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DECOMPOSITION OF THE COMPLETE DIRECTED GRAPH INTO TWO FACTORS WITH GIVEN DIAMETERS

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The subject mentioned in the title was studied for nondirected graphs in paper [2]. We prove a theorem concerning the "directed case".

We shall study directed graphs without loops and multiple edges (two edges ab and ba are allowed). All terms are used in the usual sense (cf. [3]).

Theorem. *Let $n \geqq 2$ be a cardinal number and d_1, d_2 natural numbers or symbols ∞. Then we have: if the complete directed graph with n vertices (we denote it by $\langle\langle n \rangle\rangle$) can be decomposed into two factors with given diameters d_1 and d_2, then for any cardinal number $N > n$ the complete directed graph $\langle\langle N \rangle\rangle$ can be also decomposed into two factors with diameters d_1 and d_2.*

Proof. If one of the diameters is 1 then the proof is evident. If $d_1 = d_2 = 2$ then our assertion easily follows from [2] and from the facts that $\langle\langle 3 \rangle\rangle$ and $\langle\langle 4 \rangle\rangle$ can be decomposed into two factors with diameter 2. Hence we can suppose that

$$d_1 \geqq 3 \quad \text{and} \quad d_2 \geqq 2.$$

Let us denote $G = \langle\langle N \rangle\rangle$. By the symbol F denote some complete directed subgraph of G with n vertices, i.e. $F = \langle\langle n \rangle\rangle$. Let us denote the vertex set of F by A and the set of all remaining vertices of G by B. Choose any vertex from A and denote it by v.

Let us suppose that the graph F is decomposable into two factors with diameters d_i $(i = 1, 2)$; denote these factors by F_i. Decompose G into two factors G_i $(i = 1, 2)$ as follows:

(a) G_i contains all the edges of F_i;

(b) if $v \neq u_1 \in A$, $u_2 \in B$, then the edge $u_1u_2(u_2u_1)$ belongs to G_i if and only if the edge $u_1v(vu_1)$ belongs to F_i;

(c) all the edges of the complete directed subgraph whose vertex set is $\{v\} \cup B$ belong to G_k, where $k \in \{1, 2\}$ will be specified later.

By similar considerations as in the proof of Theorem 1 in [2] we can prove that the distance of two vertices from A is the same in G_i as in F_i $(i = 1, 2)$; hence $d(G_i) \geqq d(F_i)$ for any i. Further, the distance $\varrho_{G_i}(r, s)$ $[\varrho_{G_i}(s, r)]$ of two vertices $s \in A$, $r \in B$ $(s \neq v)$ in G_i is the same as the distance $\varrho_{F_i}(v, s)$ $[\varrho_{F_i}(s, v)]$. To prove our theorem it is sufficient to show that the index $k \in \{1, 2\}$ can be chosen so that for any $x, y \in \{v\} \cup B$, $x \neq y$ we have

$$(+) \qquad \varrho_{G_1}(x, y) \leqq d_1, \quad \varrho_{G_2}(x, y) \leqq d_2 .$$

If $d_1 = \infty$, then we put $k = 2$, if $d_1 < \infty$ but $d_2 = \infty$, then we put $k = 1$ and the inequalities (+) obviously hold.

Let us suppose that $d_1, d_2 < \infty$. Denote by $C_i(D_i)$ the set of all vertices z for which the edge $vz(zv)$ belongs to F_i $(i = 1, 2)$. Then at least one of the following conditions holds:

1° $C_1 \cap D_1$ is non-empty and contains at least one vertex w;

2° $C_2 \cap D_2$ is non-empty and contains at least one vertex w;

3° $C_1 = D_2$, $D_1 = C_2$, $\{v\} \cup C_1 \cup D_1 = A$ and there exists in F_1 at least one edge mn directed from C_1 to D_1.

In the case 1° (2°) we put $k = 2$ $(k = 1)$. Then the edges xw and wy belong to $F_1(F_2)$ and hence $\varrho_{G_1}(x, y) \leqq 2[\varrho_{G_2}(x, y) \leqq 2]$. Obviously $\varrho_{G_2}(x, y) = 1[\varrho_{G_1}(x, y) = 1]$.

In the case 3° we put $k = 2$ (therefore $\varrho_{G_2}(x, y) = 1$). Then the edges xm, mn, ny belong to G_1 and hence $\varrho_{G_1}(x, y) \leqq 3$; q.e.d.

In paper [2] we proved an analogical theorem for the decomposition of a non-directed complete graph into an arbitrary number of factors. In the directed case this theorem cannot be generalized in the above mentioned way, which is evident from the following considerations.

Lemma. *Let d be a natural number. Then any directed graph of diameter d with $d + 2$ vertices contains at least $d + 4$ edges.*

Proof. Let us suppose that there exists a graph G of diameter d with $d + 2$ vertices containing less than $d + 4$ edges. Denote the vertices of G by $v_1, \ldots, v_{d+2}$. Suppose that $\varrho_G(v_1, v_{d+1}) = d$ and that $v_1 \ldots v_{d+1}$ is a path of length d, containing d edges. Hence (because d is finite), the vertex v_{d+2} is of degree 2 or 3. Let us suppose that only one edge enters into v_{d+2} (in the remaining case we could consider similarly). Obviously there exists an edge e coming out from v_{d+1}. The edge e cannot enter into v_{d+2}, because then we should have $\varrho_G(v_1, v_{d+2}) = d + 1$. Hence e enters into one of the vertices $v_1, \ldots, v_d$ and v_{d+2} must be of degree 2 (i. e. exactly one edge comes out from v_{d+2}). Into the vertex v_1 there enters at least one edge e'. The only edge coming out from v_{d+2} cannot enter into v_1, because then we should have $\varrho_G(v_{d+2}, v_{d+1}) = d + 1$. Therefore e' comes out from one of the vertices $v_2, \ldots, v_{d+1}$. Since v_{d+2} is of degree 2, we get $e = e'$. However, it is easy to verify that the graph drawn in Fig. 1 is of a diameter $> d$; q.e.d.

From our lemma it follows that the graph $\langle\langle d + 2 \rangle\rangle$ cannot be decomposed into d factors of diameter d for any $d \geq 3$. [Proof: the number of edges of all d factors of diameter d is at least $d(d + 4) > (d + 2)(d + 1)$]. However, it is easy to prove that for any even d the graph $\langle\langle d + 1\rangle\rangle$ can be decomposed

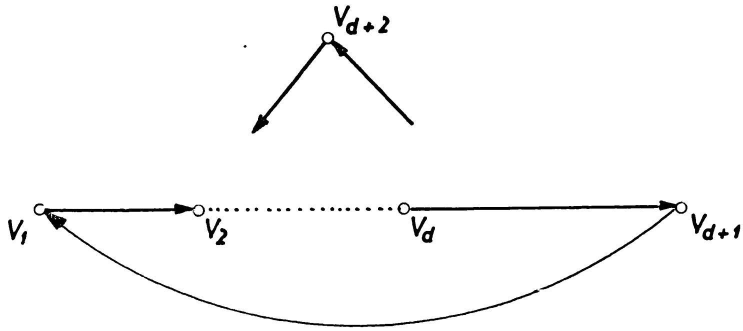

Fig. 1.

into d factors of diameter d (Hamiltonian cycles) (for the undirected case see [1], p. 188). Hence our theorem cannot be generalised for the number d of factors, where $d \geqq 4$ is even. Probably it holds only for decomposition into two and (possibly) three factors (but we cannot prove this conjecture).

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