)$.*

Proof. From the definition of the relation $s \in B(a, b, c)$ it follows

$$a \wedge b \leqslant (a \vee s) \wedge (s \vee b) = s = (a \wedge s) \vee (s \wedge b) \leqslant a \vee b$$

and similarly $b \wedge c \leqslant s \leqslant b \vee c$, $c \wedge a \leqslant s \leqslant c \vee a$. Hence the assertion.

Denote $L = [b \wedge c, b \vee c]$. There holds ([6] Lemma 1.4) that

(5) b, c is a modular pair in $S \Leftrightarrow b, c$ is a modular pair in L.

From Lemma 1, we get immediately

(6) $L = [b \wedge c, b \vee c]$, $a \in L \Rightarrow B_S(a, b, c) = B_L(a, b, c)$.

In fact, $s \in B_S(a, b, c) \Rightarrow b \wedge c \leqslant t \leqslant s \leqslant t^* \leqslant b \vee c \Rightarrow s \in L$, thus $s \in B_L(a, b, c)$. The implication $s \in B_L(a, b, c) \Rightarrow s \in B_S(a, b, c)$ is trivial.

Then from (5), (6) and Theorem 1 we obtain for the given elements $b, c \in S$ $B_S(a, b, c) \neq \emptyset$ for all $a \in S$, $a \leqslant c \Leftrightarrow b, c$ is a modular pair in $S \Leftrightarrow b, c$ is a modular pair in $L \Leftrightarrow B_L(a, b, c) \neq \emptyset$ for all $a \in S$, $b \wedge c \leqslant a \leqslant c$.

The dual proposition holds as well. We summarize as Corollary 2 of Theorem 1.

Corollary 2. *Let b, c be given elements of the lattice S. Then the folloving holds*

1. *$B(a, b, c) \neq \emptyset$ for all $a \in S$, $a \leqslant c \Leftrightarrow B(a, b, c) \neq \emptyset$ for all $a \in S$, $b \wedge c \leqslant a \leqslant c$.*

2. $B(a, b, c) \neq \emptyset$ *for all* $a \in S$, $a \geqslant c \Leftrightarrow B(a, b, c) \neq \emptyset$ *for all* $a \in S$, $b > c \geqslant$
$> a \geqslant c$.

The modular equality (2) for a, b, c is characterized in the following theorem.

Theorem 2. *Let a, b, c be elements of the lattice S. Hence*
(2) $\Leftrightarrow B(A, b, C) \neq \emptyset$, where $C = (a \vee b) \wedge (a \vee c)$, $A = a \vee [b \wedge (c \vee a)]$.
The dual assertion is true as well.

Proof. If (2) does not hold, i. e. if the following is fulfilled

$$A = a \vee [b \wedge (c \vee a)] < (a \vee b) \wedge (a \vee c) = C,$$

then $a \vee b = E > C > A > D = b \wedge (c \vee a)$, $E > b > D$, $C \parallel b \parallel A$, and E, C, A, b, D is a sublattice of S of the form of a "pentagon". For it is a matter of routine work to verify that $b \wedge C = D$ and $b \vee A = E$.

If $s \in B(A, b, C)$, then $(C \vee s) \wedge (s \vee A) = s = (C \wedge s) \vee (s \wedge A)$, thus $A \leqslant s \leqslant C$. Hence $C = E \wedge C \leqslant (b \vee s) \wedge (s \vee C) = s = (b \wedge s) \vee (s \wedge A) \leqslant$
$\leqslant D \vee A = A$, a contradiction.

Thus $B(A, b, C) = \emptyset$ holds. The converse implication is trivial, for if $A = C$, then $A \in B(A, b, C)$.

§ 2.

In section 2 the distributive equalities are dealt with. Let a, b, c be given elements of the lattice S.

Definition. *Denote by* (I) *the system of* (*six*) *equalities which originate from* (3) *and* (3*) *by permuting a, b, c. These equalities together with $t = t^*$ will be called the distributive equalities for a, b, c.*

Lemma 2.
(7) $t^* \geqslant (a \vee t) \wedge (b \vee t) \geqslant t = (a \wedge t) \vee (b \wedge t)$ *and dually*:
(7*) $t^* = (a \vee t^*) \wedge (b \vee t^*) \geqslant (a \wedge t^*) \vee (b \wedge t^*) \geqslant t$
and further four relations by permuting a, b, c.

Proof. Clearly it suffices to prove (7). The first inequality:

$$(a \vee t) \wedge (b \vee t) = [a \vee (b \wedge c)] \wedge [b \vee (c \wedge a)] \leqslant$$
$$\leqslant (a \vee b) \wedge (a \vee c) \wedge (b \vee c) \wedge (b \vee a) = t^*.$$

The second one is trivial. The equality:
$$a \wedge t = a \wedge [(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)] \geqslant (a \wedge b) \vee (a \wedge b \wedge c) \vee (a \wedge c) =$$
$$= (a \wedge b) \vee (c \wedge a);$$
similarly $b \wedge t \geqslant (b \wedge a) \vee (b \wedge c)$. Thus $t \geqslant (a \wedge t) \vee (b \wedge t) \geqslant (a \wedge b) \vee$
$(a \wedge c) \vee (b \wedge c) = t$.

Corollary. $t = (a \wedge t) \vee (b \wedge t) \vee (c \wedge t)$ *and dually*:

$$t^* = (a \vee t^*) \wedge (b \vee t^*) \wedge (c \vee t^*).$$

Remark. The equalities in (7) and (7*), Lemma 2, imply the following distributive equalities of the types (3) and (3*)

$$(a \vee t^*) \wedge (b \vee t^*) = (a \wedge b) \vee t^*$$
$$(a \wedge t) \vee (b \wedge t) = (a \vee b) \wedge t$$

and further four equalities by permuting a, b, c.

It follows from $t^* = (a \vee t^*) \wedge (b \vee t^*) \geqslant (a \wedge b) \vee t^* \geqslant t^*$. Dually the second equality.

Lemma 3. *If* (I) *holds, we have* $(a \vee t) \wedge (b \vee t) = t^*$ *and dually* $(a \wedge t^*) \vee (b \wedge t^*) = t$. *Further four equalities by permuting* a, b, c.

Proof. $a \vee t = a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, $b \vee t = b \vee (a \wedge c) = (b \vee a) \wedge (b \vee c) \Rightarrow (a \vee t) \wedge (b \vee t) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a) = t^*$.

Lemma 4. *If* (I) *and* $B(a, b, c) \neq \emptyset$ *hold, then* (4) *is true.*

Proof. By Lemma 1 $t \leqslant s \leqslant t^*$ whenever $s \in B(a, b, c)$ and by Lemma 3 $t^* = (a \vee t) \wedge (b \vee t) \leqslant (a \vee s) \wedge (b \vee s) = s \leqslant t^*$, i. e. $s = t^*$; dually, we obtain $s = t$. Hence (4) $t = t^*$.

Lemma 5. (4) *implies* $B(a, b, c) = \{t\} = \{t^*\}$.

Proof. By hypothesis $t = t^*$ and by Lemma 2

$$(a \vee t) \wedge (b \vee t) = (a \vee t^*) \wedge (b \vee t^*) = t^* = t = (a \wedge t) \vee (b \wedge t).$$

Analogously btc, cta. Hence $t \in B(a, b, c)$. By Lemma 1, one gets: $s \in B(a,b,c) \Rightarrow t^* \geqslant s \geqslant t = t^* \Rightarrow s = t$, hence $B(a, b, c) = \{t\} = \{t^*\}$.

Corollary. (I) $\Rightarrow$ card $B(a, b, c) \leqslant 1$. (*In the case of equality there is* $B(a,b,c) = \{t\} = \{t^*\}$.)

Proof. Either $B(a, b, c) = \emptyset$ holds or, by Lemmas 4 and 5, $B(a, b, c) = \{t\}(= \{t^*\})$ is true.

Lemma 6. *If* (4) *holds, we have*

$$t = t^* = [a \wedge (b \vee c)] \vee [b \wedge (c \vee a)] \tag{8}$$

$$a \vee [b \wedge (c \vee a)] = a \vee (b \wedge c) = t^* \vee a = t \vee a, \tag{9}$$

and further relations by permuting a, b, c, *and by dualisation.*

Proof. By Lemma 5, $t = t^* = [(a \vee b) \wedge (a \vee c)] \wedge (b \vee c) \geqslant \{a \vee [b \wedge (c \vee a)]\} \wedge (b \vee c) \geqslant [a \wedge (b \vee c)] \vee [b \wedge (c \vee a)] \geqslant (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) \vee (b \wedge a) = t$.

Then (8) holds and therefore

$$t \vee a = t^* \vee a = a \vee [a \wedge (b \vee c)] \vee [b \wedge (c \vee a)] = a \vee [b \wedge (c \vee a)] .$$

Finally, from the definition of t we obtain $t \vee a = a \vee (b \wedge c)$. Thus (9).

Corollary. *If* (4) *holds, the following is true for arbitrary* a, b, c*:* (2) $\Leftrightarrow$ (3), (2*) $\Leftrightarrow$ (3*) *and further equivalences by permuting* a, b, c.

Lemma 7. *If in the modular lattice for* a, b, c *one of the equalities, which originate from* (3) *and* (3*) *by permuting* a, b, c*, does not hold, then none holds.*

(Lemma 7 follows from J. v. Neumann's Theorem [4], p. 108 — see also [7] Th. 35).

Lemma 8. *From the hypothesis* (4) *it follows that the set* $\mathfrak{L} = \mathfrak{A} \cup \mathfrak{B} \cup \mathfrak{A}^*$ *is the sublattice* $\langle a,b,c \rangle$ *of* S*, where* $\mathfrak{A} = \{t \vee a, t \vee b, t \vee c; C = (a \vee b) \wedge (a \vee c), P = (a \vee c) \wedge (b \vee c), Q = (a \vee b) \wedge (b \vee c); a \vee b, b \vee c, a \vee c; a \vee b \vee c\}$, $\mathfrak{B} = \{a, t, b, c\}$, $\mathfrak{A}^*$ *is the set of the elements dual to those of* $\mathfrak{A}$.

Proof. It suffices to prove that for $\alpha, \alpha_1 \in \mathfrak{A} \setminus \{a \vee b \vee c\}$, $\beta, \beta_1 \in \mathfrak{B}$ the elements $\alpha \vee \alpha_1, \alpha \vee \beta, \alpha \vee \alpha_1^*, \alpha \wedge \alpha_1, \alpha \wedge \beta, \alpha \wedge \alpha_1^*, \beta \vee \beta_1$ belong to $\mathfrak{L}$.

We consider $\alpha \vee \alpha_1$. (We write $\alpha_1 \sim \delta$ instead of $\alpha \vee \alpha_1 = \delta$.)

$\alpha = t \vee a$:

$t \vee b \sim a \vee b$, $t \vee c \sim a \vee c$, $C \sim C$; $a \vee c \geqslant t \vee a \vee P \geqslant t \vee a \vee t \vee c = a \vee c \Rightarrow P \sim a \vee c$; analogously $Q \sim a \vee b$; $a \vee b \sim a \vee b, b \vee c \sim a \vee b \vee c$, $a \vee c \sim a \vee c$.

$\alpha = t \vee b$ or $\alpha = t \vee c$ — it suffices to interchange a and b or a and c, respectively, in the above section.

$\alpha = C$: $a \vee c \geqslant C \vee P \geqslant t \vee a \vee t \vee c = a \vee c \Rightarrow P \sim a \vee c$; analogously $Q \sim a \vee b$; $a \vee b \sim a \vee b$; $C \vee b \vee c \geqslant C \vee Q \vee C \vee P = a \vee b \vee c \Rightarrow b \vee c \sim a \vee b \vee c$.

$\alpha = P$ or $\alpha = Q$ — interchange a and c or a and b, respectively, in the foregoing section.

$\alpha = a \vee b$: $b \vee c \sim a \vee b \vee c$, $a \vee c \sim a \vee b \vee c$.

$\alpha = b \vee c$ or $\alpha = a \vee c$ — interchange a and c or b and c, respectively, in the foregoing section.

We consider $\alpha \vee \beta$.

$\alpha = t \vee a$: $a \sim t \vee a$, $t \sim t \vee a$, $b \sim a \vee b$, $c \sim a \vee c$.

$\alpha = C$: $a \sim C, t \sim C$; $a \vee b \geqslant C \geqslant a \Rightarrow a \vee b \geqslant C \vee b \geqslant a \vee b \Rightarrow b \sim a \vee b$; $a \vee c \geqslant C \vee c \geqslant a \vee c \Rightarrow c \sim a \vee c$.

$\alpha = P$ or $\alpha = Q$ — analogously.

$\alpha = a \vee b$: $a \sim a \vee b$, $t \sim a \vee b$, $b \sim a \vee b$, $c \sim a \vee b \vee c$.

$\alpha = b \vee c$ or $\alpha = a \vee c$ — analogously.

We consider $\alpha \vee \alpha_1^*$; it suffices to state that $\alpha \geqslant t^* = t \geqslant \alpha_1^*$.

We consider $\alpha \wedge \alpha_1$. (Now, $\alpha_1 \sim \delta$ stands for $\alpha \wedge \alpha_1 = \delta$.)

$\alpha = t \vee a$: $t = t^* \geqslant (t \vee a) \wedge (t \vee b) \geqslant t$ by Lemma 2, hence $t \vee b \sim t$; $t \vee c \sim t$ (analogously), $C \sim t \vee a$; $t = t^* = C \wedge P \Rightarrow P \sim t$; $Q \sim t$ (similarly), $a \vee b \sim t \vee a$; $t = t^* = C \wedge (b \vee c) \geqslant (t \vee a) \wedge (t \vee c) > t \Rightarrow b \vee c \sim t$; $a \vee c \sim t \vee a$.

$\alpha = t \vee b$ or $\alpha = t \vee c$ — similarly.

$\alpha = C : P \sim t^*$, $Q \sim t^*$, $a \vee b \sim C$, $b \vee c \sim t^*$, $a \vee c \sim C$.

$\alpha = P$ or $\alpha = Q$ — similarly.

$\alpha = a \vee b : b \vee c \sim Q$, $a \vee c \sim C$.

$\alpha = b \vee c$ or $\alpha = a \vee c$ — similarly.

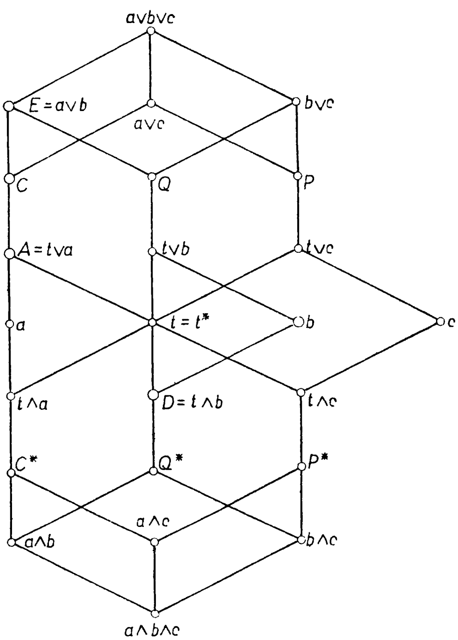
We consider $\beta \wedge \alpha$.

$\beta = a$: $t \vee a \sim a$; $a \wedge t = a \wedge (t \vee a) \wedge (t \vee b)$ (by (7*) and (4)) $\Rightarrow a \wedge t = a \wedge (t \vee b) \geqslant a \wedge t \Rightarrow t \vee b \sim t \wedge a$; similarly $t \vee c \sim t \wedge a$, $C \sim a$; by section $\alpha \wedge \alpha_1$ we have $t = (t \vee a) \wedge P$ hence $a \wedge t = a \wedge (t \vee a) \wedge P = a \wedge P = a \wedge (b \vee c) = t \wedge a \Rightarrow P \sim t \wedge a$; analogously $Q \sim t \wedge a$: $a \vee b \sim a$, $b \vee c \sim t \wedge a$, $a \vee c \sim a$.

$\beta = t : t \wedge \alpha = t$ because of $\alpha \geqslant t$.

$\beta = b$ or $\beta = c$ — analogously to the case $\beta = a$.

We consider $\alpha \wedge \alpha_1^*$.



The lattice $\mathfrak{L}$

But $\alpha \quad \alpha_1^* = \alpha_1^*$ for $\alpha \geqslant t = t^* \geqslant \alpha_1^*$.

We consider $\beta \vee \beta_1$. Clearly $\beta \vee \beta_1 \in \mathfrak{A} \subseteq \mathfrak{L}$. Thus we have proved the lemma.

If different symbols in $\mathfrak{L}$ represent different elements of S, the set $\mathfrak{L}$ stands for the lattice $\overline{\mathfrak{L}}$, the graph of which is in the figure. The sublattice $\langle a, b. c$ generated in S by a, b, c which satisfy (4) is a factor-lattice of $\mathfrak{L}$ corresponding to a suitable congruence over $\overline{\mathfrak{L}}$.

Definition. *Denote by* (II) *one of the following conditions*

(14) $\langle a, b, c\rangle$ *satisfies the upper covering condition*

(15) $\langle a, b, c\rangle$ *satisfies the lower covering condition*

(16) $\langle a, b, c\rangle$ *is semimodular.*

Remark. Since the lattice $\langle a, b, c\rangle$ is finite supposing (4), the lower covering condition is equivalent to the semimodularity ([7] Th. 65).

Theorem 3. *Let a, b, c be elements of the lattice S. Then the following conditions are equivalent.*

(a) *There hold* (4) *and* (II).

(b) *There hold* (I) *and* $B(a, b, c) \neq \emptyset$.

(c) *The lattice $\langle a, b, c\rangle$ is distributive.*

(d) *The lattice $\langle a, b, c\rangle$ is modular and one of the (seven) distributive equalities for a, b, c is true.*

Proof. (a) $\Rightarrow$ (b): Suppose that (3) is not true, i. e. that the following holds

$$(17) \qquad a \vee (b \wedge c) < (a \vee b) \wedge (a \vee c)\,.$$

By Lemma 6 (9) we have $a \vee [b \wedge (c \vee a)] < (a \vee b) \wedge (a \vee c)$. As in the proof of Theorem 2, the following holds $E = a \vee b > C = (a \vee b) \wedge (a \vee c) > A = a \vee [b \wedge (c \vee a)] = t \vee a > D = b \wedge (a \vee c) = t \wedge b$, $E > b > D$, $C \parallel b \parallel A$ and the set $\{E, C, A, D, b\}$ is a sublattice of S. In the Hasse diagram of $\mathfrak{L}$ the elements A, b, C, D, E are already marked. The lattice $\langle a, b, c\rangle$ is a congruence over $\overline{\mathfrak{L}}$ for which there holds that any two of the elements A, b, C, D, E are non-congruent. Suppose now that (II) is represented by (15) or (by the equivalent condition) (16). Then from $b \succ D = A \wedge b$ (b covers D) it follows $A \prec A \vee b = E$, while $A < C < E$, a contradiction.

Suppose that (II) is represented by (14). The elements $E > C > A \geqslant t$, $E \geqslant Q \geqslant t$ form a sublattice of S because $Q \wedge C = t^* = t$, $Q \wedge A = Q \wedge (t \vee a) = t$ (see section $\alpha \wedge \alpha_1$ of the proof of Lemma 8), $Q \vee C = a \vee b = E$ (section $\alpha \vee \alpha_1$, Lemma 8), $Q \vee A = Q \vee (t \vee a) = a \vee b = E$ (section $\alpha \vee \alpha_1$, Lemma 8). If $C \vee Q = E \succ Q$, then (14) gives $C \succ C \wedge Q = t$, which together with $C > A \geqslant t$ implies $A = t$, i. e. $t \geqslant a$. But now $Q \geqslant t \vee b \Rightarrow Q \geqslant t \vee b \vee a = a \vee b \quad E$, and $Q = E$ contradicts the hypothesis. Since there are two possibilities only (see the diagram of $\overline{\mathfrak{L}}$), $E \succ Q$ or $E = Q$ and the first one was excluded, there remains $E = Q$.

The elements $E > C > A \geqslant t$, $E(= Q) \geqslant t \vee b \geqslant t$ form a sublattice of S for $(t \vee b) \wedge C = t$ (see section $\alpha \wedge \alpha_1$ -- change a and b in the relation $(t \vee a) \wedge Q = t)$, $(t \vee b) \wedge A = (t \vee b) \wedge (t \vee a) = t$ ((7*) in Lemma 2 and (4)), $(t \vee b) \vee C \geqslant (t \vee b) \vee a = a \vee b \Rightarrow (t \vee b) \vee C = a \vee b = E$, $(t \vee b) \vee A = (t \vee b) \vee (t \vee a) = a \vee b = E$. From (14) and the supposition $E = C \vee (t \vee b) \succ t \vee b$ we obtain $C \succ C \wedge (t \vee b) = t$. Since $C > A \geqslant t$, then $t = A$ and thus $t \geqslant a$. It follows that $t \vee b \geqslant a \vee b = E$ and so $t \vee b = E$, a contradiction. Since there are only two possibilities $E = Q \succ t \vee b$ or $E = Q = t \vee b$ (see the diagram of $\overline{\mathfrak{Q}}$) and the first one has been excluded, there remains $E = Q = t \vee b$.

Finally, $E > b$, thus $E = Q = t \vee b \succ b$ (see the diagram of $\overline{\mathfrak{Q}}$). Since the elements $E > b > D$, $E > C > A > D$ form a sublattice, from the relation $E = C \vee b \succ b$ and from (14) it follows that $C \succ C \wedge b = D$ while $C > A > D$, a contradiction.

We conclude that the hypothesis (17) is false.

In an analogous way (by way of permuting a, b, c and by dualisation), one deduces from (4) and (II) all the equalities belonging to (I). By Lemma 5 $B(a, b, c) \neq \emptyset$ holds. Hence (b).

(b) $\Rightarrow$ (c): By Lemma 4 there holds (4). Thus all seven distributive equalities for a, b, c hold. Then (c) results from Corollary 2 of Theorem 2a in [5] which is formulated as follows:

The sublattice generated by three elements a, b, c of a lattice S is distributive if and only if all seven distributive equalities for a, b, c are true.

(c) $\Rightarrow$ (d) is evident.

(d) $\Rightarrow$ (a): In the modular lattice $\langle a, b, c\rangle$ there hold (14), (15) and (16) (see i. e. [7] Th. 37 and p. 136). We shall prove that (4) and thus (a) is true. In (d) one supposes (4) or one of the six equalities of (I). In the latter case (I) holds by Lemma 7. From (I) and the modularity (4) results for

$$t^* = (c \vee a) \wedge (a \vee b) \wedge (b \vee c) = [a \vee (b \wedge c)] \wedge (b \vee c) =$$
$$= [a \wedge (b \vee c)] \vee (b \wedge c) = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) = t\,.$$

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