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Representations of Topological Semigroups

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Weyl for compact groups, the existence of a complete system of characters on a locally compact Abelian group, and the Pontrjagin duality theorem. For discrete semigroups, a comprehensive survey of known results in these directions may be found in [15]. For arbitrary locally compact semigroups, generalizations have been thwarted by the lack of an analogue to the Haar integral on locally compact groups, a fact which has been proved and reproved sufficiently often in the literature to be now regarded as folklore.

It seems clear that the techniques employed in group representations may be applied to a given semigroup only if the structure of the semigroup is determined largely by some group. We describe some results which lend credence to this statement. A convex subset S of a locally convex topological linear space X is an *affine topological semigroup* if S is a topological semigroup in the relative topology inherited from X and the left and right translation functions of S into itself induced by each element of S are affine functions.

In case the set of extreme points of S forms a group, we refer to S as a *group-extremal* affine semigroup. Examples include the complex unit disk and the convolution semigroup of probability measures on a compact topological group. For further results in this area, see [17], [50], and [51].

Now, if S is a compact group-extremal semigroup, then S is the closed convex hull of $H(1)$, so that the structure of S is intimately connected to that of $H(1)$. In fact, we have

Theorem [**22**]: *A compact group-extremal affine semigroup has a complete system of affine representations by real matrices.* It would be of considerable interest to know other classes of semigroups whose structure is so largely determined by $H(1)$.

Other results on linear representations of topological semigroups are few and far between. At this point, we mention only one additional.

Theorem [**9**]: *A compact simple semigroup of finite inductive dimension in which each maximal group is a Lie group and in which the idempotents form a semigroup has a faithful representation as (hence is iseomorphic to) a semigroup of real matrices.* The proof of this result, it should be noted, depends more on the Rees-Wallace theorem [30] than on the techniques of group representations.

It is well known that the circle group does not play the same role in semigroup theory that it does in group theory. The first attempts at modifications turned naturally to the complex unit disk. In [52], Schwarz initiated the study of the relation between the structure of a compact commutative semigroup and the continuous homomorphisms of it into the disk. Such maps he called characters, but more recent works have used the term *semicharacters*. The class of semigroups which admit a complete system of semicharacters —

indeed, which admit any non-trivial semicharacters — has not yet been satisfactorily described. It is known to include the following types of semigroups:

1. Compact commutative group-extremal affine semigroups [22];
2. Compact commutative uniquely divisible semigroups of finite inductive dimension, having totally disconnected idempotent set [10] (see section 3);
3. Locally compact commutative continuous inverse semigroups having an identity and totally disconnected idempotent set [4].

Denote by D the complex unit disk semigroup, and by S^* the set Hom (S, D) with the constant zero map deleted. Clearly, point-wise multiplication of members of Hom (S, D) makes this set into an algebraic semigroup; the subset S^* need not be a subsemigroup. However, if it is, then it is a topological semigroup in the compact-open topology. The evaluation mapping is a continuous homomorphism of S into S^{**}; if this map is, in fact, an onto iseomorphism, then *duality holds for S*. Using methods of Austin [3], Shneperman [44] has proved

Theorem: *Let S be a compact commutative semigroup. Then duality holds for S if and only if S is a Clifford semigroup with identity and totally disconnected idempotent set.*

2. Representations of Lattices and Semilattices. *A topological lattice* is a topological space which is also a lattice in which both cup and cap operations are jointly continuous. *A topological* (lower) *semilattice* is, similarly, a space which has defined a jointly continuous „cap" operation under which it becomes a lower semilattice. Equivalently, a topological semilattice is a commutative idempotent topological semigroup.

The cannonical example of a compact topological lattice is the real interval [0,1] with $a \wedge b = \min \{a, b\}$, $a \vee b = \max \{a, b\}$. In the sequel, we shall refer to this example as I, whether considered as a lattice or as a lower semilattice.

In [20], Dyer and Shields posed the following problem: if L is a compact, connected, distributive topological lattice, does Hom (L, I) separate points· Here, of course, members of Hom (L, I) must be continuous *lattice* homomorphisms, preserving both cup and cap operations. If L is also totally disconnected, then a well known result of Numakura [43] states that L is the inverse limit of finite lattices and hence Hom (L, I) certainly separates points. This much having been done, attention naturally turned to the antipodal case, in which L is connected.

Definition: *A lattice L has breadth less than or equal to n provided for every finite subset M of L, there is a subset $N \subseteq M$,* card $N \leqq n$, *such that* $\bigwedge M = \bigwedge N$, *where $\bigwedge M$ is the cap product of all elements of M. A lattice has breadth n if it has breadth $\leqq n$ and it does not have breadth $\leqq n-1$.* Since breadth depends only on the cap operation, it is clear that the same notion exists for semilattices. In [1], Anderson proved that if L is a compact connected distributive lattice with finite breadth, then $\mathrm{Hom}(L, I)$ separates points. Having already established [2] the result that the codimension of such a lattice bounds its breadth above, Anderson was thus able to report solution of the Dyer-Shields problem for the subclass of finite dimensional continua, and was even able to weaken the compactness to local compactness in the hypothesis.

Up to this point, semilattice representations had not been studied separately to any significant degree, primarily because of the following.

Theorem: *Every compact topological semilattice S is iseomorphically embeddable* (as a lower semilattice) *in a compact connected distributive lattice L.* This theorem is of the folk variety, although an outline of the construction is presented in [40]. In brief, S is first connectified by coneing it with the semilattice I. The lattice L is then the space of closed semilattice ideals of this connected semilattice, with cap and cup being intersection and union, respectively, and with the Vietoris topology on L. Unfortunately, the lattice L is very rarely finite dimensional, and, as Anderson's results defied generalization (with good reason — see below), attention was, at length turned to the representation theory of topological semilattices.

Definition: (Lawson [39]) *A topological semilattice S has small semilattices at p if, for any open set U containing p, there exists a subsemilattice K of S such that $p \in K^0 \subseteq K \subseteq U$. If S has small semilattices at every point, then S has small semilattices.*

This property is both hereditary and productive; also, if S is compact, then any continuous homomorphic image of S will have small semilattices if S does [39]. In particular, I, and thus any subsemilattice of an arbitrary Cartesian product of copies of I, has small semilattices. If follows immediately that if S is a compact topological semilattice and Hom (S, I) separates points, then S has small semilattices. The very surprising fact that the converse holds is a direct consequence of the following

Theorem: (Lawson [39]) *Let S be a locally compact topological semilattice having small semilattices. Let A be a closed ideal of S and let $b \in S \backslash A$. Then there exists $f \in \mathrm{Hom}(S, I)$ such that $f(A) = 0$ and $f(b) = 1$. Indeed, by a suitable modification of the above remarks,* Lawson *has proved* [36] *that if L is a compact distributive topological lattice, then* $\mathrm{Hom}(L, I)$ *separates points if and only if L has small semilattices under each of its operations.*

The „small semilattices" condition gives rise to a sequence of elegant theorems, all due to Lawson [39]. We join these into one

Theorem: *Let S be a locally compact topological semilattice. If S has any of the following properties, then S has small semilattices, and hence* $\operatorname{Hom}(S, I)$ *separates points*; (1) *S is totally disconnected*; (2) *S is locally connected, codimension $S \leqq n$*; (3) *breadth $S \leqq n$*. For a discussion of codimension, see [16].

Conditions (2) and (3) are distinct here, since there exist finite semilattices of arbitrarily large breadth. The coneing technique used above (which is unavailable for lattice constructions) permits the construction of semilattices on one dimensional continua having arbitrarily large finite breadth.

There is one final result worthy of note in this direction, again due to Lawson [37]. Recall that a set $B \subseteq S$ is *order dense* if $x, y \in B$, $x < y$, implies there exists $z \in B$ such that $x < z < y$.

Theorem: *If S is a compact topological semilattice of finite codimension, and if each element of S has an order dense neighborhood, then S has finite breadth.*

Contemporal with these results are the papers of Davies [18] and Strauss [46], which contribute more affirmative results in the case that S is a lattice with various extra conditions on it. We mention in passing also the papers of Clark and Eberhart [14] and Stralka [45].

In spite of the preponderence of evidence in the favorable direction, examples have recently been discovered — predictably, by Lawson [40] — which show that the answer to the Dyer-Shields problem is, in general, no.

Theorem: *There exists a one dimensional continuum topological semilattice S and an infinite dimensional continuum distributive lattice L such that* $\operatorname{Hom}(S, I)$ and $\operatorname{Hom}(L, I)$ *consist only of trivial maps.*

The details of these examples are too lengthy to be set forth here. The underlying space of S is a subcontinuum of a Cantor fan; L is the lattice of closed ideals of S.

The remaining problem of precisely what *topological* conditions on a semilattice S are necessary and sufficient for $\operatorname{Hom}(S, I)$ to separate points is of interest to us with regard to the material in the next section.

3. Representations of Divisible Semigroups. A semigroup S is *(uniquely) divisible* if for each $x \in S$ and for each positive integer n, there exists a (unique) $y \in S$ such that $y^n = x$. Divisible groups have been studied extensively; an excellent account of the Abelian family is given in [33]. The general case has been examined in depth by Baumslag in several papers [5—8]. The algebraic theory of divisible semigroups suffers by comparison, and we cite only three references germane to the area [12, 23, 47].

There is a characterization of uniquely divisible, commutative (UDC) semigroups [12], which has motivated much of the topological work done in this direction. If V is a rational (or real) vector space, then a *cone* K of V is a non-empty additive subsemigroup of V closed under multiplication by positive scalars and disjoint from its image under multiplication by negative scalars. Note that, in contrast with some other references, we do not require that the identity of V belong to K.

Theorem A: *Any* UDC *semigroup is a semilattice of subsemigroups, each of which is isomorphic to the direct sum of a rational vector space and a cone of a rational vector space.*

In the theory of compact topological groups, it is well known that connectivity and divisibility are equivalent [42]. The reliance on character theory used in establishing this result is, unfortunately, incidental to our version of representation theory. The theorem itself, however, is indicative of the wealth of the category of compact divisible semigroups.

Divisibility has often invaded the classical study of topological semigroups, although usually as an anonymous property. It is particularly helpful in the classification of threads [21, 41]. Indeed, any algebraically irreducible clan (irreducible hormos) is easily seen to be divisible. Other works utilizing divisibility explicitly include fundamental papers of Anne Hudson [31], Hofmann [29], and Hildebrant [24—28]. See also Day [19] and Keimel [34]. Most of these references have influenced our investigations, and some, our results; however, since we treat this subject at length elsewhere [53], we reduce further exposition on their content to the minimum needed to state our theorems.

Representations are achieved by first characterizing special members of the class to be represented; we adhere to this pedagogical rule of thumb and immediately restrict our attention to compact UDC semigroups. Since this class still includes all compact semilattices, for which ,,ordinary" semicharacters are clearly insufficient, we specialize further to semigroups having exactly two idempotents, an identity and a zero. In this case, $S\backslash\{0\}$ is easily seen to be a subsemigroup of S satisfying the cancellation law. The algebraic structure of $S\backslash\{0\}$ if $H(1) = \{1\}$, must therefore by Theorem A be that of a cone in a rational vector space. This removes any element of surprise from the conclusion of

Theorem B [10]: *Let S be a compact* UDC *semigroup having finite inductive dimension, $E(S) = \{0, 1\}$, and $H(1) = \{1\}$. Then S is iseomorphic to the one point compactification of a closed cone of a finite dimensional real vector space, in which the ideal point acts as zero.* In particular, S must topologically be an n-cell. The additional restriction of finite dimensionality is necessary to permit usage of a concept due to C. E. Clark [13]. In a commutative semigroup S, let $\{T_i\}$, $i = 1, \ldots, n$, be a finite family of subsemigroups, each iseomorphic to U, the interval $[0, 1]$ under real multiplication. Suppose the $T_{i'}$ s have a common identity, $\{e\}$ and a common zero, $\{z\}$. The family $\{T_i\}$ is *algebraically independent* provided that, for *any* decomposition of $\{1, \ldots, n\}$ into disjoint, non-empty sets A and B, the intersection of the subsemigroups generated by $\{T_i\}$, $i \in A$ and $\{T_i\}$, $i \in B$, is precisely $\{e, z\}$.

Next by some careful manuevering, we are able to remove one undesired condition:

Theorem C [**10**]: *Let S be as in Theorem B, without the requirement that $H(1)$ be trivial. Then S is iseomorphic to the Rees quotient of $S/H(1) \times H(1)$ modulo its minimal ideal.* Here $S/H(1)$ is the orbit space of $H(1)$ acting on S by translation. With some finagling, $S/H(1)$ may be shown to be finite dimensional; it clearly has all remaining properties required to satisfy the hypotheses of Theorem B. Again, algebraically, the subsemigroup $S \backslash \{0\}$ of Theorem C must be isomorphic to the direct sum of a rational vector space and a cone, „predicting“ Theorem C.

We are now in a position to state a representation theorem:

Theorem D [**10**]: *Let S be a compact* UDC *semigroup having finite inductive dimension. Let $E(S)$ be totally disconnected. Then S has enough „ordinary“ semicharacters to separate points.* The proof of D involves the fundamental theorem of Koch [35] on compact partially ordered spaces having no local minima, Theorem C, and the pleasant properties of commutativity. Theorem D may be varied to permit separating S by homomorphisms into a UDC semigroup formed by the Rees quotient of the Cartesian product of U with the character group of the discrete additive rational numbers modulo its minimal ideal.

Theorem D is „best possible“ with respect to the idempotent condition on S, since the complex disk has only two idempotents. Because the idempotents of a compact UDC semigroup form a compact semilattice, a range space containing a copy of I would permit usage of the theorems of section 2. Such an example is readily available. Let T be the Rees quotient of $U \times I$ modulo the ideal of elements having a least one zero coordinate. Ursell [48] first noted that T has the pathological property of possessing no one dimensional continuous homomorphic images. It is easily seen that any non-trivial *uniquely* divisible continuous homomorphic image of T is iseomorphic to T. It follows that any candidate for a range space for a generalized notion of semicharacters on compact divisible semigroups must contain a copy of T. Of course, T has trivial groups, but this deficiency is easily remedied.

Specifically, let C be the circle group and let B be the Rees quotient of $U \times I \times C$ modulo the ideal of elements having a zero in at least one of its U or I coordinates. Note that B contains both a copy of T and a copy of the complex disk. Our ultimate theorem on representations of UDC semigroups is

Theorem E [**11**]: *Let S be a compact* UDC *semigroup having finite inductive dimension. Suppose* $\mathrm{Hom}(E(S), I)$ *separates points. Then* $\mathrm{Hom}(S, B)$ *separates points*. The proof of this result is quite involved, and we refer the reader to [11] for the details. As in Theorem D, the circle may be replaced in Theorem E by the character group of the discrete additive rational numbers to obtain a similar result.

There is a direct relationship between arbitrary divisible commutative semigroups and UDC semigroups. Indeed, Hildebrant [26] has shown:

Theorem: *If S is a compact divisible commutative semigroup, then there is a compact* UDC *semigroup X of the same dimension as S, such that S is the continuous homomorphic image of X and such that* $\mathrm{Hom}(X, S)$ *separates points.* One might hope, with this result and a suitable modification of the semigroup B above, that a representation theory could be established for arbitrary compact divisible commutative semigroups. The following examples show that a finite dimensional analogue of the semigroup B cannot exist for this purpose.

Let T_n be the semigroup obtained by one-point compactifying the non-negative cone of n dimensional real linear space in such a way that the ideal point acts as a zero. Define $F : T_n \to [0, \infty]$ by $F(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i$, $F(\omega) = \infty$, where ω is the ideal point of T_n. Note that, if $[0, \infty]$ is given the semigroup structure induced by real addition, with ∞ acting as a zero element, then F is a continuous homomorphism. Let $J_1 = \{x \in T_n : x = \infty \text{ or } x = (x_1, \ldots, x_n), x_i \geqslant 1 \text{ for some } i\}$; J_1 is a closed ideal in T_n. Set $d_0 = (1/n) + (n - 1)$ and $p_0 = ((n - 1)/n, \ldots, (n - 1)/n)$, and let $J_2 = \{x \in T_n + p_0 : F(x) \geqslant d_0\}$; then J_2 is also a closed ideal of T_n. Finally, define a relation ϱ on T_n by $a\varrho b$ if and only if (1) $a = b$, or (2) $a, b \in J_1 \cup J_2$, or (3) $a, b \in T_n + p_0$ and $F(a) = F(b) < d_0$. The relation ϱ is a closed congruence; denote by S_n the quotient semigroup T_n/ϱ. Let η represent the natural map of T_n onto S_n. Let δ be the point in S_n which is the image under η of the point in T_n, each of whose coordinates is equal to $[2n(n - 1) + 1]/2n^2$.

Example 1: *S_n is a compact divisible commutative semigroup with $E(S_n) = \{0, 1\}$. The semigroup S_n is topologically an n-cell* [13]. *If S' is a compact semigroup, of inductive dimension less than n, and if $f \in \mathrm{Hom}(S_n, S')$, then $f(\delta) = f(0)$. In particular,* $\mathrm{Hom}(S_n, S')$ *does not separate points in this case.*

Example 2: *Let S be the Cartesian product of the family of S_n's constructed above, $n \geqslant 2$. Then, if S' is any finite dimensional semigroup, then* $\mathrm{Hom}(S, S')$ *will not separate points.* The details of these examples will appear elsewhere.

We close this section with a remark on linear representation of compact, non-commutative uniquely divisible semigroups. Recall that, for any idempotent e, the *Core* of e is the set of elements for which e is a two-sided zero. We state an embryonic result.

Theorem F: *Let S be a compact uniquely divisible semigroup such that*
(i) $M(S)$ *is a left trivial idempotent semigroup on an* $(n - 1)$ *cell*;
(ii) $E(S) = M(S) \cup \{1\}$;
(iii) $H(1) = \{1\}$;
(iv) *for each* $e \in M(S)$, *Core* e *is iseomorphic to the semigroup* U;
(v) $S \backslash M(S)$ *is a cancellative subsemigroup of* S. *Then* $S \backslash M(S)$ *is iseomorphically embeddable in a closed subgroup of* $Gl_n(R)$, *the group of invertible* $n \times n$ *real matrices. Moreover, if* $n = 2$, *then this embedding can be extended from all of* S *into* $M_n(R)$, *and* S *is iseomorphic to the semigroup of* 2×2 *matrices*

$$\left\{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : 0 \leqq x, y, x + y \leqq 1\right\}.$$

Of course if $n = 1$, then S is iseomorphic to U. The details of the proof of this theorem will also appear elsewhere. It is interesting to note that, in contrast to the expected situation, it is precisely the violent non-commutativity of such semigroups that permits their characterization in this manner. Much stronger results in this direction appear to be within short reach.

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