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GENERALIZATION OF SOME RESULTS FOR EXACTLY COVERING SYSTEMS

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We call a system of residual classes

$$(1) \qquad a_i (\mathrm{mod}\, n_i), \ 0 \leqq a_i < n_i, \ i = 1, 2, \ldots, k$$

exactly covering if every integer belongs to exactly one of these classes.

In this paper we shall study exactly covering systems with exactly one m-tuple of the residual classes with respect to the same module whereas the remaining modules are distinct. The cases $m = 2, 3$ are investigated in [3] and [4], and there the following results are proved:

Lemma 1. *Let* (1) *be an exactly covering system. Let there in* (1) *appear one couple of the residual classes with respect to the same module and the other ones are distinct. Then* $n_i = 2^i$ *for* $i = 1, 2, \ldots, k - 2$; $n_{k-1} = n_k = 2^{k-1}$. (see [3])

Lemma 2. *Let there in the exactly covering system* (1) *exist one triple of the residual classes with respect to the same module and the other modules are distinct. Then we have* $n_i = 2^i$ *for* $i = 1, 2, \ldots, k - 3$; $n_{k-2} = n_{k-1} = n_k = 3 \,.\, 2^{k-1}$. (see [4])

We shall study the cases $m = 4, 5$, and partly 7.

We can assume in (1) that $n_1 \leqq n_2 \leqq \ldots \leqq n_k$. In [1] it is proved that in every exactly covering system we have $n_{k-1} = n_k$. Hence we can suppose

$$n_1 < n_2 < \ldots < n_{k-m} < n_{k-m+1} = \ldots = n_k .$$

If (1) is an exactly covering system and z is a complex number, with $|z| < 1$, then

$$\sum_{i=0}^{\infty} z^i = \sum_{\substack{i \equiv a_1(n_1) \\ i \geqq 0}} z^i + \sum_{\substack{i \equiv a_2(n_2) \\ i \geqq 0}} z^i + \ldots + \sum_{\substack{i \equiv a_k(n_k) \\ i \geqq 0}} z^i .$$

In the case of $n_1 < n_2 < \ldots < n_{k-m} < n_{k-m+1} = \ldots = n_k$ we get

$$\frac{1}{1-z} = \frac{z^{a_1}}{1-z^{n_1}} + \frac{z^{a_2}}{1-z^{n_2}} + \ldots + \frac{z^{a_{k-m}}}{1-z^{n_{k-m}}} + \frac{z^{a_{k-m+1}} + \ldots + z^{a_k}}{1-z^{n_k}} .$$

Let us suppose $z \to e^{\frac{2\pi i}{n_k}}$, $|z| < 1$. If

$$\lim_{\substack{z \to e^{\frac{2\pi i}{n_k}} \\ |z|<1}} (z^{a_{k-m+1}} + \ldots + z^{a_k}) \neq 0,$$

then the last fraction of the right-hand side tends to infinity but the remainig ones are finite. This is a contradiction and therefore

$$\lim_{\substack{z \to e^{\frac{2\pi i}{n_k}} \\ |z|<1}} (z^{a_{k-m+1}} + \ldots + z^{a_k}) = 0,$$

i. e.

$$e^{\frac{2\pi i}{n_k} a_{k-m+1}} + e^{\frac{2\pi i}{n_k} a_{k-m+2}} + \ldots + e^{\frac{2\pi i}{n_k} a_k} = 0.$$

In the following we shall need the following lemma.

Lemma 3. *Let* $p, b_1, \ldots, b_p, m$ *be integers with* $2 \leqq p \leqq 5$, $0 \leqq b_1 < \ldots < b_p < m$. Let

$$e^{\frac{2\pi i}{m} b_1} + e^{\frac{2\pi i}{m} b_2} + \ldots + e^{\frac{2\pi i}{m} b_p} = 0.$$

Let no partial sum of this sum vanish. Then

a) *the residual class* $b_1 \pmod{m/p}$ *contains exactly those integers which belong to the system*

$$b_1 (\mathrm{mod}\, m),\ b_2 (\mathrm{mod}\, m), \ldots, b_p (\mathrm{mod}\, m),$$

if $p = 2, 3,$ *and* 5.

b) *the case* $p = 4$ *is impossible.*

Proof. Let us assign to the complex numbers $e^{\frac{2\pi i}{m} b_1}, \ldots, e^{\frac{2\pi i}{m} b_p}$ the vectors $\mathbf{z}_1, \ldots, \mathbf{z}_p$ in the usual way.

a) If $p = 2$, then the vectors $\mathbf{z}_1, \mathbf{z}_2$ are conversely oriented and so

$$\frac{2\pi}{m} b_1 + \pi = \frac{2\pi}{m} b_2.$$

We have from here

$$b_1 + \frac{m}{2} = b_2.$$

In case $p = 3$ (resp. $p = 5$) we get according to Theorem 6 in [2] that the vectors $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ (resp. $\mathbf{z}_1, \ldots, \mathbf{z}_5$) form a regular triangle (resp. a regular pentagon). Then

$$\frac{2\pi}{m} b_{i-1} + \frac{2\pi}{3} = \frac{2\pi}{m} b_i, \quad i = 2, 3,$$

resp.

$$\frac{2\pi}{m} b_{i-1} + \frac{2\pi}{5} = \frac{2\pi}{m} b_i, \quad i = 2, 3, 4, 5.$$

After some modifications we get

$$b_i = b_{i-1} + \frac{m}{3}, \quad i = 2, 3,$$

resp.

$$b_i = b_{i-1} + \frac{m}{5}, \quad i = 2, 3, 4, 5.$$

We see that the residual class $b_1 (\mathrm{mod}\, m/p)$ consequently contains those numbers which are contained in the system

$$b_1 (\mathrm{mod}\, m),\ b_2 (\mathrm{mod}\, m), \ldots, b_p (\mathrm{mod}\, m)$$

and not any other if $p = 2, 3$ and 5.

b) From the mentioned Theorem 6 in [2] it follows that the case $p = 4$ is impossible. (We can obtain this statement also if we consider that the vectors $\mathbf{z}_1, \ldots, \mathbf{z}_4$ form a parallelogram.)

Now we shall prove some theorems analogous to Lemmas 1 and 2.

Theorem 1. *Let* (1) *be an exactly covering system. Let there exist exactly one 4-tuple of the residual classes with respect to the same module and the remaining ones are distinct. Then*

$$n_i = 2^i \text{ for } i = 1, 2, \ldots, k-4;\ n_{k-3} = n_{k-2} = n_{k-1} = n_k = 2^{k-2}$$

or

$$n_i = 2^i \text{ for } i = 1, 2, \ldots, k-5;\ n_{k-4} = 3 \,.\, 2^{k-5},\ n_{k-3} = \ldots = n_k = 3.2^{k-4}.$$

Proof. As we have pointed out we can order the residual classes in (1) so that $n_1 < n_2 < \ldots < n_{k-3} = n_{k-2} = n_{k-1} = n_k$. We know that the following holds for (1)

$$e^{\frac{2\pi i}{n_k} a_{k-3}} + e^{\frac{2\pi i}{n_k} a_{k-2}} + e^{\frac{2\pi i}{n_k} a_{k-1}} + e^{\frac{2\pi i}{n_k} a_k} = 0.$$

Let us decompose this sum into partial sums so that none of them contains any vanishing subsum. This is possible in a unique way (with a suitable changing of the indices of a_i s), i. e.

$$e^{\frac{2\pi i}{n_k} a_{k-3}} + e^{\frac{2\pi i}{n_k} a_{k-2}} = e^{\frac{2\pi i}{n_k} a_{k-1}} + e^{\frac{2\pi i}{n_k} a_k} = 0.$$

Then we can modify the system (1) into the form (see Lemma 3)

(2) $\quad a_i \pmod{n_i},\ i = 1, \ldots, k-4;\quad a_{k-3} \pmod{n_k/2},\ a_{k-1} \pmod{n_k/2}.$

If $n_k/2 < n_{k-4}$, the system (2) is exactly covering with exactly one residual class with respect to the greatest module, which is impossible owing to the result in [1].

Let $n_k/2 = n_{k-4}$. Then the exactly covering system (2) contains exactly one triple of the residual classes with respect to the same module and the other modules are distinct. Owing to Lemma 2 we have

$$n_i = 2^i,\ i = 1, \ldots, k-5;\quad n_{k-4} = \frac{n_k}{2} = 3 \,.\, 2^{k-5}$$

and after some modification

$$n_i = 2^i,\ i = 1, \ldots, k-5;\quad n_{k-4} = 3 \,.\, 2^{k-5},\ n_{k-3} = n_{k-2} = n_{k-1} = n_k = 3 \,.\, 2^{k-4}.$$

In case $n_k/2 > n_{k-4}$ we obtain an exactly covering system with one couple of the residual classes with respect to the greatest module. From Lemma 1 we have

$$n_i = 2^i,\ i = 1, \ldots, k-4;\quad n_{k-3} = n_{k-2} = n_{k-1} = n_k = 2^{k-2},$$

which proves our theorem.

Theorem 2. *Let* (1) *be an exactly covering system. Let* (1) *contain exactly one 5-tuple of the residual classes with respect to the same module and the others are distinct. Then*

$$n_i = 2^i,\ i = 1, \ldots, k-5;\ n_{k-4} = n_{k-3} = n_{k-2} = n_{k-1} = n_k = 5.2^{k-5}.$$

Proof. We can suppose similarly as in the preceding case that $n_{k-4} = n_{k-3} = n_{k-2} = n_{k-1} = n_k$ and

(3) $$e^{\frac{2\pi i}{n_k} a_{k-4}} + e^{\frac{2\pi i}{n_k} a_{k-3}} + e^{\frac{2\pi i}{n_k} a_{k-2}} + e^{\frac{2\pi i}{n_k} a_{k-1}} + e^{\frac{2\pi i}{n_k} a_k} = 0.$$

Let us distinguish the following cases (with a suitable changing of indices of a_i, s):

a) $$e^{\frac{2\pi i}{n_k} a_{k-4}} + e^{\frac{2\pi i}{n_k} a_{k-3}} + e^{\frac{2\pi i}{n_k} a_{k-2}} = e^{\frac{2\pi i}{n_k} a_{k-1}} + e^{\frac{2\pi i}{n_k} a_k} = 0,$$

where no of these two sums contains a vanishing partial sum. By Lemma 3 the system

(4) $$a_i \pmod{n_i},\ i = 1, \ldots, k-5;\quad a_{k-4} (\bmod\ n_k/3),\ a_{k-1} (\bmod\ n_k/2)$$

is exactly covering, too. Since an exactly covering system contains at least two residual classes with respect to the greatest module, we have $\frac{n_k}{2} = n_{k-5}$.

Since $\frac{n_k}{3} \neq \frac{n_k}{2}$, (4) is an exactly covering system with two residual classes with respect to the greatest module. From Lemma 3 it follows that

$$a_i \pmod{n_i},\ i = 1, \ldots, k-6;\quad a_{k-4} (\bmod\ n_k/3),\ a_{k-1} (\bmod\ n_k/4)$$

is an exactly covering system. Evidently $n_k/3 > n_k/4$. Owing to the same consideration we have $n_{k-6} = n_k/3$ and therefore the system

$$a_i (\bmod\ n_i),\ i = 1, \ldots, k-7;\quad a_{k-4} (\bmod\ n_k/6),\ a_{k-1} (\bmod\ n_k/4)$$

is exactly covering. Again $\frac{n_k}{6} \neq \frac{n_k}{4}$.

Thus we can reduce the number of residual classes (4) to two

(5) $$a_{k-4} \left(\bmod\ \frac{n_k}{2^\alpha . 3}\right),\ a_{k-1} \left(\bmod\ \frac{n_k}{2^\beta}\right),\ \alpha, \beta \text{ are natural numbers}$$

and this system is exactly covering. For arbitrary natural numbers α, β we have $2^\alpha . 3 \neq 2^\beta$. Therefore (5) is an exactly covering system with two distinct modules, which is a contradiction with the result in [1], i. e. this case is impossible.

b) The sum (3) contains no vanishing partial sum. Then we can modify the system (1) by Lemma 3 into the form

$$a_i (\bmod\ n_i),\ i = 1, \ldots, k-5;\quad a_k (\bmod\ n_k/5).$$

This is an exactly covering system with one couple of the residual classes with respect to the same module. We get from Lemma 1 that

$$n_i = 2^i,\ i = 1, \ldots, k-5;\quad n_{k-4} = n_{k-3} = n_{k-2} = n_{k-1} = n_k = 5 . 2^{k-5}.$$

There are no other possibilities of the decomposition of the sum (3) into sums with no vanishing partial sum and therefore the proof is finished.

The cases with $m = 2, 3$ and 5 indicate the following statement: Let m be a prime number. Let exactly one m-tuple of residual classes with respect to the same module appear in the exactly covering system (1) and let the remaining modules be distinct. Then the exactly covering system (1) is uniquely determined. However, the following holds:

Theorem 3. *Let the exactly covering system* (1) *contain exactly one* 7*-tuple of the residual classes with respect to the same module whereas the other modules are distinct. Then at least the following three cases can hold*

$n_i = 2^i,\ i = 1, \ldots, k-9;$

$$n_{k-8} = 2^{k-7},\ n_{k-7} = 3 \,.\, 2^{k-8},\ n_{k-6} = \ldots = n_k = 3 \,.\, 2^{k-7}$$

$n_i = 2^i,\ i = 1, \ldots, k-10;$

$$n_{k-9} = 3 \,.\, 2^{k-10},\ n_{k-8} = 3 \,.\, 2^{k-9},\ n_{k-7} = 3^2 \,.\, 2^{k-10},$$

$$n_{k-6} = \ldots = n_k = 3^2 \,.\, 2^{k-9}$$

$n_i = 2^i,\ i = 1, \ldots, k-7;$

$$n_{k-6} = n_{k-5} = \ldots = n_k = 7 \,.\, 2^{k-7}.$$

Proof. We get similarly as before that $n_{k-6} = \ldots = n_k$ and

$$e^{\frac{2\pi i}{n_k} a_{k-6}} + e^{\frac{2\pi i}{n_k} a_{k-5}} + \ldots + e^{\frac{2\pi i}{n_k} a_k} = 0.$$

Let

$$e^{\frac{2\pi i}{n_k} a_{k-6}} + e^{\frac{2\pi i}{n_k} a_{k\ 5}} = e^{\frac{2\pi i}{n_k} a_{k-4}} + e^{\frac{2ni}{n_k} a_{k-3}} = e^{\frac{2\pi i}{n_k} a_{k-2}} + e^{\frac{2\pi i}{n_k} a_{k-1}} + e^{\frac{2\pi i}{n_k} a_k} = 0.$$

Then we can modify the system (1) into the form

$a_i(\mathrm{mod}\ n_i),\ i = 1, \ldots, k-7;\ a_{k-2}(\mathrm{mod}\ n_k/3),\ a_{k-4}(\mathrm{mod}\ n_k/2),\ a_{k-6}(\mathrm{mod}\ n_k/2).$

Let $\frac{n_k}{2} = n_{k\ 7}$ and $\frac{n_k}{3} = n_{k-8}$. Then from the latest system we get the following one

$$a_i(\mathrm{mod}\ n_i),\ i = 1, \ldots, k-9;\quad a_{k-2}(\mathrm{mod}\ n_k/6),\ a_{k-4}(\mathrm{mod}\ n_k/6),$$

where either $n_{k-9} = \frac{n_k}{6}$ or $n_{k-9} \neq \frac{n_k}{6}$. In the first case we get, owing to Lemma 2,

$$n_i = 2^i,\ i = 1, \ldots, k-10;\quad n_{k-9} = \frac{n_{k-8}}{2} = \frac{n_{k-7}}{3} = \frac{n_k}{6} = 3 \,.\, 2^{k-10},$$

and in the second by Lemma 1

$$n_i = 2^i, \ i = 1, \dots, k - 9; \quad \frac{n_{k-8}}{2} = \frac{n_{k-7}}{3} = \frac{n_k}{6} = 2^{k-8},$$

i. e. the first and the second possibility of our theorem.

The third possibility is obtained if we take an exactly covering system of Lemma 1 with $k - 6$ residual classes and we cover one of the couple of residual classes with seven residual classes modulo $7 \, . \, 2^{k-7}$. (To this possibility leads the case where the vectors $\mathbf{z}_1, \dots, \mathbf{z}_7$ assigned to the complex numbers

$$e^{\frac{2\pi i}{n_k} a_{k-6}}, \dots, e^{\frac{2\pi i}{n_k} a_k}$$

form a regular heptagon.)

Remark. It seems there are no more possibilities of exactly covering systems with one 7-tuple of the residual classes with respect to the same module as in Theorem 3.

REFERENCES

[1] ERDÖS, P.: Egy kongruenciarendszerekröl szóló problémáról. Mat. lapok, *3*, 1952, 122—128.

[2] MANN, H. B.: On linear relations between roots of unity. Mathematika, *12*, 1965, 107—117.

[3] STEIN, S. K.: Unions of arithmetical sequences. Math. Ann., *134*, 1958, 289—294.

[4] ZNÁM, Š.: On exactly covering system of arithmetic sequences. Math. Ann., *180*, 1969, 227—232.

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