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# EXPLICIT SOLUTIONS OF SEVERAL KUMMER'S NONLINEAR DIFFERENTIAL EQUATIONS

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Dedicated to Professor Otakar BORŮVKA on the occasion of his 75th birthday

In this paper we shall deduce relations among the elliptic, hyperbolic and parabolic phases of the linear differential equation

$$(q) \qquad y'' = q(t)y, \quad (y' = \mathrm{d}y/\mathrm{d}t)$$

the carrier $q$ of which belongs to the class $C_0(j)$, where $j = (a, b)$ is an open interval, $-\infty \leqq a < b \leqq \infty$.

Note that the symbol $C_k(M)$, where $k$ is a nonnegative integer, denotes the set of all real functions with the continuous derivates of the $k$-th order.

*A function $\alpha_e$, resp. $\alpha_h$, resp. $\alpha_p$, defined in the interval $j_e \subset j$, resp. $j_h \subset j$, resp. $j_p \subset j$ is called an elliptic, resp. hyperbolic, resp. parabolic phase of the differential equation (q) if*

1. *it belongs to the class $C_3$ in a corresponding interval of the definition,*
2. *it has the 1st derivative $\neq 0$ for all points from its interval of definition,*
3. *it satisfies the differential equation*

$$-\{\alpha_e, t\} - \alpha_e'^2 = q_e(t), \quad t \in j_e, \ resp.$$

$$-\{\alpha_h, t\} + \alpha_h'^2 = q_h(t), \quad t \in j_h, \ resp.$$

$$-\{\alpha_p, t\} = q_p(t), \qquad t \in j_p,$$

*where the function $q_e$, resp. $q_h$, resp. $q_p$ is a partial function from $q$ defined in the interval $j_e$, resp. $j_h$, resp. $j_p$.*

A symbol $\{X, t\} = (X''/2X')' - (X''/2X')^2$ denotes the *Schwarz derivative* of the function $X \in C_3(j_0 \subset j)$ in a point $t \in j_0$ such that $X'(t) \neq 0$.

The importance of the introduced kinds of phases of (q) is evident from the transformation of the solutions of the differential equation

$$(Q) \qquad \ddot{Y} = Q(T)Y, \quad (\dot{Y} = \mathrm{d}Y/\mathrm{d}T)$$

$Q \in C_0(J = (A, B))$, into solutions of (q), which transformation is realized by solutions $X$ of the nonlinear differential equation of the 3rd order

$(Q, q)$ $$-\{X, t\} + Q(X)X'^2 = q(t),$$

called *Kummer's differential equation.* A relation between solutions of $(Q)$ and $(q)$ is such that if a function $X$ is a solution of $(Q, q)$ defined in the interval $i \subset j$, $X'(t) \neq 0$ for all $t \in i$, and $Y$ is any solution of $(Q)$ defined in the interval $I = X(i) \subset J$, then the composed function

$$y(t) = Y[X(t)]/\sqrt{|X'(t)|}$$

is a solution of $(q)$ defined in the interval $i$. The function $X$ is called a *kernel of the transformation* of solutions of $(Q)$ into solutions of $(q)$.

From the point of view of the transformations the elliptic, hyperbolic and parabolic phases, respectively, are kernels of the transformations of solutions of $(Q)$ with constant carriers $-1$, $1$, $0$, respectively, into solutions of $(q)$

Further we shall need the following result from the theory of the transformation of solutions of Kummer's differential equation $(Q, q)$ [2, III., § 23.1., pg. 186]:

*If a function $X(t)$, $t \in i$, is a solution of $(Q, q)$, then the function $x(T)$, $T \in I$ inverse to $X$ is a solution of Kummer's differential equation*

$(q, Q)$ $$-\{x, T\} + q(x)\dot{x}^2 = Q(T).$$

*Kummer's differential equation* $(-1, q)$ is studied in detail in the book [2], where an explicit expression of Kummer's differential equation $(-1, -1)$ is given in the form:

$$X(t) = \operatorname{arctg} [(c_{11} \operatorname{tg} t + c_{12})/(c_{21} \operatorname{tg} t + c_{22})],$$

where $c_{ik}$, $i, k = 1, 2$, are arbitrary constants and the determinant $|c_{ik}| \neq 0$

Similarly in paper [4] *Kummer's differential equation* $(1, q)$ is investigated and solutions of Kummer's differential equation $(1, 1)$ are given in the form

$$X(t) = \operatorname{argtgh} [(c_{11} \operatorname{tgh} t + c_{12})/(c_{21} \operatorname{tgh} t + c_{22})],$$

where constants $c_{ik}$, $i, k = 1, 2$ have the same meaning as above.

From [2, I., § 1,8., pg. 5] it is evident that *Kummer's differential equation* $(0,0)$ has a solution given by the formula:

$$X(t) = (c_{11}t + c_{12})/(c_{21}t + c_{22}),$$

where constants $c_{ik}$, $i, k = 1, 2$, have the foregoing meaning.

In paper [5] a connection between parabolic and hyperbolic phases of $(q)$ is investigated. We shall need a result of Theorem 2 from [5]; this is given in the following lemma.

**Lemma.** *Let $\alpha_p$ be a parabolic phase of $(q)$ defined in the interval $j_p$. Then the function*

(1) $$\alpha_h = (\lambda/2) \log |(c_{11}\alpha_p + c_{12})/(c_{21}\alpha_p + c_{22})|,$$

(*where* $\lambda = \pm 1$ *and* $c_{ik}$, $i, k = 1, 2$, *are arbitrary constants*, $|c_{ik}| \neq 0$) *defined in the interval* $j_h \subset j_p$, *is a hyperbolic phase of* $(q)$

Especially if $\alpha_p \equiv t$, we obtain from (1) an explicit solution of *Kummer's differential equation* (1,0) in the form:

$$X(t) = (\lambda/2) \log |(c_{11}t + c_{12})/(c_{21}t + c_{22})|,$$

from which, calculating an inverse function, we get a solution of *Kummer's differential equation* (0,1):

$$X(t) = [c_{11} \exp (2\lambda t) + c_{12}]/[c_{21} \exp (2\lambda t) + c_{22}],$$

where the meaning of the constants $\lambda, c_{ik}$, $i, k = 1, 2$ is similar to that in lemma.

Now we are going to investigate a connection between elliptic and hyperbolic, resp. parabolic phases of $(q)$ and give — as corollary — explicit solutions of Kummer's differential equations obtained taking variations of the second class with a repetition of constant carriers: $-1, 1, 0$, of differential equations $(q)$, $(Q)$. Such results have not yet been published.

**Theorem 1.** *Let* $\alpha_e$ *be an elliptic phase of* $(q)$ *defined in the interval* $j_e$. *Then the function*:

(2) $$\alpha_h = (\lambda/2) \log |(c_{11} \operatorname{tg} \alpha_e + c_{12})/(c_{21} \operatorname{tg} \alpha_e + c_{22})|,$$

($\lambda = \pm 1$, $c_{ik}$, $i, k = 1, 2$, *are arbitrary constants*, $|c_{ik}| \neq 0$) *defined in the interval* $j_h \subset j_e$, *is a hyperbolic phase of* $(q)$.

Proof. One can easily see that if the function $\alpha_e$ is an elliptic phase of $(q)$ in the interval $j_e$, then the function $\alpha$ defined by relation:

$$\operatorname{tg} \alpha = (c_{11} \operatorname{tg} \alpha_e + c_{12})/(c_{21} \operatorname{tg} \alpha_e + c_{22}),$$

($c_{ik}$, $i, k = 1, 2$, have the above-mentioned meaning) is also an elliptic phase of the same differential equation $(q)$ and is defined in an interval $j_1 \subset j_e$. Take the function:

$$\alpha_h = (\lambda/2) \log |\operatorname{tg} \alpha|, \quad \lambda = \pm 1,$$

defined in the interval $j_h \subset j_1$ and its derivative:

$$\alpha_h' = \lambda\alpha'/\sin 2\alpha, \ \alpha_h'' = (\lambda\alpha''/\sin 2\alpha) - (2\lambda\alpha'^2 \operatorname{cotg} 2\alpha)/\sin 2\alpha,$$

from which we obtain

$$(\alpha_h''/2\alpha_h')' = (\alpha''/2\alpha')' - (\alpha' \operatorname{cotg} 2\alpha)',$$

$$(\alpha_h''/2\alpha_h')^2 = (\alpha''/2\alpha')^2 - \alpha'' \operatorname{cotg} 2\alpha + \alpha'^2 \operatorname{cotg}^2 2\alpha,$$

$$\alpha_h'^2 \quad \alpha'^2 \sin^2 2\alpha .$$

hence for all $t \in j_h$ we have

$$-\{\alpha_h, t\} + \alpha_h'^2 - \quad \{\alpha, t\} \quad \alpha'^2$$

and the theorem is proved.

**Corollary.** *Kummer's differential equation* (1, —1), *resp.* ( 1, 1) *has a solution*:

$$X(t) \quad (\lambda\ 2) \log\ (c_{11} \operatorname{tg} t + c_{12})\ (c_{21} \operatorname{tg} t + c_{22})\ ,$$

*resp.*

$$X(t) \quad \operatorname{arctg} ([c_{11} \exp (2\lambda t) \quad c_{12}] / [c_{21} \exp (2\lambda t) - c_{22}]),$$

*where the constants* $\lambda$, $c_{ik}$, $i, k$ 1, 2, *have the same meaning as in Theorem* 1

Proof. We obtain an explicit expression of a solution of Kummer's differential equation (1, 1) from the relation (2) by a special choice of the elliptic phase, namely $\alpha_e \equiv t$ and a solution of Kummer's differential equation ( 1, 1 as the inverse function to the solution of (1, 1).

**Theorem 2.** *Let* $\alpha_e$ *be an elliptic phase of* ($q$) *defined in the interval* $j$ . *Then the function*:

$$\text{(3)} \qquad \alpha_p \quad (c_{11} \operatorname{tg} \alpha_e + c_{12})\ (c_{21} \operatorname{tg} \alpha_e + c_{22}) .$$

($c_{ik}$, $i$, $k$ 1, 2, *are arbitrary constants*, $c_{ik}$ 0) *defined in an interval* $j_p \subset j$ *is a parabolic phase of* ($q$).

Proof. We can obtain assertion of the theorem either in such a way that we verify calculating the validity of the relation:

$$\{\alpha_p, t\} = -\{(c_{11} \operatorname{tg} \alpha_e + c_{12})/(c_{21} \operatorname{tg} \alpha_e + c_{22}), t\} \quad -\{\alpha_e, t\} \quad \alpha_e'^2$$

for all $t \in j_p$, or we could obtain (3) by comparing the relations (1) and (2)

**Corollary.** *Kummer's differential equation* (0, 1), *resp.* ( 1, 0) *has a solution*:

$$X(t) \quad (c_{11} \operatorname{tg} t \quad c_{12})\ (c_{21} \operatorname{tg} t \quad c_{22}) .$$

*resp.*

$$X(t) \quad \operatorname{arctg} [(c_{11}t + c_{12})/(c_{21}t + c_{22})] .$$

*where* $c_{ik}$, $i$, $k$ — 1, 2, *are arbitrary constants with the property* $c_{ik} \neq 0$.

The idea of the proof is the same as in the proof of the corollary of Theorem 1

Remark. From the relation (2), resp. (3) we can find an expression of the elliptic phase by means of the hyperbolic, resp. the parabolic phase. From the explicit connections among the mentioned phases we can see (similarly as in paper [1]) the priority of the elliptic phases, the theory of which was systematically built up by O. Borůvka in [2], first of all for their universality. The hyperbolic, resp. parabolic phases are convenient for the study of disconjugacy of $(q)$, especially in the investigation of the pure, resp. special disconjugate differential equation $(q)$. These phases are also convenient for the study of the asymptotic properties of the nonoscillatory differential equation (see e.g. [3], resp. [5]).

Results of this paper can be used to construct Kummer's differential equations of the type $(-1, q)$, $(1, q)$, $(0, q)$ with an explicitely given solution and also Kummer's equations $(q, -1)$, $(q, 1)$, $(q, 0)$. We shall give an example.

Example. By direct calculation we verify easily that the function:

(4) $$X(t) = (c_{11}t^\beta + c_{12})/(c_{21}t^\beta + c_{22}).$$

($c_{ik}$, $i, k = 1, 2$, are arbitrary constants, $|c_{ik}| \neq 0$, $\beta \neq 0$ real number) satisfies Kummer's differential equation $(0, (\beta^2 - 1)/4t^2)$, i.e. it is a parabolic phase of the differential equation:

(5) $$y'' = [(\beta^2 - 1)/4t^2]y$$

By using relation (1) we get from the relation (4)

(6) $$X(t) = (\lambda/2) \log (c_{11}t^\beta + c_{12})/(c_{21}t^\beta + c_{22}), \quad \lambda = +1,$$

and this is a solution of Kummer's differential equation $(1, (\beta^2 - 1)/4t^2)$, i.e. a hyperbolic phase of (5). From the relation (3) and from (4) we get the function $X$ in the form:

(7) $$X(t) = \operatorname{arctg} [(c_{11}t^\beta + c_{12})/(c_{21}t^\beta + c_{22})].$$

and this is a solution of Kummer's differential equation $(-1, (\beta^2 - 1)/4t^2)$ i.e. an elliptic phase of (5).

The inverse functions to the functions (4), resp. (6), resp. (7) are solutions of Kummer's differential equations: $((\beta^2 - 1)/4t^2, 0)$, resp. $((\beta^2 - 1)/4t^2, 1)$, resp. $((\beta^2 - 1)/4t^2, -1)$.

Finally we remark that if we choose in this example $\beta = \pm 1$, we obtain the above mentioned solution of Kummer's differential equation: $(0, 0)$, $(0, 1)$, $(0, -1)$, $(1, 0)$, $(-1, 0)$.

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