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QUASIGROUPS AND FACTORISATION OF COMPLETE DIGRAPHS

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In the present paper the results of CAYLEY and FRUCHT (quoted in [2]) on groups will be transferred onto quasigroups and loops. The graph-theoretical terminology is that of ORE [2], the algebraic terminology is the translation of the terminology of BELOUSOV [1]. The term "loop" will be used here in two quite different senses: in the algebraic sense (a quasigroup with a two-side unit element) and in the graph-theoretical sense (an edge joining a vertex with this vertex itself). For avoiding misunderstandings due to this homonymy, after the word "loop" we shall always put either "a. s." (algebraic sense), or "g. s." (graphtheoretical sense) in brackets. We shall consider digraphs without multiple edges, but with loops (g. s.) and with pairs of edges joining the same pair of vertices, but differently directed.

In [2] the Cayley colour graph of a group is described. Here we shall generalize this concept for quasigroups.

Let Q be a quasigroup of the order n (n can be also infinite). Take a complete digraph with loops (g. s.) with n vertices. (This is a digraph in which any two distinct vertices are joined by a pair of differently directed edges and at each vertex there is a loop (g. s.).) We put the elements of Q and the vertices of this graph into a one-to-one correspondence. Then we colour the edges of the graph by n colours which are also in a one-to-one correspondence with the elements of Q so that for any two elements $x \in Q$, $y \in Q$ the edge outgoing from the vertex corresponding to x into the vertex corresponding to y obtains the colour corresponding to the element $y \setminus x$. Then the resulting graph with the described colouring of edges is called the Cayley colour graph of Q and is denoted by $C(Q)$. Here $y \setminus x$ denotes the element $z \in Q$ for which $xz = y$ holds. As Q is a quasigroup, this element is uniquely determined for any x and y of Q.

It is easy to prove that the edges coloured by the same colour in $C(Q)$ form a linear factor of this digraph. In fact, let us take a vertex x of $C(Q)$; the element of Q corresponding to it will be denoted also by x. An edge coloured by the colour corresponding to some $y \in Q$ outgoing from x leads into xy in $C(Q)$. The element xy is exactly one for any x and y of Q, therefore exactly one edge of the colour corresponding to y goes out from the vertex x for any x and y of Q. An edge of the colour corresponding to $y \in Q$ incoming into x goes out from the vertex corresponding to $x/y \in Q$; here x/y denotes the element $z \in Q$ for which $zy = x$ holds. For any x and y of Q the element x/y is exactly one, therefore exactly one edge of the colour corresponding to y comes also into x.

Thus the Cayley colour graph $C(Q)$ can be considered as an ordered pair $\langle \mathscr{F}, \xi \rangle$, where $\mathscr{F}$ is a decomposition of the complete digraph $\vec{K}_n$ with n vertices with loops (g. s.) into edge-disjoint linear factors and ξ is a one-to-one mapping of the vertex set of $\vec{K}_n$ onto the set of factors of $\mathscr{F}$. (As well-known, if $\vec{K}_n$ is decomposed into linear factors, the number of these factors is exactly n.)

Theorem 1. *Let $\vec{K}_n$ be the complete digraph with n vertices with loops (g. s.), where n is a finite or infinite cardinal number. Any ordered pair $\langle \mathscr{F}, \xi \rangle$, where $\mathscr{F}$ is a decomposition of $\vec{K}_n$ into edge-disjoint linear factors and ξ is a one-to-one mapping of the vertex set of $\vec{K}_n$ onto the set of factors of $\mathscr{F}$, determines a quasigroup Q such that the Cayley colour graph $C(Q)$ of Q can be considered as $\langle \mathscr{F}, \xi \rangle$, as described above.*

Proof. Let the set of elements of Q be the set of vertices of $\vec{K}_n$. If $x \in Q$, $y \in Q$, then xy is the terminal vertex of the edge outgoing from the vertex x and belonging to the factor $\xi(y) \in \mathscr{F}$. As $\xi(y)$ is a linear factor of $\vec{K}_n$, this edge is exactly one. The element $x \setminus y$ is such an element $z \in Q$ that the edge going from y to x belongs to $\xi(z)$. The element x/y is the initial vertex of the edge whose terminal vertex is x and which belongs to the factor $\xi(y)$. These elements are determined uniquely, thus we have obtained a quasigroup.

Evidently the Cayley colour graphs of two quasigroups Q_1 and Q_2 are isomorphic, if and only if Q_1 and Q_2 are isomorphic. (Here we mean the isomorphism preserving colours of edges.) We shall consider isotopies of quasigroups. As defined in [1], an isotopy of a quasigroup Q_1 onto a quasigroup Q_2 is an ordered triple $\langle \alpha, \beta, \gamma \rangle$ of one-to-one mappings of Q_1 onto Q_2 such that for any three elements x, y, z of Q_1 the equality $\alpha(x)\beta(y) = \gamma(z)$ in Q_2 is equivalent to the equality $xy = z$ in Q_1.

Theorem 2. *Let Q_1 and Q_2 be two quasigroups on the same set M of n elements. The following two assertions are equivalent:*

(1) *The Cayley colour graphs of Q_1 and Q_2 can be considered as pairs $\langle \mathscr{F}, \xi_1 \rangle$ and $\langle \mathscr{F}, \xi_2 \rangle$, respectively, $\mathscr{F}$ being the same in both pairs.*

(2) *There exists an isotopy of Q_1 onto Q_2 of the form $\langle \varepsilon, \beta, \varepsilon \rangle$, where ε is the identical mapping of the set M.*

Remark. The pairs of the form $\langle \mathscr{F}, \xi \rangle$ are defined above.

Proof. (1) $\Rightarrow$ (2). We can identify the vertices of Cayley colour graphs of Q_1 and Q_2 with the set M. Let $\beta = \xi_2^{-1}\xi_1$; this is a permutation of M. Let x, y, z be three arbitrary elements of M. If $xy = z$ in Q_1, then an edge goes from x into z with the colour (of the factor) $\xi_1(y)$ in $C(Q_1)$. In $C(Q_2)$ the edge going from x into z must then be of the colour $\xi_2(t)$, where t is some element of Q_2, i. e. of M. We have $xt = z$ in Q_2. Therefore the factors $\xi_1(y)$, $\xi_2(t)$ of $\mathscr{F}$ are equal. We have $\xi_1(y) = \xi_2(t)$, which means $t = \xi_2^{-1}\xi_1(y) = \beta(y)$. Thus $xy = z$ in Q_1 implies $\varepsilon(x)\beta(y) = \varepsilon(z)$ in Q_2; analogously we can prove the inverse implication and thus the equivalence of these equalities. We have proved that $\langle\varepsilon, \beta, \varepsilon\rangle$ is an isotopy of Q_1 onto Q_2.

(2) $\rightarrow$ (1). Let there exist an isotopy of the form $\langle\varepsilon, \beta, \varepsilon\rangle$ of Q_1 onto Q_2. Then $xy = z$ in Q_1 is equivalent to $x\beta(y) = z$ in Q_2 for any three elements x, y, z, of M. Let us have two arbitrary edges of $\vec{K}_n$, one going from x_1 to y_1, another going from x_2 to y_2, where x_1, y_1, x_2, y_2 are some vertices of K_n, i. e. elements of M. These two edges have the same colour in $C(Q_1)$ if and only if $y_1 \setminus x_1 = y_2 \setminus x_2$ in Q_1, i. e. if $x_1z = y_1, x_2z = y_2$ for some z. But this is equivalent to $x_1\beta(z) = y_1, x_2\beta(z) = y_2$ in Q_2, which means that $y_1 \setminus x_1 = y_2 \setminus x_2$ also in Q_2 and these edges have the same colour also in $C(Q_2)$. Therefore the factorisation $\mathscr{F}$ is the same for both quasigroups Q_1, Q_2.

In this theorem we have considered two quasigroups Q_1 and Q_2 with the same set M of elements. We have done this for the sake of simplicity. But these considerations can be transferred to the case of two quasigroups with distinct sets of elements, obviously with equal cardinalities. Then we have an isotopy of the form $\langle\alpha, \beta, \alpha\rangle$, where α and β are one-to-one mappings of Q_1 onto Q_2.

We have here considered Cayley colour graphs as pairs $\langle\mathscr{F}, \xi\rangle$. In the following it will be more convenient, if we consider them again as complete digraphs with loops (g. s.) with some colouring. We shall give a definition of isotopy of these graphs (compare [3]).

Let G_1 and G_2 be two digraphs whose edges are coloured in some way. A colour-preserving isotopy of G_1 onto G_2 is an ordered triple $\langle f_1, f_2, \varphi\rangle$, where f_1 and f_2 are one-to-one mappings of the vertex set V_1 of G_1 onto the vertex set V_2 of G_2 and φ is a one-to-one mapping of the set of colours of edges of G_1 onto the set of colours of edges of G_2 such that for any two vertices u, v of G_1 the existence of the edge $\overrightarrow{uv}$ in G_1 is equivalent to the existence of the edge $\overrightarrow{f_1(u)f_2(v)}$ in G_2 and if $\overrightarrow{uv}$ in G_1 exists and has the colour c, then $\overrightarrow{f_1(u)f_2(v)}$ in G_2 has the colour $\varphi(c)$.

A colour-preserving isotopy of a digraph G onto itself is called a colour-preserving autotopy of G. If moreover φ is an identical mapping of the colour set of G, this autotopy is called strongly colour-preserving.

Theorem 3. *Let Q_1, Q_2 be two quasigroups, let there exist an isotopy of Q_1*

onto Q_2. Then there exists a color-preserving isotopy of $C(Q_1)$ onto $C(Q_2)$ and vice versa.

Proof. Let $\langle\alpha, \beta, \gamma\rangle$ be an isotopy of Q_1 onto Q_2. If $xy = z$ in Q_1, then $\alpha(x)\beta(y) = \gamma(z)$ in Q_2. In $C(Q_1)$ the edge outgoing from x and incoming into z has the colour corresponding to y (we may say shortly that it has the colour y). In $C(Q_2)$ the edge outgoing from $\alpha(x)$ and incoming into $\gamma(z)$ has the colour $\beta(y)$. If $xy \neq z$, then this evidently does not hold. Therefore $\langle\alpha, \gamma, \beta\rangle$ is the corresponding colour-preserving isotopy of $C(Q_1)$ onto $C(Q_2)$. On the other hand, let there exist a colour-preserving isotopy $\langle f_1, f_2, \varphi\rangle$ of $C(Q_1)$ onto $C(Q_2)$. If $xy = z$ in Q_1, then in $C(Q_1)$ an edge of the colour y goes from x into z. Therefore in $C(Q_2)$ an edge of the colour $\varphi(y)$ goes from $f_1(x)$ into $f_2(z)$, which means that $f_1(x)\varphi(y) = f_2(z)$ in Q_2 holds. Thus $\langle f_1, \varphi, f_2\rangle$ is an isotopy of Q_1 onto Q_2.

Now we define the colour-preserving isomorphism of G_1 onto G_2 (where G_1 and G_2 are again digraphs with coloured edges) as an isotopy $\langle f_1, f_2, \varphi\rangle$, where $f_1 \equiv f_2$. Colour-preserving and strongly colour-preserving automorphisms are defined analogously. After defining these concepts we can express a further theorem.

Theorem 4. *Let Q_1, Q_2 be two quasigroups, let there exist an isotopy of Q_1 onto Q_2 of the form $\langle\alpha, \beta, \alpha\rangle$. Then there exists a colour-preserving isomorphism of $C(Q_1)$ onto $C(Q_2)$.*

The proof follows from Theorems 2 and 3.

The aim for which Cayley colour graphs of groups were defined was to construct a graph whose group of colour-preserving automorphisms is isomorphic to a given group. If H is a group, then the group of strongly colour preserving automorphisms of $C(H)$ is isomorphic to H. We shall investigate the group of strongly colour-preserving automorphisms of $C(Q)$, where Q is a quasigroup.

Theorem 5. *Let Q be a quasigroup, let $C(Q)$ be its Cayley colour graph. The group of strongly colour-preserving automorphisms of $C(Q)$ is isomorphic to the group of all autotopies of Q having the form $\langle\alpha, \varepsilon, \alpha\rangle$, where ε is the identical mapping of Q.*

Proof. Let $\langle f, f, \varepsilon_0\rangle$ be a strongly colour-preserving automorphism of $C(Q)$, where ε_0 is the identical mapping of the colour set of $C(Q)$. If x, y, z are three elements of Q, then the edge outgoing from x and incoming into z has the colour y if and only if $xy = z$. If and only if this holds, the edge outgoing from $f(x)$ into $f(z)$ has also the colour y and thus $f(x)y = f(z)$. If we put $\alpha = f$, we have an autotopy of Q of the form $\langle\alpha, \varepsilon, \alpha\rangle$. On the other hand, let us have an autotopy $\langle\alpha, \varepsilon, \alpha\rangle$ of Q. Then the edges $\overrightarrow{xz}$ and $\overrightarrow{\alpha(x)\alpha(z)}$ must have the same colour corresponding to $z \setminus x$ and therefore $\langle\alpha, \alpha, \varepsilon_0\rangle$ is a strongly colour-preserving automorphism of $C(Q)$. Thus we have obtained a one-to-one correspondence between strongly colour-preserving automorphisms of $C(Q)$ and autotopies of Q of the form $\langle\alpha, \varepsilon, \alpha\rangle$. This correspondence is formed by a simple exchange of two co-ordinates in ordered triples, thus it is easy to prove that it is preserved by the superposition of these strongly colour-preserving automorphisms or autotopies, and the assertion holds.

Theorem 6. *Let Q be a quasigroup, let $C(Q)$ be its Cayley colour graph. The group of colour-preserving automorphisms of $C(Q)$ is isomorphic to the group of all autotopies of Q having the form $\langle\alpha, \beta, \alpha\rangle$.*

The proof is analogous to the proof of Theorem 5 and Theorem 2 is used in it.

We have proved some theorems concerning Cayley colour graphs of quasigroups in general. Now we shall study special cases of quasigroups — loops (a. s.) and groups.

A loop (a. s.) is a quasigroup containing a two-side unit element, i. e. an element e such that $ex = xe = x$ for each element x of this quasigroup.

Theorem 7. *Let L be a loop (a. s.). Then its Cayley colour graph $C(L)$ can be considered as a pair $\langle\mathscr{F}, \xi\rangle$, where $\mathscr{F}$ is a decomposition of $\vec{K}_n$ into edge-disjoint linear factors and ξ is a one-to-one mapping of the vertex set of $\vec{K}_n$ onto the set of factors of $\mathscr{F}$ and which has the following properties:*
(1) One of the factors of $\mathscr{F}$ is formed by all loops (g. s.) of $\vec{K}_n$.
(2) There exists a vertex of $\vec{K}_n$ such that any edge outgoing from it belongs to the factor $\xi(v)$, where v is its terminal vertex.

Any pair $\langle\mathscr{F}, \xi\rangle$ with the described properties determines a Cayley colour graph $C(L)$ of some loop (a. s.) L.

Proof. Let e be the unit element of L. Then $xe = x$ for each $x \in L$, therefore any edge of $C(L)$ of the colour corresponding to e is a loop (g. s.). Thus $\xi(e)$ is the factor consisting of all loops (g. s.) of $\vec{K}_n$. But also $ex = x$ for each $x \in L$, therefore any edge outgoing from e and incoming to some x has the colour corresponding to x. On the other hand, let $\langle\mathscr{F}, \xi\rangle$ be some pair with the above described properties. We construct the loop (a. s.) L so that the vertex set of $\vec{K}_n$ is taken as the set of elements of L, the vertex from (2) is taken as e and the factor from (1) is taken as $\xi(e)$. For two elements x and y we define xy as the terminal vertex of the edge outgoing from x and belonging to the same factor of F as the edge outgoing from e and incoming into y. Analogously as in the proof of Theorem 1 we can prove that L is a quasigroup and the equalities $ex - x$, $xe - x$ follow directly from the properties (1) and (2).

Corollary. *To any decomposition $\mathscr{F}_0$ of a complete digraph with n vertices without loops (g. s.) into pairwise edge-disjoint linear factors and for any* *arbitrarily chosen vertex v of it there exists a loop (a. s.) L such that the pair* $\langle\mathscr{F}, \xi\rangle$, *where $\mathscr{F}$ is obtained from $\mathscr{F}_0$ by adjoining a linear factor consisting of loops (g. s.) at each vertex and ξ is a suitable one-to-one mapping of the vertex set of this graph onto the set of factors of $\mathscr{F}$, is its Cayley colour graph and the vertex v corresponds to the unit element of L.*

As defined in [1], the left kernel of a quasigroup Q is the set of elements a of Q such that $(ax)y = a(xy)$ for any two elements x and y of Q. The left kernel of a loop L is a group under the multiplication in L.

Theorem 8. *The group of strongly colour-preserving automorphisms of the Cayley colour graph $C(L)$ of a loop (a. s.) L is isomorphic to the left kernel of L.*

Proof. According to Theorem 5 the group of strongly colour-preserving automorphisms of $C(L)$ is isomorphic to the group of all autotopies of L of the form $\langle\alpha, \varepsilon, \alpha\rangle$. Let us have some autotopy of this form. Let e be the unit element of L, let x be some element of L. We have $ex = x$, thus $\alpha(e)\varepsilon(x) = \alpha(e)x = \alpha(x)$. As x was chosen arbitrarily, we have $\alpha(x) = \alpha(e)x$ for any $x \in L$. If we denote $\alpha(e)$ as a, we have $\alpha(x) = ax$. Therefore each autotopy of L of the form $\langle\alpha, \varepsilon, \alpha\rangle$ has the property that α is a left translation [1] by some element $a \in L$. Now let $x \in L$, $y \in L$. We have $\alpha(x)y = \alpha(xy)$, which means $(ax)y = a(xy)$ and a belongs to the left kernel of L. Wow let us have two autotopies $\langle\alpha, \varepsilon, \alpha\rangle$, $\langle\beta, \varepsilon, \beta\rangle$ of L. We have $\alpha(x) = ax$, $\beta(x) = bx$ for some a and b of the left kernel of L and for each $x \in L$. Then $\alpha\beta(x) = a(bx)$ and this is equal to $(ab)x$, as a is in the left kernel of L. Thus if we assign to any autotopy of the form $\langle\alpha, \varepsilon, \alpha\rangle$ the element a of the left kernel of L such that $\alpha(x) = ax$ for each $x \in L$, this assigning is an isomorphism between the set of autotopies of L of the form $\langle\alpha, \varepsilon, \alpha\rangle$ and the left kernel of L.

Now we come to groups.

Theorem 9. *Let H be a group. Then its Cayley colour graph $C(H)$ can be considered as a pair $\langle\mathscr{F}, \xi\rangle$, where $\mathscr{F}$ is a decomposition of $\vec{K}_n$ into edge-disjoint linear factors and ξ is a one-to-one mapping of the vertex set of $\vec{K}_n$ into the set of factors of $\mathscr{F}$ and which has the properties* (1) *and* (2) *from Theorem* 7 *and a further property:*
(3) *In each acyclically directed triangle T of $\vec{K}_n$ the factors of $\mathscr{F}$ to which two edges of T belong determine uniquely the factor of $\mathscr{F}$ to which the third edge of T belongs.*

Any pair $\langle\mathscr{F}, \xi\rangle$ with the described properties determines a Cayley colour graph $C(H)$ of some group H.

Proof. The pair $\langle\mathscr{F}, \xi\rangle$ must have the properties (1) and (2), because every group is a loop (a. s.). Let us have an acyclically directed triangle T in $C(H)$. The vertices of T can be totally ordered by the ordering determined by the orientation of T; let x be the first of them in this ordering. If the edge outgoing from x and incoming to the second vertex of T belongs to the factor $\xi(y) \in \mathscr{F}$, this second vertex is xy. If the edge outgoing from the second vertex of T and incoming to the third belongs to the factor $\xi(z) \in \mathscr{F}$, the third vertex is xyz. Thus the edge going from the first vertex of T into the third must belong to the factor $\xi(yz)$; this factor is uniquely determined by y and z independently on x. Now if the edge going from the first vertex of T into the second belongs to $\xi(y)$ and the edge going from the first vertex of T into the third belongs to $\xi(z)$, then the second vertex of T is xy, the third is xz and the edge going from the second vertex of T into the third belongs to $\xi(y^{-1}z)$; this is also uniquely determined by y and z independently on x. Analogously we could prove this for the situation when the factors of $\mathscr{F}$ to which the edge going from the first vertex of T to the third and the edge going from the second vertex of T to the third belong. Thus (3) is satisfied. On the other hand, let us have $\langle \mathscr{F}, \xi \rangle$ satisfying (1), (2), (3). According to Theorem 7 it determines a Cayley colour graph of some loop (a. s.) L. Let us have an acyclically directed triangle T in $\vec{K}_n$, let its first vertex be x, let the edge going from the first vertex of T into the second belong to $\xi(y)$, let the edge going from the second vertex of T to the third belong to $\xi(z)$. Then the second vertex of T is xy, the third is $(xy)z$. The edge going from the first vertex of T into the third belongs to the factor $\xi(t)$, where t is uniquely determined by y and z (independently on x). Therefore the third vertex of T is xt and we have $xt = (xy)z$ for each $x \in L$. Especially for $x = e$, where e is the unit element of L, we have $et = (ey)z$, which means $t = yz$. Therefore $x(yz) = (xy)z$ for any three elements x, y, z of L and L is a group.

The group of strongly colour-preserving automorphisms of $C(H)$ is well-known (see [2]); it is isomorphic to H itself. In [2] also from a Cayley colour graph of a group its Frucht graph is derived by substituting the edges by suitable graphs, edges of the same colour being substituted by the same graphs. To strongly colour-preserving automorphisms of the Cayley colour graph there correspond automorphisms (without further conditions) of the Frucht graph. The reader who knows the last chapter of [2] can easily make himself these considerations for quasigroups as well.

We have proved some theorems on quasigroups and factorisations of complete digraphs. These results are no surprising discoveries, but they show the interrelations between quasigroups and these factorisations, which, maybe, could be useful in a further investigation of these topics.

Finally we shall give an example of the Cayley colour graph of a quasigroup. Let Q be the quasigroup whose Cayley table is in [1], p. 13.

	a	b	c	d	e
a	c	a	d	b	e
b	e	b	c	a	d
c	a	d	b	e	c
d	d	e	a	c	b
e	b	c	e	d	a

The factors of the Cayley colour graph of this quasigroup Q are shown in Fig. 1.

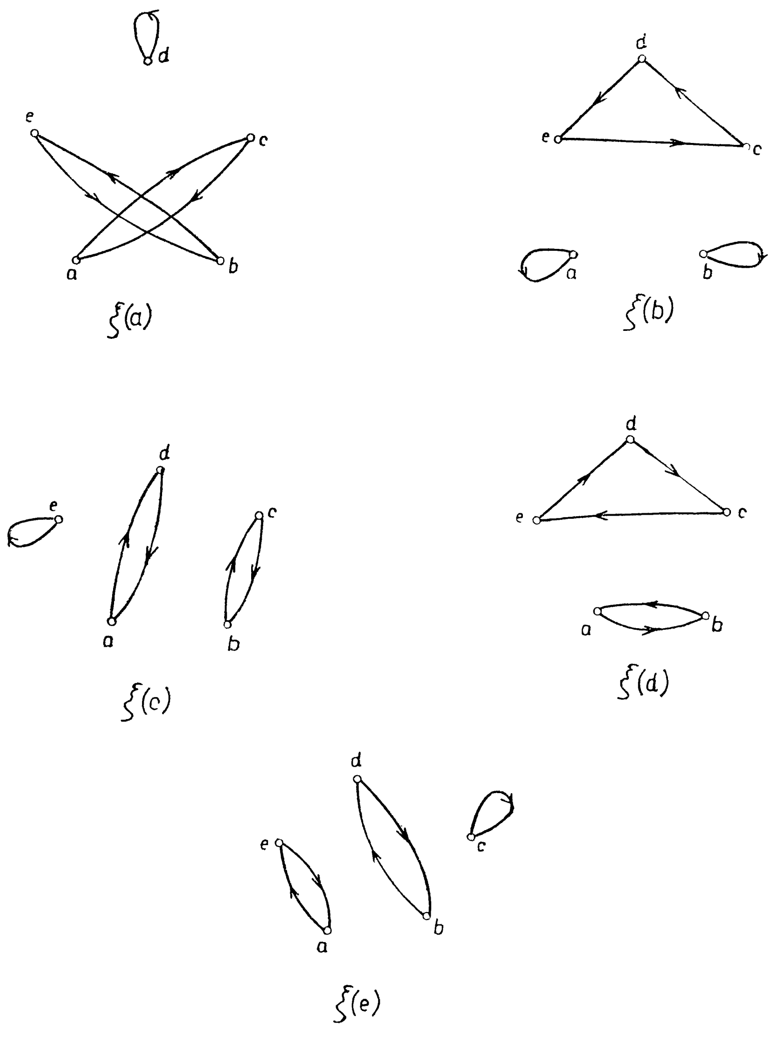

Fig. 1.

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