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ON A PROBLEM IN MAP COLOURING

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The number of papers dealing with graph colouring is very large. In all of them, the problems of colouring faces (regions), edges or vertices (common name: elements) of graphs and maps on various surfaces are dealt with separately. To our knowledge, except Ringel [3], there has not been studied so far the simultaneous colouring of elements of graphs and maps of different kinds in such a way that mutually incident elements have different colours.

The objects studied in this note will be maps on closed surfaces of an arbitrary Euler characteristic (in the following maps) which have the following properties: The graph of such a map is connected and finite. Every vertex and every face has a degree $\leqq 3$. Two faces have at most one edge in common and every edge is incident at exactly two faces. If two faces have two vertices A, B in common then there exists an edge AB. — Clearly, the vertices, edges and faces of a polyhedron of an arbitrary genus form such a map.

We shall study the problem of the minimal number $\varphi(P)$ of colours necessary for colouring all elements of a map P in such a way that

(a) two adjacent faces have different colours,

(b) two adjacent vertices have different colours,

(c) two edges incident with the same vertex have different colours,

(d) mutually incident elements have different colours.

In our note we give an upper bound for $\varphi(P)$ if P is a regular trivalent and quadrivalent map respectively. We use the well-known concepts of a Hamilton or Euler circuit of P by which is meant a closed path of the graph of P which passes through every vertex or every edge of P respectively exactly once. The linear or quadratic factor of P is a subgraph P^* of the graph P which contains all vertices of P in such a way that every vertex of P^* has in P the degree one or two, respectively.

Theorem 1. *For a regular trivalent map P we have*

$$\varphi(P) \leqq \chi_2(P) + 3$$

where $\chi_2(P)$ is the minimal number of colours necessary for a colouring of faces of P which satisfies condition (a).

Proof. The theorem holds for a map whose graph is K_4, the complete graph with four vertices. Suppose that P is not such a map. By Brooks [1] the vertices of P may be coloured with the colours 1, 2, 3. Let all faces be coloured with the colours 4, 5, ..., $\chi_2(P) + 3$. A known theorem (see Ringel [2]) states that any finite trivalent graph without bridges may be decomposed into a linear and a quadratic factor. We colour first the edges of the components of the quadratic factor in the following way: The component is oriented and the edge $\overrightarrow{AB}$ receives the same colour as that face which is incident with the vertex A but not with the vertex B. Adjacent edges of the quadratic factor have different colours, for they have the colours of two adjacent faces. The edge AM of the linear factor gets that colour from among 1, 2, 3, which is neither the colour of A nor of B. This colouring satisfies all conditions (a), (b), (c) and (d).

Theorem 2. *If for a regular trivalent map P holds $\chi_2(P) = 3$, then $\varphi(P) = 5$.*

Proof. Considerations analogous to those of Theorem 1 show that for colouring faces and edges three colours 1, 2, 3 are sufficient. Every vertex is incident with faces coloured by the colours 1, 2, 3. Moreover, in adjacent vertices the directions of rotation from face with colour 1 through 2 to 3 are opposite. The vertices with one direction of rotation are coloured with the colour 4, the others with 5. Thus five colours are sufficient, and clearly also necessary.

Theorem 3. *For a regular quadrivalent map P we have*

$$\varphi(P) \leqq 10.$$

Proof. The faces of the map P may be coloured with two colours 1, 2. Suppose that the graph of P is not the complete graph K_5. Then it is possible to colour its vertices with four more colours 3, 4, 5 and 6 (by the cited theorem of Brooks [1]). The graph of P has an Euler circuit (see Ringel [2]) with at least one triple of adjacent vertices A, B, C coloured with different colours, for instance A has the colour 3, B the colour 4 and C the colour 5. We colour the edges of the oriented Euler circuit starting with $\overrightarrow{BC}$ which is coloured by 3. Each further edge $XY \neq AB$ is coloured by one of the colours 3, ..., 10 so that neither X nor Y are coloured by that colour and it is not the colour of any edge incident with X which is left out nor the colours of (at most two) edges incident with Y. Thus for colouring any edge at most seven colours from among eight colours 3, ..., 10 are unsuitable. These colours are sufficient also for the colouring of the last edge $\overrightarrow{AB}$ of the Euler circuit as the vertex A and the edge BC have the same colour. The same is applicable to a complete graph with five vertices, where five colours are needed to colour the vertices.

Remarks 1. Trivially, $\varphi(P) \geqq 5$ for any map P. However, a discussion of possible colourings of a tetrahedron shows that in general it is impossible to improve the bound in Theorem 1 by simply reducing the additive constant. Six colours are not enough to colour all vertices, edges and faces of a tetrahedron in accordance with (*a*), (*b*), (*c*), (*d*).

2. The problem of colouring elements of different kinds of a map P in accordance with (*a*), (*b*), (*c*) and (*d*) may be narrowed by considering only the elements of two different kinds. From the proofs of our Theorems 1 and 3 if follows that the faces and edges of a regular trivalent map P or the edges and vertices of a regular quadrivalent map may be coloured by $\chi_2(P) + 1$ or 8 colours respectively. If $\chi_2(P) \geqq 5$ (e. g. in the case of a toroidal polyhedron — see Ringel [2]) for a regular trivalent map P, then the edges and faces of P are colourable with $\chi_2(P)$ colours.

3. After finishing this paper I heard of H. Izbicki's communication at the Colloquium on graph theory in Oberwolfach 1967 mentioning the unpublished work of M. Neuberger and W. Gmeiner which dealt in a more general form with the simultaneous colouring of elements of regular trivalent maps on the surface of a sphere. W. Gmeiner proved that for such a map P holds $\varphi(P) \leqq 7$ independently from the four-colour conjecture. — The results which form the contents of this paper have been obtained by the author independently.

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